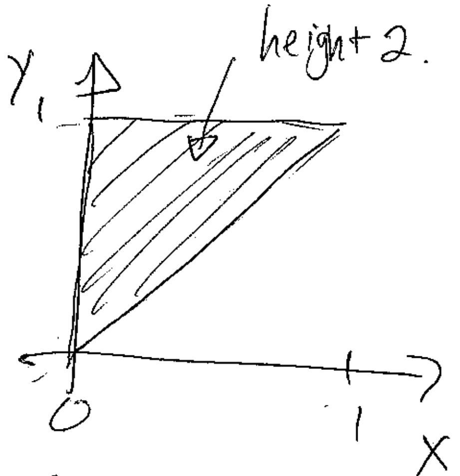
6.2.2: Find the PDF of $W = X+Y$ when X, Y have joint pdf

$$f_{XY}(x,y) = \begin{cases} 2 & 0 \le x \le y \le 1 \\ 0 & \text{else.} \end{cases}$$



X, Y are NOT independent!

Cannot use MGF, or convolution or any nice tricks.

Since continuous, find CDF first ~~$F_{XY}(x,y)$ then take~~ $F_W(w)$ then find $f_W(w) = \frac{d}{dw} F_W(w)$.

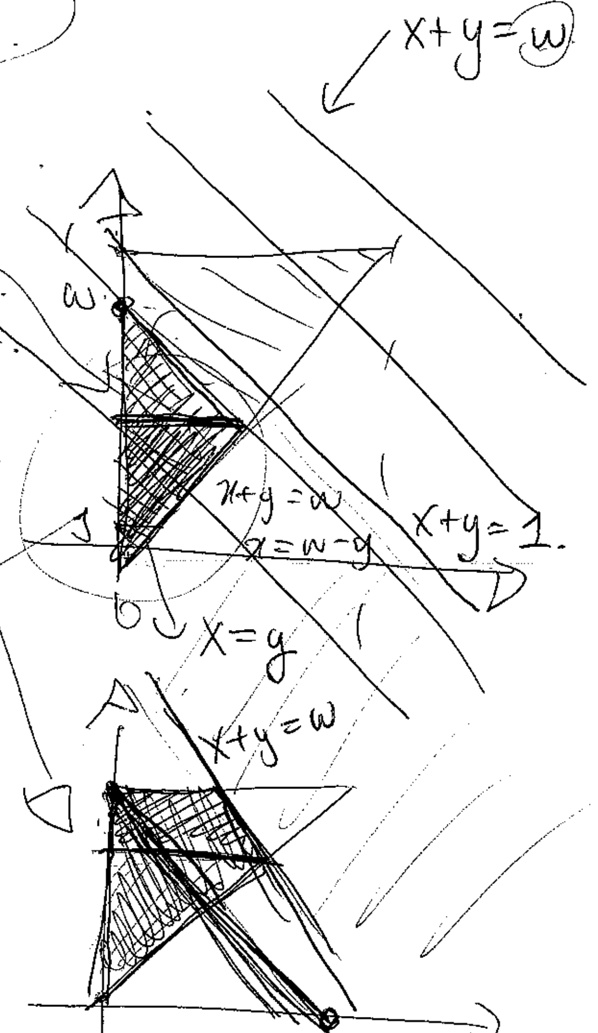
$$F_W(w) = P[W \le w] = P[X+Y \le w]$$

$$= \begin{cases} 0 & \text{for } w \le 0 \\ & 0 < w < 1 \\ & 1 < w < 2 \\ 1 & \text{for } w \ge 2. \end{cases}$$

for $w \in [0,2]$:

$$\cancel{P[X+Y \le w] = \iint_{X+Y \le w} 2 \, dx\,dy}$$

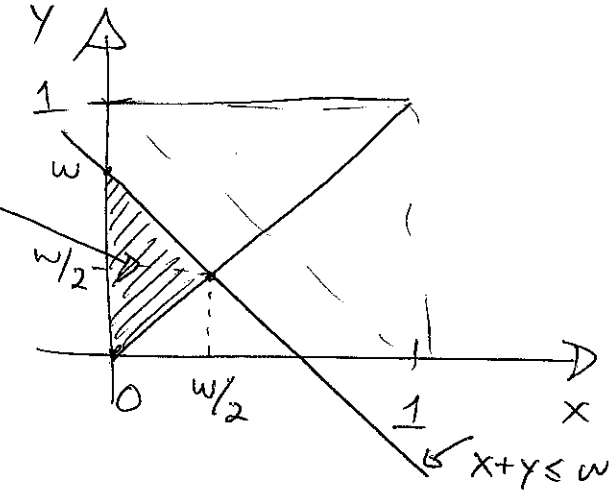
$$\cancel{= (2)\int_{y=0}^{w} \int_{x=0}^{w-y} 2 \, dx\,dy.}$$



See next page.

For $0 < w < 1$ the region looks like:

$P[X+Y \le w]$ = area under this



$= 2 \times \left[\frac{\text{base} \times \text{height}}{2}\right]$

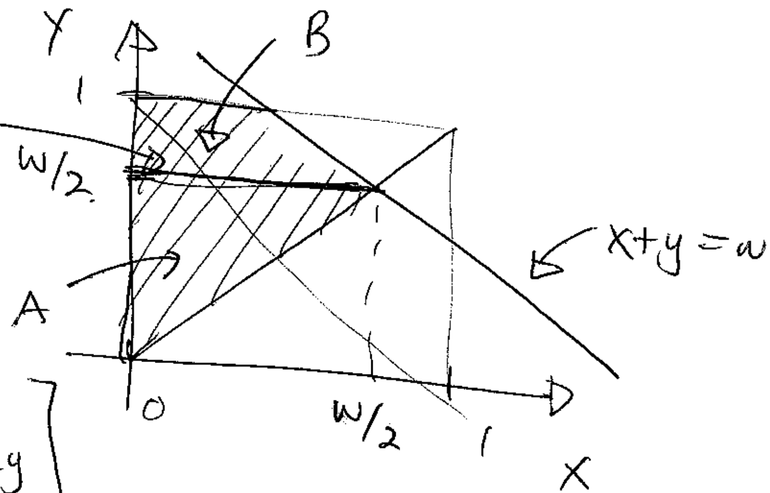
$f_{XY}(x,y)$

$= 2 \times \left[\frac{w \times \frac{w}{2}}{2}\right] = \frac{w^2}{2}.$

For $1 < w < 2$ The region looks like

$P[X+Y \le w]$ = area under this

f_{XY}



$= \left[\text{area}(A) + \text{area}(B)\right] \times 2$

triangle

$= 2 \times \left[\frac{\frac{w}{2} \times \frac{w}{2}}{2} + \int_{y=w/2}^{1} \int_{x=0}^{x=w-y} dx\, dy\right]$

$= 2 \times \left[\frac{w^2}{8} + \int_{w/2}^{1} (w-y)\, dy\right] = 2 \times \left[\frac{w^2}{8} + wy - \frac{y^2}{2}\Big|_{w/2}^{1}\right]$

$= 2 \times \left[\frac{w^2}{8} + w - \frac{1}{2} - \frac{w^2}{2} + \frac{w^2}{8}\right] = 2 \times \left[\frac{-w^2}{4} + w - \frac{1}{2}\right].$

$= -\frac{w^2}{2} + 2w -$

So, overall, we have

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^2/2 & 0 \leq w \leq 1 \\ \frac{-w^2}{2} + 2w - 1 & 1 < w < 2 \\ 1 & w \geq 2 \end{cases}$$

Then, taking derivative wrt. w

$$f_W(w) = \begin{cases} w & 0 \leq w \leq 1 \\ 2 - w & 1 < w < 2. \\ 0 & \text{else.} \end{cases}$$