ECE 534: Elements of Information Theory, Fall 2013
Homework 3
Out: September 11, due September 18, 2013
Solutions

Problem 3.1 *Markov's inequality and Chebyshev's inequality*

1. *(Markov's inequality)* For any nonnegative random variable X and any $t > 0$, show that

$$Pr\{X \geq t\} \leq \frac{E[X]}{t}.$$

   Exhibit a random variable that achieves this inequality with equality.

2. *(Chebyshev's inequality)* Let Y be a random variable with mean μ and variance σ^2. By letting $X = (Y - \mu)^2$, show that for any $\epsilon > 0$,

$$Pr\{|Y - \mu| > \epsilon\} \leq \frac{\sigma^2}{\epsilon^2}$$

3. *(Weak law of large numbers)* Let $Z_1, Z_2, \ldots, Z_n$ be a sequence of i.i.d. random variables with mean μ and variance σ^2. Let $\bar{Z}_n = \frac{1}{n}\sum_{i=1}^{n} Z_i$ be the sample mean. Show that

$$Pr\{|\bar{Z}_n - \mu| > \epsilon\} \leq \frac{\sigma^2}{n\epsilon^2}.$$

   Thus, $Pr\{|\bar{Z}_n - \mu| > \epsilon\} \to 0$ as $n \to \infty$. This is known as the weak law of large numbers.

Problem 3.3 *Piece of cake.* A cake is sliced roughly in half, the largest piece being chosen each time, the other pieces discarded. We will assume that a random cut creates pieces of proportions

$$P = \begin{cases} (\frac{2}{3}, \frac{1}{3}) & \text{with probability } \frac{3}{4} \\ (\frac{2}{5}, \frac{3}{5}) & \text{with probability } \frac{1}{4} \end{cases}$$

Thus, for example, the first cut (and choice of largest piece) may result in a piece of size $\frac{3}{5}$. Cutting and choosing from this piece might reduce it to size $\frac{3}{5} \cdot \frac{2}{3}$ at time 2, and so on. How large, to first order in the exponent, is the piece of cake after n cuts?

Problem 3.6 *AEP-like limit.* Let $X_1, X_2, \cdots$ be drawn i.i.d. according to probability mass function $p(x)$. Find

$$\lim_{n\to\infty} (p(X_1, X_2, \cdots X_n))^{\frac{1}{n}}.$$

Problem 3.7 *AEP and source coding.* A discrete memoryless source emits a sequence of statistically independent binary digits with probabilities $p(1) = 0.005$ and $p(0) = 0.995$. The digits are taken 100 at a time and a binary codeword is provided for every sequence of 100 digits containing three or fewer 1's.

1. Assuming that all codewords are the same length, find the minimum length required to provide codewords for all sequences with three or fewer 1's.

2. Calculate the probability of observing a source sequence for which no codeword has been assigned.

3. Use Chebychev's inequality to bound the probability of observing a source sequence for which no codeword has been assigned. Compare this bound with the actual probability computed in part 2.

Problem 3.13 *Calculation of typical set.* To clarify the notion of a typical set $A_\epsilon^{(n)}$ and the smallest set of high probability $B_\delta^{(n)}$, we will calculate the set for a simple example. Consider a sequence of i.i.d. binary random variables $X_1, X_2, \cdots X_n$, where the probability that $X_i = 1$ is 0.6 (and therefore the probability that $X_i = 0$ is 0.4).

1. Calculate $H(X)$.
2. With $n = 25$ and $\epsilon = 0.1$, which sequences fall in the typical set $A_\epsilon^{(n)}$? What is the probability of the typical set? How many elements are there in the typical set? (This involves computation of a table of probabilities for sequences with k 1's, $0 \leq k \leq 25$, and finding those sequences that are in the typical set.)
3. How many elements are there in the smallest set that has probability 0.9?
4. How many elements are there in the intersection of the sets in parts 2. and 3.? What is the probability of this intersection?