Sums of Random Variables (Ch. 6)

Say we have 2 RVs, X and Y with joint pdf $f_{XY}(x,y)$ and marginals $f_X(x)$ and $f_Y(y)$.

<u>Question</u>: What is the pdf $f_W(w)$ of the new RV

$$W = X + Y \quad ?$$

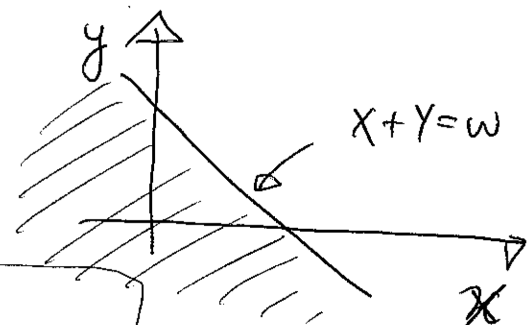
To find the pdf $f_W(w)$, we use the "usual approach":

1) Find the CDF $F_W(w) = P[W \leq w]$
2) Take derivative of CDF to find PDF, $f_W(w) = \frac{d}{dw} F_W(w)$.

Applying to $W = X + Y$:

1) $F_W(w) \triangleq P[W \leq w] = P[X+Y \leq w] = \underset{\{X+Y\leq w\}}{\iint} f_{XY}(x,y)\,dx\,dy.$

$$= \int_{-\infty}^{+\infty} \underbrace{\int_{-\infty}^{y=w-x} f_{XY}(x,y)\,dy}\,dx$$



2) $$\boxed{f_W(w) = \frac{d}{dw}\left[F_W(w)\right] = \int_{-\infty}^{+\infty} f_{XY}(x, w-x)\,dx.} \; *$$

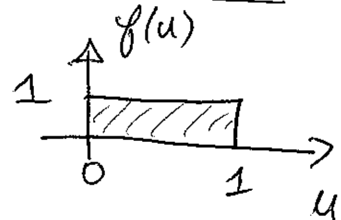
This is true in GENERAL, for all X, Y even those that are <u>NOT</u> independent!

If X and Y are independent then $f_{XY}(x,y) = f_X(x) f_Y(y)$ by definition of independence.

This means $f_{XY}(x, w-x) = f_X(x) f_Y(w-x)$. and so

$$\boxed{f_W(w) = \int_{-\infty}^{+\infty} f_X(x) f_Y(w-x)\,dx. \qquad \text{CONVOLUTION of PDFs of } X \text{ and } Y! \\ = f_X * f_Y}$$

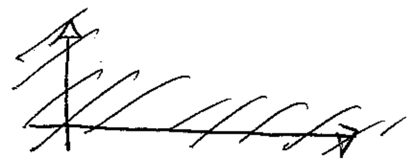
This is only true for X, Y independent.

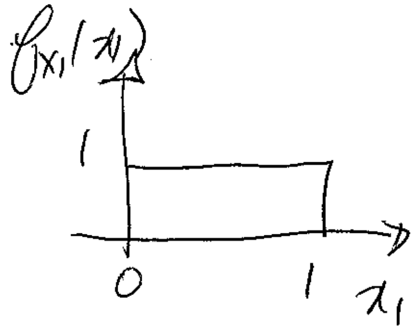


E.g: Let X_1, X_2 be independent and uniform on $[0,1]$.

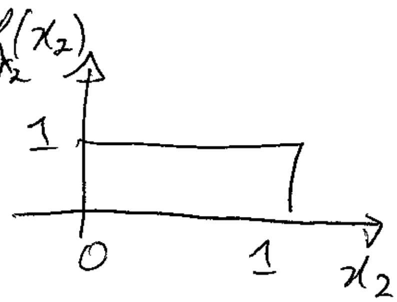
$$\Rightarrow \quad f_{X_1}(x_1) = f_{X_2}(x_2) = f(u) = \begin{cases} 1 & \text{for } 0 \le u \le 1 \\ 0 & \text{else} \end{cases}$$

What is the pdf of $W = X_1 + X_2$??

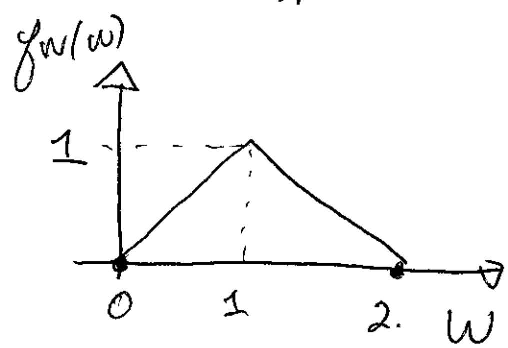




$*$

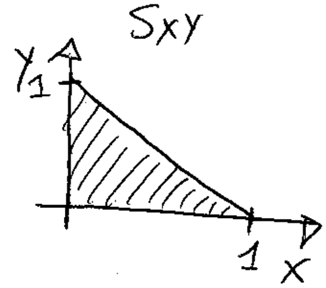


$=$



E.g. Find the PDF of $W = X + Y$ when X, Y have the joint PDF

$$f_{XY}(x,y) = \begin{cases} 2 & 0 \le y \le 1,\ 0 \le x \le 1,\ x+y \le 1 \\ 0 & \text{else.} \end{cases}$$



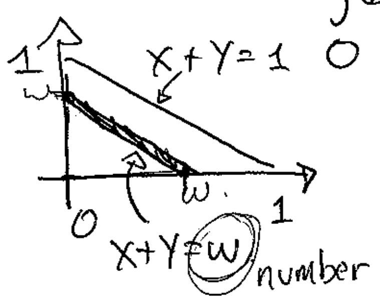
Since X, Y are not necessarily independent, use the general formula:

$$f_W(w) = \int_{-\infty}^{+\infty} f_{XY}(x, w-x)\, dx.$$

What is S_W? $= [0,1]$ so $f_W(w) = \begin{cases} 0 & w < 0 \text{ or } w > 1 \\ \sim & w \in [0,1]. \end{cases}$

For $w \in [0,1]$, $w = x + y \Rightarrow y = w - x$.

$$f_W(w) = \int_{-\infty}^{+\infty} f_{XY}(x, w-x)\, dx = \int_0^w \underbrace{f_{XY}(x, w-x)}_{2}\, dx = \int_0^w 2\, dx = 2w.$$



integrate along the line $x + y = w$

Problem 6.2.6 (◇)

Show that the pdf of two independent Poisson RVs is again a Poisson RV.

Let J and K be Poisson RVs with expected values α, β respectively.
Show that $N = J + K$ is again Poisson with ($\alpha = \lambda T$)
expected value $\alpha + \beta$.

Proof: Since J, K are independent, we can use the discrete convolution formula for the pdf of $N = J + K$.

$$P_N(n) = \sum_{k=-\infty}^{+\infty} P_J[J = n-k]\, P_K[K = k] = \sum_{k=0}^{n} \frac{\alpha^{(n-k)} e^{-\alpha}}{(n-k)!} \frac{\beta^k e^{-\beta}}{k!}$$

TRICK

$$\boxed{\overset{\text{WANT}}{=} \frac{(\gamma)^n e^{-\gamma}}{n!} \quad \text{for some } \gamma \text{ to be determined}}$$

$$= \frac{(\alpha+\beta)^n e^{-(\alpha+\beta)}}{n!} \underbrace{\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(\frac{\alpha}{\alpha+\beta}\right)^{n-k} \left(\frac{\beta}{\alpha+\beta}\right)^{k}}_{1}$$

$$= \frac{(\alpha+\beta)^n e^{-(\alpha+\beta)}}{n!}$$

as sum of a binomial RV over all possible values

$$P[b = k] = \frac{n!}{k!(n-k)!} p^{n-k} (1-p)^k$$

binomial corresponds to # heads in n Bernoulli trials. with prob(heads) = $1-p$.

$\Rightarrow$ Poisson for $\gamma = \alpha + \beta$.