EX: Outdoor measurements of temperature at a weather station indicate that the temperature at 6am, noon, 6pm are jointly Gaussian distributed, denoted as X_1, X_2, X_3 with variance $16^\circ F$ and expected values $50^\circ F, 62^\circ F, 58^\circ F$. The covariance matrix of the 3 measurements is: (1)

$$C_{\underline{X}} = \begin{bmatrix} 16.0 & 12.8 & 11.2 \\ 12.8 & 16.0 & 12.8 \\ 11.2 & 12.8 & 16.0 \end{bmatrix}$$

(a) Write the joint pdf of X_1 and X_2 in algebraic notation. (x_1, x_2 not vectors).

1) X_1, X_2 are jointly Gaussian (any subset of jointly Gaussian random vectors is again Gaussian)

2) $$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[\frac{-\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} - \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right]$$

Thus, we read off: $\mu_1 = 50$, $\mu_2 = 62$, $\sigma_1 = \sigma_2 = 4$, $\rho = \frac{cov(X_1,X_2)}{\sigma_1\sigma_2} = \frac{12.8}{16} = 0.8$

Plug these into

(b) Write the joint pdf of X_1, X_2 in vector notation.

Let $\underline{W} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $\mu_{\underline{W}} = \begin{bmatrix} 50 \\ 62 \end{bmatrix}$, $C_{\underline{W}} = \begin{bmatrix} 16.0 & 12.8 \\ 12.8 & 16.0 \end{bmatrix}$, $\det(C_W) = 92.16$

$|\det(C_W)|^{1/2} = 9.6$

then $$f_{\underline{W}}(\underline{w}) = \frac{1}{60.3} \exp\left(-\frac{1}{2}(\underline{w}-\mu_{\underline{w}})^T C_{\underline{W}}^{-1}(\underline{w}-\mu_{\underline{w}})\right)$$

with the values as above.

Thm: $\underline{X} \sim \mathcal{N}(\mu_{\underline{X}}, C_{\underline{X}})$ $(n\times 1)$.

and $\underline{Y} = A\underline{X} + b$ where A is $m\times n$, b is $m\times 1$

then $\underline{Y} \sim \mathcal{N}\left(\mu_{\underline{Y}} = A\mu_{\underline{X}} + b,\ C_{\underline{Y}} = A\, C_{\underline{X}}\, A^T\right)$

(in 1-D this is like $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ if $Y = aX + b$ then
$Y \sim \mathcal{N}(a\mu_X + b,\ a^2\sigma_X^2)$)

Continuing the temperature example ....
Let's convert everything to °C. $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ is temp. in °F
what is the the pdf of $\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$ where $\underline{Y}$ is temp in °C ??

Recall that $Y_i = \left(\frac{5}{9}\right)(X_i - 32)$

1) $\underline{Y}$ is again jointly Gaussian as we're taking linear combination of jointly Gaussian!

2) $\underline{Y} \sim \mathcal{N}\left(\mu_{\underline{Y}} = A\mu_{\underline{X}} + \underline{b} = \begin{bmatrix} 10 \\ 50/3 \\ 130/9 \end{bmatrix},\ C_{\underline{Y}} = A\, C_{\underline{X}}\, A^T\right)$

$$\underline{Y} = A\underline{X} + b = \underbrace{\begin{bmatrix} 5/9 & 0 & 0 \\ 0 & 5/9 & 0 \\ 0 & 0 & 5/9 \end{bmatrix}}_{A} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \underbrace{\begin{bmatrix} -160/9 \\ -160/9 \\ -160/9 \end{bmatrix}}_{b}$$

Then $\mu_{\underline{Y}} = A\mu_{\underline{X}} + b = \begin{bmatrix} 10 \\ 50/3 \\ 130/9 \end{bmatrix}$, $C_{\underline{Y}} = A\, C_{\underline{X}}\, A^T = (5/9)^2\, C_{\underline{X}}$

Ch. 6 Sum of RVs

Consider X, Y RVs which are not necessarily independent.
(This means we know their joint pdf $f_{XY}(x,y)$ which does not necessarily factor as $f_X(x) f_Y(y)$.)

Define a new RV, $W = X + Y$

E.g.: Let X be dice roll and Y be dice roll, independent.

Then $f_{XY}(x,y) = \frac{1}{36}$ for each of the 36 possible pairs

$$(X,Y) \in \underbrace{\{(1,1), (1,2), \ldots, (1,6), (2,1), (2,2), \ldots, (2,6), \ldots, (6,1), (6,2), \ldots, (6,6)\}}_{S_{XY}}$$

→ What is the pmf of W, $P_W(w)$?

1) $S_W = \{2, 3, \ldots, 12\}$

$P_W(2) = P[(1,1)] = \frac{1}{36}$

$P_W(3) = P[(1,2) \cup (2,1)] = P[(1,2)] + P[(2,1)] = \frac{2}{36}$.

$P_W(4) = P[(1,3) \cup (2,2) \cup (3,1)] = \frac{3}{36}$

$P_W(5) = \ldots = 4/36$

$P_W(6) = \ldots = 5/36$.

$P_W(7) = \ldots = 6/36 = 1/6$.

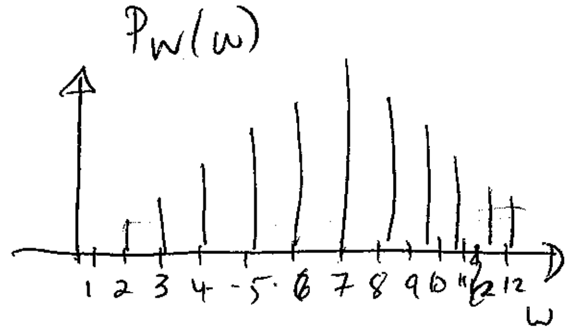
$P_W(8) = \ldots = 5/36$

$P_W(9) = \ldots = 4/36$

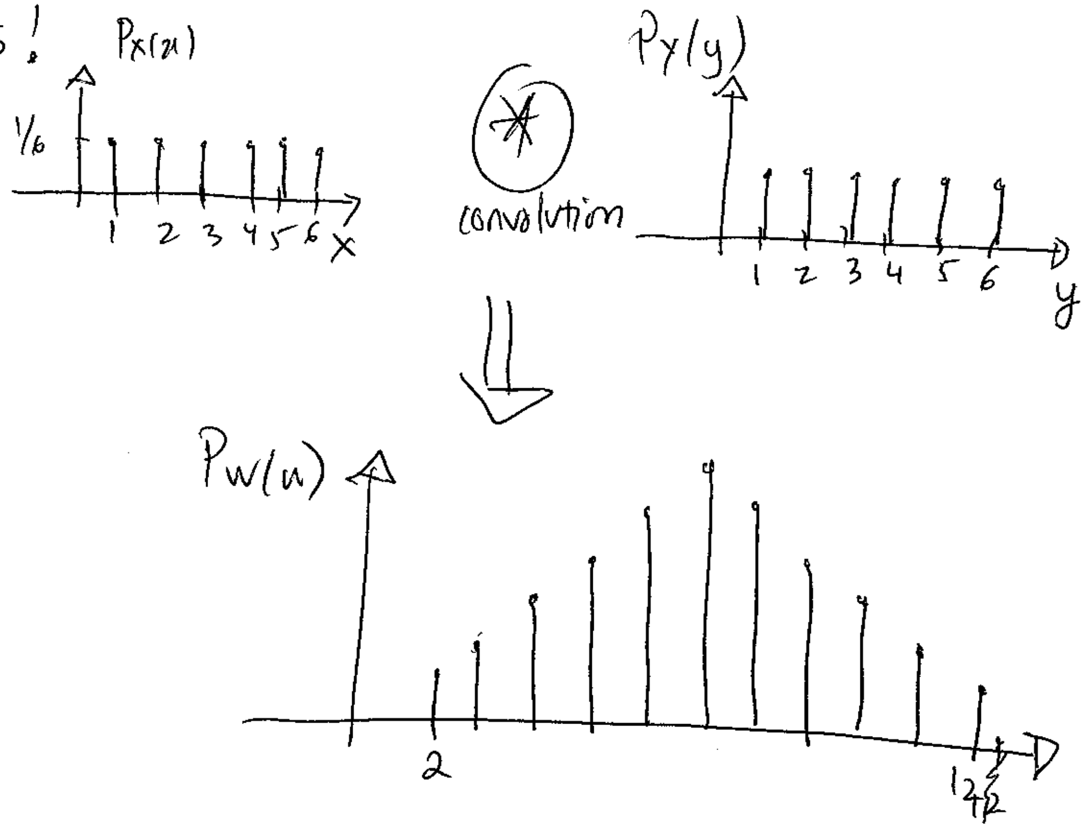
$P_W(10) = \ldots = 3/36$

$P_W(11) = \ldots = 2/36$

$P_W(12) = 1/36$



Notice that $P_W(w)$ looks like the convolution of 2 rectangular pdfs!



NOTE: When X, Y are independent then pdf of W = X + Y ~~has 2 pdf~~ is the convolution of the pdfs of X and Y.