ECE 534 Information Theory - FINAL EXAM

Dec.6, 2017, 4-5:15pm in LH 107.

- You will be given the full 1.25 hours. **Use it wisely!** Many of the problems have short answers; try to find shortcuts. Do questions that you think you can answer correctly first.
- You may bring and use three 8.5x11" double-sided crib sheets.
- No other notes or books are permitted.
- No calculators are permitted.
- Talking, passing notes, copying (and all other forms of cheating) is forbidden.
- Make sure you explain your answers in a way that illustrates your understanding of the problem. Ideas are important, not just the calculation.
- Partial marks will be given.
- Write all answers directly on this exam.

Your name: ____________________

Your UIN: ____________________

Your signature: ____________________

The exam has 4 questions, for a total of 100 points.

Question:	1	2	3	4	Total
Points:	26	29	25	20	100
Score:					

1. *Multiple access channels.*

   Consider the following multiple access channel, with inputs $X_1 \in \{2,4\}$, $X_2 \in \{1,2\}$ and output

$$Y = X_1^{X_2}.$$

   (a) (13 points) Find the capacity region of this multiple access channel.

   (b) (13 points) Suppose the range of X_1 is $\{1,2\}$ instead. Find the capacity region (expressed in terms of parameters) now.

   HINT: For both parts, you may guess the capacity region with some arguments, or you may find it useful to express the outer bound in parametric form and write what equations you would need to solve.

Points earned: ____________ out of a possible 26 points

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Points earned: ________ out of a possible 0 points

2. *Short answers.*

(a) (5 points) When compressing a source, when is it useful to group together source symbols and code over larger blocks?

(b) (5 points) Suppose I give you a $(2^{nR_1}, 2^{nR_2}, n)$ code for the multiple access channel with discrete inputs X_1, X_2 and discrete output Y, whose probability of error is less than some $\epsilon > 0$. What can we say about the conditional entropy $H(X_1^n|Y^n, X_2^n)$? Justify any steps.

(c) (4 points) True or false (no need to justify unless you are unsure):

(i) All typical sequences in $A_\epsilon^{(n)}$ have the same probability.

(ii) The typical set $A_\epsilon^{(n)}$ is defined as the smallest set of sequences with $\Pr\{A_\epsilon^{(n)}\} \geq 1 - \epsilon$.

(iii) The number of sequences in $A_\epsilon^{(n)}$ may be bounded as $|A_\epsilon^{(n)}| \leq 2^{-n(H(X)-\epsilon)}$.

(iv) If x^n is a *typical* sequence drawn i.i.d. according to $p(x^n) = \prod_{i=1}^n p(x_i)$ and y^n is a *typical* sequence drawn i.i.d. according to $p(y^n) = \prod_{i=1}^n p(y_i)$ then (x^n, y^n) are *jointly typical* according to $\prod_{i=1}^n p(x_i)p(y_i)$.

Points earned: ______________ out of a possible 14 points

(d) (5 points) Find the 4-ary Huffman code ($D = 4$) for the source with probability mass function $(\frac{8}{36}, \frac{7}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36})$.

(e) (5 points) Consider all probability mass functions of the form $p(u, x, y) = p(u, x)p(y|x)$. Is $I(U; Y) \leq I(X; Y)$? If true give a short derivation, if not give a counter-example why not.

(f) (5 points) Prove the following inequality and find conditions for equality.

$$I(X; Z|Y) \geq I(Z; Y|X) - I(Z; Y) + I(X; Z).$$

3. *Bandlimited channel capacity.*

   (a) (5 points) Consider the discrete time AWGN channel $Y = gX + Z$ for Z i.i.d. Gaussian noise with 0 mean and variance σ^2, $g \in \mathbb{R}$ *a known constant,* and X the input is subject to the constraint $E[|X|^2] \le P$. What is the capacity of this channel? What input achieves capacity?

   (b) (5 points) Consider the following continuous-time channel $Y(t) = h(t) * (gX(t) + Z(t))$ where $Z(t)$ is the noise waveform from a white Gaussian noise process with power spectral density $N_0/2$, and $h(t)$ is the impulse response of an ideal bandpass filter, which cuts out all frequencies greater than W. Suppose the input has a power constraint P. $g \in \mathbb{R}$ is a known constant. What is the capacity of this channel? (No need to derive it)

   (c) (5 points) As the bandwidth $W \to \infty$, what does the capacity tend to? How can you interpret this result?

Points earned: ______________ out of a possible 15 points

(d) (5 points) Now consider an additive white Gaussian noise *multiple access channel* defined by $Y(t) = h(t) * (X_1(t) + X_2(t) + Z(t))$ for $Z(t)$ the noise waveform from a white Gaussian noise process with power spectral density $N_0/2$, and $h(t)$ is the impulse response of an ideal bandpass filter, which cuts out all frequencies greater than W. Suppose input 1 has a power constraint P_1 and input 2 has a power constraint P_2. Find the capacity region of this channel.

(e) (5 points) As the bandwidth $W \to \infty$, what does the capacity region tend to? How can you interpret this result?

Points earned: ____________ out of a possible 10 points

4. *Rate-distortion.*

   (a) (5 points) In 2 sentences describe the rate-distortion function (meaning not just equation) and when it may be useful.

   (b) (5 points) What is the rate-distortion function for a $\mathcal{N}(\mu, \sigma^2)$ Gaussian source? *(note the mean μ!)*

   (c) (10 points) Now let us consider 2 independent Gaussian sources which we hope to compress: $X_1 \sim \mathcal{N}(0, 4)$ and $X_2 \sim \mathcal{N}(0, 9)$, subject to an overall distortion constraint D. *RECALL: The rate is the sum of the rates on the individual channels and the distortion is the sum of the distortions on the individual channels.* For what values of D do we only assign bits to one of the sources?

Points earned: ____________ out of a possible 20 points