ECE 534 Information Theory - MIDTERM 2

11/16/2018, LH 210.

- You will be given the full 50 minutes. **Use it wisely!** Many of the problems have short answers; try to find shortcuts. Do questions that you think you can answer correctly first.
- You may bring and use two 8.5x11" double-sided crib sheets.
- No other notes or books are permitted.
- No calculators are permitted.
- Talking, passing notes, copying (and all other forms of cheating) is forbidden.
- Make sure you explain your answers in a way that illustrates your understanding of the problem. Ideas are important, not just the calculation.
- Partial marks will be given.
- Write all answers directly on this exam.

Your name: ____________________

Your UIN: ____________________

Your signature: ____________________

The exam has 4 questions, for a total of 100 points.

Question:	1	2	3	4	Total
Points:	20	20	20	40	100
Score:					

1. *Short answers / calculations*

   (a) (5 points) How many codewords does the $(7, 4)$ Hamming code have?

   (b) (5 points) If I have a telephone line with bandwidth 10kHz and an SNR of 20dB. What is the channel capacity in bits per second?

   (c) (5 points) Given an example of a difference between discrete entropy $H(X)$ and differential entropy $h(X)$ (just mention one difference).

   (d) (5 points) Can (1, 2, 2) be the codeword lengths of a binary Huffman code? Explain why/why not.

Points earned: ____________ out of a possible 20 points

2. *Maximum entropy distributions*

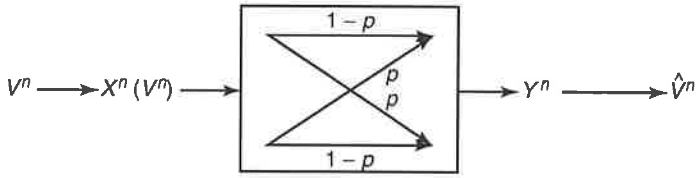
   (a) (10 points) Find the discrete probability mass function on alphabet $\{-5, -2, 0, 2, 5\}$ that maximizes the entropy $H(X)$ subject to a constraint on the mean, $E[X] = 0$.

   (b) (10 points) Find the distribution $f(x)$ of random variable X on support $\mathcal{S}_X = (-\infty, \infty)$ that maximizes the mutual information $I(X;Y)$ if $Y = 4X + N$, where $N \sim \mathcal{N}(0, 20)$ subject to the constraint that $E[|X|^2] \leq 1000$, where X and N are independent.

Points earned: __________ out of a possible 20 points

3. (20 points) *Source channel separation.*

We wish to encode a source that outputs a string of bits (0's and 1's), V^n, i.i.d. according to the Bernoulli(α) distribution ($P(V_i = 0) = 1 - \alpha, P(V_i = 1) = \alpha$) and send this over a binary symmetric channel with crossover probability p, as shown below.



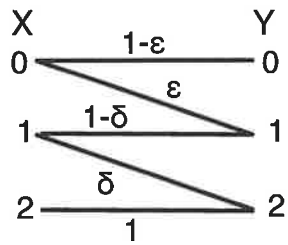
Find conditions on α and p so that the probability of error $P(\hat{V}^n \neq V^n)$ can be made to go to zero as $n \to \infty$.

Points earned: ______________ out of a possible 20 points

4. *Capacity of several simple channels.*

   (a) (20 points) Consider 2 parallel Gaussian channels $Y_1 = X_1 + Z_1$ and $Y_2 = X_2 + Z_2$ where Z_1 and Z_2 are independent, zero mean additive white Gaussian noise $Z_1 \sim \mathcal{N}(0, \sigma_1^2 = 3)$ and $Z_2 \sim \mathcal{N}(0, \sigma_2^2 = 7)$ subject to a total power constraint of $P = 10$. Find the capacity of this channel AND the input distribution which achieves this capacity.

   (b) (20 points) Find the equation(s) to be solved to obtain the capacity of the following channel as functions of ϵ and δ and any other parameters you might introduce (do not solve it!):



Points earned: ________________ out of a possible 40 points

Points earned: __________ out of a possible ?? points

Midterm 2, 2018 solutions

1. (a) (7,4) means (n, k) where n = # encoded bits
k = # unencoded bits

so $2^4 = 16$ codewords.

(b) $C = W\log_2(1 + SNR) = W\log_2\left(1 + \frac{P}{N_0 W}\right)$

Here $W = 10{,}000$ and $SNR_{dB} = 20\,dB \Rightarrow 10\log_{10}(SNR) = 20$

$\Rightarrow SNR = 10^2 = 100$ 1

So,

$C = 10{,}000 \log_2(1 + 100)$

(c) There are many... $H(X) = -\sum_{x\in\mathcal{X}} p(x)\log p(x)$

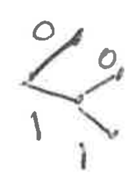
$h(X) = -\int_{\mathcal{X}} f(x)\log f(x)\,dx$

$H(X) \geq 0$ but $h(X)$ can be negative!

(d) Check this via the Kraft inequality $\sum_i D^{-l_i} \leq 1$
for a D-ary code with codeword lengths $\{l_i\}_i$.

$2^{-1} + 2^{-2} + 2^{-2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$ so yes, possible.

If Kraft ineq. satisfied can find an instantaneous code.



2. (a) Alphabet has size $|\mathcal{X}| = 5$, so the uniform distribution maximizes this. Since the uniform would have $E[X]=0$ anyways, this imposes no extra constraints. So,

$p(-5) = p(-2) = p(0) = p(2) = p(5) = \frac{1}{5}$ will maximize entropy and has $E[X]=0$.

(b) $Y = 4X + N$, $I(X;Y) = h(Y) - h(Y|X)$

N: $\mathcal{N}(0,20)$

$= h(Y) - h(N)$

$= h(Y) - \frac{1}{2}\log 2\pi e \cdot (20)$

This is maximized by taking X Gaussian so that Y will be Gaussian, as $Y = 4X + N$. Take

$X \sim \mathcal{N}(0, 1000)$ since $E[|X|^2] \leq 1000$.

3. By the source-channel separation Theorem, a source $\{V^n\}$ can be sent over a channel with capacity C if and only if its entropy rate is less than capacity, i.e.

$$H(V) < C$$ 5

In this problem,

$$H(V) = H(\alpha) = -\alpha \log \alpha - (1-\alpha) \log (1-\alpha)$$ 5

↑

binary entropy function

and

↓

$$C = 1 - H(p) = 1 + p \log p + (1-p) \log (1-p)$$ 5

So, we need α and p to satisfy

$$H(\alpha) < 1 - H(p)$$ 5

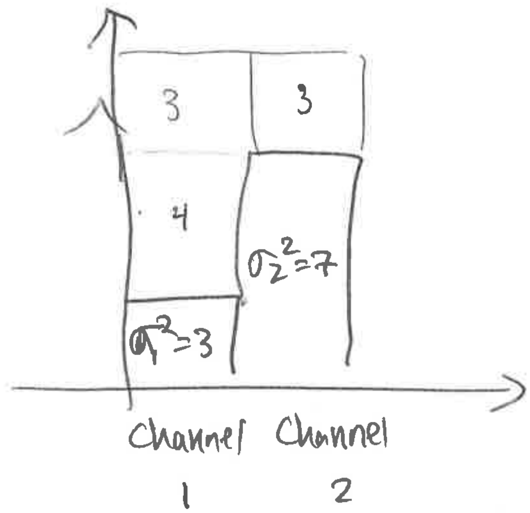
or $H(\alpha) + H(p) < 1$

4.

(a) $Y_1 = X_1 + Z_1 \qquad Z_1 \sim N(0, 3)$

$Y_2 = X_2 + Z_2 \qquad Z_2 \sim N(0, 7)$

$E[|X_1|^2] + E[|X_2|^2] \le 10$



This is a waterfilling problem.
+5

So, channel 1 gets power 7

channel 2 gets power 3

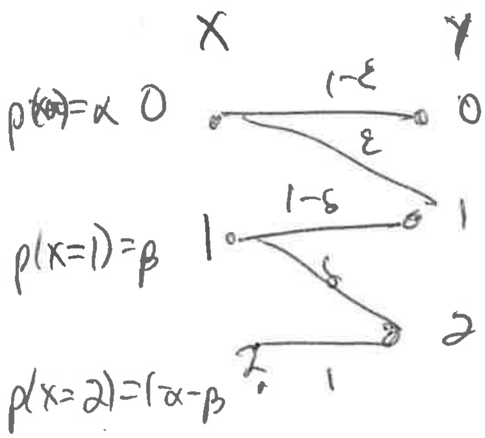
$X_1 \sim N(0,7)$, $X_2 \sim N(0,3)$ ← capacity achieving input distribution.
+5 +5

$$C = \frac{1}{2}\log\left(1+\frac{7}{3}\right) + \frac{1}{2}\log\left(1+\frac{3}{7}\right) = \frac{1}{2}\log\left(\frac{10}{3}\right) + \frac{1}{2}\log\left(\frac{10}{7}\right)$$

+5

(b) $C = \max_{p(x)} I(X;Y) = \max_{\alpha,\beta} H(X) - H(X|Y)$ 1+2

$= \max_{\alpha,\beta} H(Y) - H(Y|X)$



$p(y=0) = \alpha(1-\varepsilon)$

$p(y=1) = \alpha\varepsilon + \beta(1-\delta)$

$p(y=2) = \beta\cdot\delta + 1-\alpha-\beta.$

+8

So, $H(Y) = -\alpha(1-\varepsilon)\log\alpha(1-\varepsilon)$

$- [\alpha\varepsilon + \beta(1-\delta)]\log[\alpha\varepsilon + \beta(1-\delta)]$

$- [\beta\cdot\delta + 1-\alpha-\beta]\log[\beta\cdot\delta + 1-\alpha-\beta]$

Then $H(Y|X) = H(Y|X=0)p(x=0)$

$+ H(Y|X=1)p(x=1)$

$+ H(Y|X=2)p(x=2)$

$= \alpha\cdot H(\varepsilon) + \beta\cdot H(\delta) + 0$

+8

So,

$C = \max_{\alpha,\beta} H(Y) - H(Y|X)$ where as above ↑

+2

So set derivate of $\frac{\partial I(X;Y)}{\partial\alpha} = 0$ $\frac{\partial I(X;Y)}{\partial\beta} = 0$

and solve for α, β