Independent RVs

For 2 RVs, X, Y are independent if $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$.
For 2 events A, B are independent if $P[A \cap B] = P[A] \cdot P[B]$.

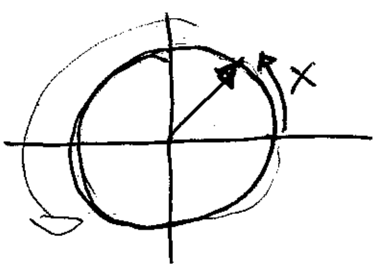
For n RVs $X_1, X_2, \ldots, X_n$ are independent if

- discrete PMF $P_{X_1 X_2 \cdots X_n}(x_1, x_2, \ldots, x_n) = P_{X_1}(x_1) P_{X_2}(x_2) \cdots P_{X_n}(x_n)$
- continuous PDF $f_{X_1 X_2 \cdots X_n}(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)$.

In the example from last class, X_1, X_2, X_3 were independent.

EX: Consider an experiment that consists of spinning a pointer on a wheel of circumference 1 m, n time (n independent spins) and we observe Y_n meters, the maximal position of the pointer in the n spins. Assume pointer is equally likely to land anywhere on the circle and that spins are independent. Find the CDF and PDF of Y_n.

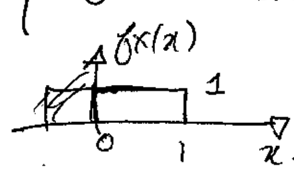
Sol:



PDF/CDF of one spin, RV X.

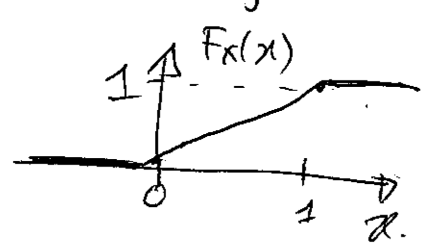
$$S_X = [0,1], \quad f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else.} \end{cases}$$

PDF of X



X Uniform RV on $[0,1]$!!

CDF of X is $F_X(x) \overset{\text{ALWAYS}}{=} \begin{cases} \int_{-\infty}^{x} f_X(u)\,du = \int_0^x 1 \cdot du = x & \text{for } 0 \le x \le 1 \\ 0 & x < 0 \\ 1 & x \ge 1. \end{cases}$



Let X_i = pointer location of spin i for $1 \le i \le n$.
These are n RVs $X_1, X_2, \ldots, X_n$.
For n independent spins $Y_n = \max(X_1, X_2, \ldots, X_n)$.
Want the PDF and CDF of Y_n.
First find CDF of Y_n, then take derivative to find PDF of Y_n.

$$F_{Y_n}(y) \overset{\text{DEF}}{=} P[Y_n \le y] = P[\max(X_1, X_2, \ldots, X_n) \le y]$$

dummy variable.

(Thm 5.7)

$$= P[X_1 \le y, X_2 \le y, \cdots, X_n \le y]$$
$$= P[X_1 \le y]\,P[X_2 \le y] \cdots P[X_n \le y]$$
$$= \left(P[X \le y]\right)^n$$

since all X_i's are distributed in the same way.

all X_i's are uniform on $[0,1]$.

$$= \left(F_X(y)\right)^n$$

$$= \begin{cases} 0 & y < 0 \\ 1 & y > 1 \\ y^n & 0 \le y \le 1 \end{cases}$$

CDF ✓

PDF

$$f_{Y_n}(y) = \frac{d}{dy}\left[F_{Y_n}(y)\right] = \begin{cases} ny^{n-1} & 0 \le y \le 1 \\ 0 & \text{else.} \end{cases}$$

This example showed how to find the PDF and CPF of <u>functions</u> of n RVs, i.e. first get CDF then ~~th~~ take derivative to get PDF.

Expected values:

Let $\bar{X} \triangleq \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ $(n \times 1)$ (my convention is to let $\bar{X}$ be a column vector).

$\bar{X}^T \triangleq [X_1 \cdots X_n]$. $(1 \times n)$ (T is "transpose").

$$E[\bar{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix}$$ is an $(n \times 1)$ vector.

$$E[g(\bar{x})] = \sum_{x_1 \in S_{X_1}} \sum_{x \in S_{X_2}} \cdots \sum_{x \in S_{X_n}} g(\bar{x})\, P_{\bar{X}}(\bar{x}) \quad \text{(discrete)}$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g(\bar{x})\, f_{\bar{X}}(\bar{x})\, d\bar{x} \quad \text{(continuous)}.$$

Theormen 5.11: If $\bar{X}$ is a continuous Random Vector and A is an invertible matrix, then $\bar{Y} = A\bar{X} + b$ has pdf.

$$f_{\bar{Y}}(\bar{y}) = \frac{1}{|\det(A)|} f_{\bar{X}}\left(A^{-1}(\bar{y} - b)\right)$$

(Result of the change of variables in integration formula).

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$\det(A)$ = determinant of $A = ad - bc$

$$A^{-1} = \text{inverse of } A = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Expected value + correlation matrix

- $E[\bar{X}]$ is a vector of ~~length~~ dimension $(n \times 1) = \mu_{\bar{X}}$
- Correlation of a random vector $\bar{X}$ is an $n \times n$ matrix, $R_{\bar{X}}$ whose (i,j) th element $R_{\bar{X}}(i,j) \triangleq E[X_i X_j]$

In vector notation.

(for 3×3, $\bar{X} = [X_1\ X_2\ X_3]^T$)

$$R_{\bar{X}} = E[\bar{X}\bar{X}^T] = \begin{bmatrix} E[X_1^2] & E[X_1 X_2] & E[X_1 X_3] \\ E[X_2 X_1] & E[X_2^2] & E[X_2 X_3] \\ E[X_3 X_1] & E[X_3 X_2] & E[X_3^2] \end{bmatrix}$$

($n \times n$ matrix)

example.

symmetric matrix meaning

$$R_{\bar{X}}^T = R_{\bar{X}}$$