Probability model: experiment ↗ procedure ↘ outcomes.

Axioms of probability

~~Kolmo~~ Kolmogorov approach: A "probability space" = a triple (S, F, P).

→ S = sample space (all possible outcomes)

→ F = collection of all events (all subsets of S)

→ P = a "probability measure" on F

A probability measure $P[\cdot]$ is a function from $F \to \mathbb{R}$ s.t.

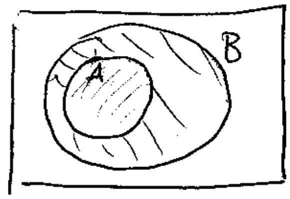
1. for any event $A \in F$, $P[A] \geq 0$.
2. $P[S] = 1$
3. for mutually exclusive events $A_1, A_2, \ldots$

$$P[A_1 \cup A_2 \cup \cdots] = P[A_1] + P[A_2] + \cdots$$

Consequences of the axioms of probabity (for A, B events in F)

- $P[\phi] = 0$
- $P[A \cup B] = P[A] + P[B]$ if $A \cap B = \phi$
- $P[A^c] = 1 - P[A]$
- $P[A \cup B] = P[A] + P[B] - P[A \cap B]$

- if $A \subset B$, then $P[A] \leq P[B]$



Proof: $P[B] = P[A] + \underbrace{P[A^c \cap B]}_{\geq 0 \text{ axiom 1}}$

$\Rightarrow \quad P[A] \leq P[B]$

- A is any event $\in \mathcal{F}$, $\{B_1, B_2, \ldots, B_m\}$ is any partition/event space, then

$$P[A] = \sum_{i=1}^{m} P[A \cap B_i]$$ "Theorem of total probability"

which follows from the event space decomposition

$$A = \underbrace{(A \cap B_1)} \cup \underbrace{(A \cap B_2)} \cup \cdots \cup \underbrace{(A \cap B_m)}$$

mutually exclusive

Notation: $P[\cdot]$ = probability of an event.
Denote events in $\{\cdot\}$

$$P[A \cap B] = P[A, B] = P[AB]$$

EX: Telephone company classifies calls as long (L), or brief (B) and observes whether calls are voice (V), data (D) or fax (F).
Experiment: observe a call + note whether it's L or B, and V, D or F.
Sample space has how many outcomes?

$S = \{LV, BV, LD, BD, LF, BF\}$

	V	D	F
L	0.3	0.12	0.15
B	0.2	0.08	0.15

Prob. brief data call?
0.08

$P[L] = 0.57 = P[LV] + P[LD] + P[LF]$.

$P[F] = P[LF] + P[BF] = 0.3$.

<u>Conditional probability</u>: for events $A, B \in \mathcal{F}$

$$\underbrace{P[A|B]}_{\text{prob. of event A given the occurence of event B.}} \triangleq \frac{P[A \cap B]}{P[B]} = \frac{P[A,B]}{P[B]} = \frac{P[AB]}{P[B]}$$

only defined if $P[B] > 0$

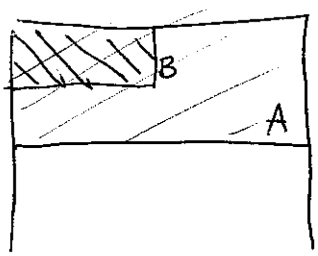
Similarly, $P[B|A] = \dfrac{P[AB]}{P[A]}$

$$\boxed{\text{Bayes' theorem: } P[B|A] = P[A|B] \times \frac{P[B]}{P[A]}}$$

$$= \frac{P[AB]}{P[A]} = \frac{P[A|B] \cdot P[B]}{P[A]}$$

<u>Example</u>:

S



<u>Given</u>: $P[A] = 1/2$, $P[B] = 1/8$

$B \subset A$. $B \subset A$, so $A \cap B = B$.

a) $P[A|B] = 1$ $\quad P[A|B] \overset{\text{def.}}{=} \dfrac{P[A,B]}{P[B]} = \dfrac{P[B]}{P[B]} = 1$.

b) $P[B|A] = \dfrac{1}{4}$ $\quad P[B|A] = \dfrac{P[AB]}{P[A]} = \dfrac{P[B]}{P[A]} = \dfrac{1/8}{1/2} = 1/4$

EX: Flip coin A → outcomes H_A, T_A, this outcome is going to determine ~~the when~~ how the next coin flip B is biased.

$P[H_A] = P[T_A] = 1/2$. (1st coin flip is fair).

If A is H_A, then $P[B = \text{Heads}] = 2/3$ (H_B = "B is heads"

A is T_A, then $P[B = \text{Tails}] = 2/3$. T_B = "B is tails")

$\therefore$ $P[H_B | H_A] = 2/3$, $P[T_B | H_A] = 1/3$.

$P[T_B | T_A] = 2/3$, $P[H_B | T_A] = 1/3$. $P[H_B|H_A] = \dfrac{P[H_A H_B]}{P[H_A]}$

① What is $P[H_A, H_B]$? $P[H_A, H_B] = P[H_A]\,P[H_B | H_A]$.

$= \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$.

Similarly $P[H_A, T_B] = 1/6$

$P[T_A, H_B] = 1/6$

$P[T_A, T_B] = 1/3$

Can draw a table:

	H_A	T_A
H_B	1/3	1/6
T_B	1/6	1/3

$P[H_A H_B]$.

$P[T_A T_B]$.

② What is $P[T_A]$?

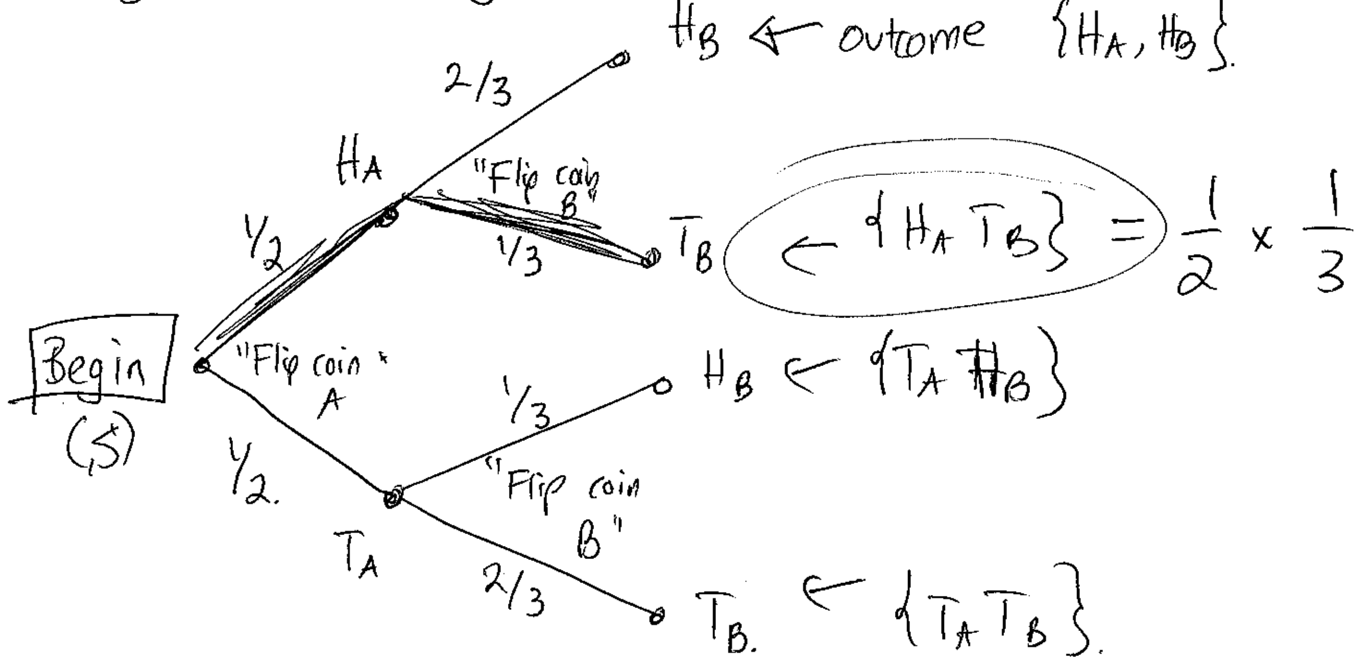
$1/2 = 1/6 + 1/3$

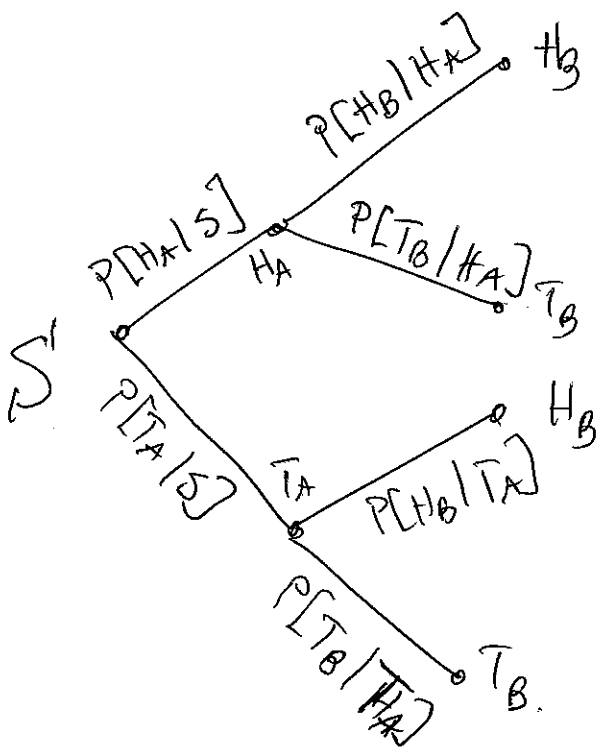
$P[T_A] = P[T_A \cap H_B] + P[T_A \cap T_B]$

$= 1/6 + 1/3$

as $\{H_B, T_B\}$ form an event space on the "B" toss!!

An alternative way of representing such a sequential process through a tree diagram:





Find $P[H_A, T_B]$?

Theorem of Total Probability

Let $\mathcal{E} = \{B_1, B_2, \ldots\}$ be a partition of S (or an event space)

Since $B_1, B_2, \ldots$ are mutually exclusive

$A \cap B_1, A \cap B_2, \ldots$ are " for any event A.

Then

$$P[A] = \sum_K P[A \cap B_K]$$

$$= \sum_K P[A \mid B_K]\, P[B_K].$$

EX: 1.19 of Rashid Ansari's slides

3 machines.

Independent events

Events A, B are independent iff $P[AB] = P[A] \cdot P[B]$.

Thus,

$$P[A \mid B] = P[A] = \frac{P[AB]}{P[B]} = \frac{P[A]\,P[B]}{P[B]}.$$

!! Independent $\neq$ disjoint !!

not in general

$A \cap B = \emptyset$, then $P[A \cap B] = 0 \neq P[A] \cdot P[B]$.

Ex. RN, Roll a dice twice:

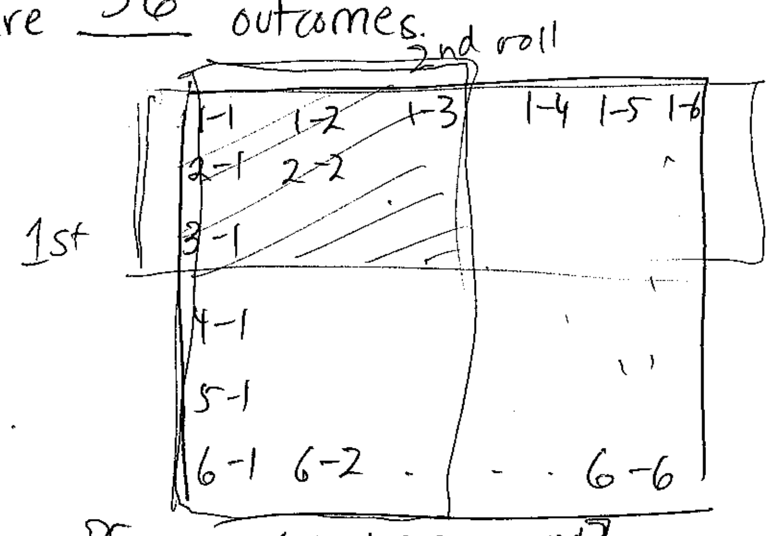
Roll a dice twice, so there are 36 outcomes.

Event A: First die 1, 2 or 3.

B: First die 3, 4 or 5

C: Sum of values is 9.

D: Second die is 1, 2, or 3.



$P[1,2,3 \text{ on 1st}, 1,2,3 \text{ on 2nd}]$

Are A, B independent? NO.

$P[A] = 1/2$, $P[B] = 1/2$, $P[A \cap B] = P[\{3\}] = 1/6$. (on first)

Are A, D independent? YES. $P[A] = 1/2$, $P[D] = 1/2$

$P[A \cap D] = 1/4 = P[A] \cdot P[D]$

Independence of 3 events:

A, B, C are independent $\iff$

① $P[A \cap B \cap C] = P[A] \cdot P[B] \cdot P[C]$.

② A, B ind. $(P[AB] = P[A] P[B])$

③ A, C ind.

④ B, C ind.

These are needed!!

EX: Take $S = \{1, 2, 3, 4\}$, take 4 equiprobable events $P[\{i\}] = 1/4$.

Let $A_1 = \{1, 3, 4\}$, $A_2 = \{2, 3, 4\}$, and $A_3 = \phi$

Are these independent?

$$P[A_1 A_2 A_3] = P[\phi] = 0 = P[A_1] \cdot P[A_2] \underbrace{P[A_3]}^{0}$$

But A_1, A_2 are dependent!

So A_1, A_2, A_3 are not independent!

Counting methods / Combinatorics

Consider selecting K objects from a given n distinguishable objects.
(e.g. $n=52$ cards, pick $K=2$ cards).

Q: How many possible selections?

- with/without replacement
- ordered / unordered selections.

With repl., orderded : 52×52

With repl., unordered : $\frac{52 \times 52}{2}$

Without replacement, ordered : 52×51

Without replacement, unordered : $\frac{52 \times 51}{2}$.

generalizing This is our goal!

a "permutation" = ordered sequence
e.g. THTTH as a result of flipping coins.

a "combination" = unordered set of outcomes
e.g. (3 Tails, 2 Heads)