Linear MMSE

Linear estimation of one RV based on knowledge of another RV and their joint statistics.

joint pdf (continuous)
joint pmf (discrete).

mean X, std X

Let's assume that RVs X and Y are described by (μ_X, σ_X), (μ_Y, σ_Y) and ρ_{XY} (correlation coeff. btw. X, Y)

mean Y, std Y

If we know X, can we estimate Y?
We can approximate samples of Y as the following linear combination of samples of X:

approx. as

$$Y \approx \hat{Y} = aX + b$$

estimate of Y. constants a, b.

Question: Find a, b that minimize the mean squared error.
Linear (MMSE)

$$MSE = E[(Y-\hat{Y})^2] = E[(Y-(aX+b))^2] \triangleq \varepsilon.$$

To find optimal a, b take derivative of ε wrt. a, b and set to 0.

~~$\frac{\partial \varepsilon}{\partial a} =$~~ Let's expand ε.

linearity of expectation

$$\varepsilon = E[Y^2 - 2Y(aX+b) + (aX+b)^2]$$
$$= E[Y^2] - 2aE[XY] - 2bE[Y] + a^2E[X^2] + 2abE[X] + E[b^2]$$

$$\frac{\partial \varepsilon}{\partial a} = -2E[XY] + 2aE[X^2] + 2bE[X] \overset{\text{WANT}}{=} 0$$

$$\frac{\partial \varepsilon}{\partial b} = -2E[Y] + 2aE[X] + 2b \overset{\text{WANT}}{=} 0$$

2 linear equations in 2 unknowns so can solve This!

Solve this for a, b.

$$\Rightarrow \quad b = E[Y] - aE[X] = \mu_Y - a\mu_X \quad \text{(sub into 1st equation).}$$

$$\Rightarrow \quad a = \frac{E[XY] - E[Y]E[X]}{E[X^2] - (E[X])^2} = \frac{\text{Cov}(X,Y)}{\text{var}(X)}$$

$$= \frac{\rho_{XY}\sqrt{\text{var}(X)\text{var}(Y)}}{\text{Var}(X)} = \frac{\rho_{XY}\sqrt{\text{var}(Y)}}{\sqrt{\text{var}(X)}} = \frac{\rho_{XY}\,\sigma_Y}{\sigma_X}$$

Thus,

$$\hat{Y} = \rho_{XY}\left(\frac{\sigma_Y}{\sigma_X}\right)X + \left(\mu_Y - \rho_{XY}\left(\frac{\sigma_Y}{\sigma_X}\right)\mu_X\right)$$

$$\boxed{\hat{Y} = \rho_{XY}(X - \mu_X)\left(\frac{\sigma_Y}{\sigma_X}\right) + \mu_Y}$$

LINEAR MMSE

Ch. 5 Random Vectors

We saw 1 RV X, pdf $f_X(x)$, cdf $F_X(x)$,

2 RVs, X, Y with joint pdf $f_{XY}(x,y)$, joint cdf $F_{XY}(x,y)$.

Now we look at 3 or more RVs $X_1, X_2, \ldots, X_n$ with

- joint pdf (continuous) $f_{X_1 X_2 \cdots X_n}(x_1, x_2, \ldots, x_n) = \dfrac{\partial^n F_{X_1 \cdots X_n}(x_1, x_2, \ldots, x_n)}{\partial x_1\, \partial x_2 \cdots \partial x_n}$

where

$$F_{X_1 X_2 \cdots X_n}(x_1, x_2, \ldots, x_n) \triangleq P[X_1 \le x_1, \underset{\text{AND}}{} X_2 \le x_2, \underset{\text{AND}}{} \ldots, X_n \le x_n]$$

joint CDF
cdf

- joint pmf (discrete)

$$P_{X_1 X_2 \cdots X_n}(x_1, x_2, \ldots, x_n) \triangleq P[X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n].$$

<u>Notation</u>: $P_{X_1 \cdots X_n}(x_1, \ldots, x_n) \triangleq P_{\bar{X}}(\bar{x})$ $\quad (X_1, \ldots, X_n \triangleq \bar{X})$

$f_{X_1 \cdots X_n}(x_1, \ldots, x_n) \triangleq f_{\bar{X}}(\bar{x})$

<u>Properties</u>: $P_{\bar{X}}(\bar{x}) \geq 0$, $\displaystyle\sum_{\bar{x} \in S_{\bar{X}}} P_{\bar{X}}(\bar{x}) = 1$. (PMF)

$f_{\bar{X}}(\bar{x}) \geq 0$, $\displaystyle\int_{\bar{x} \in S_{\bar{X}}} f_{\bar{X}}(\bar{x})\, d\bar{x} = 1$ (PDF)

EX: $f_{X_1 X_2 X_3}(x_1, x_2, x_3) = \begin{cases} c & 0 \le x_1 \le 1,\ 0 \le x_2 \le 3,\ 0 \le x_3 \le 4 \\ 0 & \text{else} \end{cases}$

- Find c: $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X_1 X_2 X_3}(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 \overset{\text{WANT}}{=} 1$

$$\int_{x_3=0}^{4} \int_{x_2=0}^{3} \int_{x_1=0}^{1} c \cdot dx_1\, dx_2\, dx_3 = c \cdot (1)(3)(4) = c \cdot 12$$

$$\Rightarrow \boxed{c = \frac{1}{12}}$$

- Find $P[A]$ where $A = [X_1 \le 1,\ X_2 \le 1,\ X_3 \le 1]$.

$$P[A] = \iiint_{(x_1, x_2, x_3 \in A)} f_{X_1 X_2 X_3}(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 \qquad \text{(in general)}$$

$$= \int_{x_1=0}^{1} \int_{x_2=0}^{1} \int_{x_3=0}^{1} \frac{1}{12}\, dx_3\, dx_2\, dx_1 = \frac{1}{12}.$$

- Marginal densities $f_{X_1}(x_1)$, $f_{X_2}(x_2)$, $f_{X_3}(x_3)$

$$f_{X_1}(x_1) = \underbrace{\int_{x_n=-\infty}^{+\infty} \cdots \int_{x_2=-\infty}^{+\infty}}_{\text{all RVs except the one interested in!}} f_{X_1 X_2 \cdots X_n}(x_1, x_2, \ldots, x_n)\, dx_2 \cdots dx_n$$

(in general)

In our example,

$$f_{X_1}(x_1) = \int_{x_3=0}^{4} \int_{x_2=0}^{3} \frac{1}{12}\, dx_2\, dx_3 = 1 \quad \text{for} \quad \underline{\underline{0 \le x_1 \le 1}}$$

$$= \begin{cases} 1 & 0 \le x_1 \le 1 \\ 0 & \text{else.} \end{cases}$$

Can find

$$f_{X_2}(x_2) = \begin{cases} 1/3 & 0 \le x_2 \le 3 \\ 0 & \text{else} \end{cases}$$

$$f_{X_3}(x_3) = \begin{cases} 1/4 & 0 \le x_3 \le 4 \\ 0 & \text{else.} \end{cases}$$