A Survey of Approximability and Inapproximability Results for Social Welfare Optimization in Multiagent Resource Allocation*

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Abstract

We survey recent approximability and inapproximability results on social welfare optimization in multiagent resource allocation, focusing on the two most central representation forms for utility functions of agents, the *bundle form* and the *k-additive form*. In addition, we provide some new (in)approximability results on maximizing egalitarian social welfare and social welfare with respect to the Nash product when restricted to certain special cases.

Introduction

Multiagent resource allocation (MARA) is a vibrant research area within the emerging field of computational social choice. In particular, MARA has applications in multiagent systems, a subfield of (distributed) artificial intelligence, and is also closely related to other areas of computer science as well as to economics and social choice theory. MARA models the following situation. We are given a set of agents and a set of resources. The resources are assumed to be indivisible and nonshareable. Each of the agents has utilities for all (bundles of) resources. The task is to find an allocation of resources that is "optimal" in terms of social welfare. Let us first, in the remainder of this section, describe how to formalize such MARA situations, how to represent the agents' utility functions, what the most important notions of social welfare are, and how the associated optimization problems can be defined. Then we provide some background of approximation theory that is needed to give a short overview of known approximability and inapproximability results. Finally, we will present some related new results for certain special cases (to appear in part in (Nguyen et al. 2012)), and we conclude with some open problems for future research.

MARA Settings

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a *set of n agents* and let $R = \{r_1, r_2, \ldots, r_m\}$ be a *set of m resources*. We assume that the resources are indivisible and nonshareable. Every agent a_i has a utility function, u_i, expressing a_i's utilities for all bundles of resources. Formally, each *utility function* u_i is a mapping from 2^R to the numerical set $\mathbb{F}$, where $\mathbb{F}$ typically is one of $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}^+$, $\mathbb{Z}$, $\mathbb{N}$, or simply $\{0,1\}$, and where 2^R is the set of all subsets of R. Let $U = \{u_1, u_2, \ldots, u_n\}$ denote the set of all utility functions of the agents. We call such a triple (A, R, U) a *MARA setting*. The utility agent a_i can realize depends on which resources he or she receives in an allocation of resources. An *allocation for a MARA setting* (A, R, U) is a mapping $X : A \to 2^R$ such that $X(a_i) \cap X(a_j) = \emptyset$ for $i \neq j$ and $\bigcup_{a_i \in A} X(a_i) = R$. Let $\Pi_{A,R}$ denote the set of all allocations for (A, R, U); note that $\|\Pi_{A,R}\| = n^m$.

Representation of Utility Functions

An important issue in a MARA model is the representation of utility functions, as choosing this representation may affect the computational complexity of finding an optimal allocation. At least for human agents, the most natural way to represent utility functions is to enumerate all the bundles of resources and attach a utility to these bundles. By convention, we skip bundles with a utility of zero. This way of representing utilities is called the *bundle form* or the *bundle enumeration*. Although this form is fully expressive, the size of its representation can be exponential in number of resources (for instance, if an agent has nonzero utilities for all possible bundles). Another form of representing a utility function is the *k-additive form*, which can be more succinct than the bundle form when k is small. However, the k-additive form is fully expressive only if k is large enough. These forms are formally defined as follows.

- The *bundle form*: Every utility function u_i is represented as the set of pairs $(R', u_i(R'))$, where $u_i(R') \in \mathbb{F}$ for each bundle $R' \subseteq R$ of resources with $u_i(R') \neq 0$.
- The *k-additive form*, for a fixed positive integer k: Every utility function u_i is represented by unique coefficients $\alpha_i^T \in \mathbb{F}$ for each bundle $T \subseteq R$ with $\|T\| \leq k$ such that for each bundle $S \subseteq R$, we have
$$u_i(S) \quad = \quad \sum_{T \subseteq S,\ \|T\| \leq k} \alpha_i^T.$$
Intuitively, the coefficients α_i^T express the "synergetic" value of agent a_i owning all resources from bundle T that contains at most k items.

There are other ways of representing utilities as well. For example, utility functions can be represented via *straight line programs*, which are based on boolean circuits, but

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Social Welfare	Bundle Form	k-Additive Form	Reference
Utilitarian	NP-c	NP-c, $k \geq 2$	Chevaleyre et al. (2008)
Egalitarian	NP-c	NP-c, $k \geq 1$ [¶]	Roos and Rothe (2010), [¶] implicitly shown by Lipton et al. (2004)
Nash Product	NP-c [†]	NP-c, $k \geq 1$	Roos and Rothe (2010), [†] also due to Ramezani and Endriss (2010)

Table 1: Complexity of decision problems for social welfare optimization. Key: NP-c means "NP-complete."

since we do not consider them in this paper, we will not define them formally and we refer to (Chevaleyre et al. 2006; Dunne, Wooldridge, and Laurence 2005) instead.

Social Welfare

In social choice theory and economics, the notion of *social welfare* measures the quality of an allocation in some sense. The survey by Chevaleyre et al. (2006) reviews some types of social welfare in detail. In this paper, however, we focus on the three most basic types of social welfare only: *utilitarian*, *egalitarian*, and *Nash product social welfare*. While utilitarian and Nash product social welfare are the sum and the product, respectively, of the agents' individual utilities, egalitarian social welfare equals the smallest utility among the agents in a given allocation, i.e., the utility of an agent who is worst off. These notions of social welfare are defined as follows.

Definition 1 (Social welfare) *For a MARA setting (A,R,U) and an allocation X, define*

1. *the* utilitarian social welfare of X *as*
$$sw_u(X) = \sum_{a_i \in A} u_i(X);$$
2. *the* egalitarian social welfare of X *as*
$$sw_e(X) = \min_{a_i \in A}\{u_i(X)\};$$
3. *the* Nash product social welfare of X *as*
$$sw_N(X) = \prod_{a_i \in A} u_i(X).$$

As an additional notation, for $S \in \{u,e,N\}$, denote the *maximum utilitarian/egalitarian/Nash product social welfare* of a MARA setting $M = (A,R,U)$ by
$$\max_S(M) = \max\{sw_S(X) \mid X \in \Pi_{A,R}\}.$$

Three maximization problems can be defined for these types of social welfare. For example, the maximization problem for utilitarian social welfare is formally defined as:

$\mathbb{F}$-Maximum Utilitarian Social Welfare$_{\text{form}}$	
Input:	A MARA setting $M = (A,R,U)$, where form indicates how every $u_i : 2^R \to \mathbb{F}$ in U is represented.
Output:	$\max_u(M)$.

As a shorthand, write $\mathbb{F}$-MAX-USW$_{\text{form}}$. Using sw_e and sw_N instead of sw_u, we obtain $\mathbb{F}$-MAXIMUM EGALITARIAN SOCIAL WELFARE$_{\text{form}}$ (or $\mathbb{F}$-MAX-ESW$_{\text{form}}$) and $\mathbb{F}$-MAXIMUM NASH PRODUCT SOCIAL WELFARE$_{\text{form}}$ (or $\mathbb{F}$-MAX-NPSW$_{\text{form}}$) accordingly.

For the related decision problems (where we ask whether for a given MARA setting there *exists* an allocation whose utilitarian, egalitarian, or Nash product social welfare is equal to or exceeds some given bound B), NP-hardness is known; see, in particular, the survey by Chevaleyre et al. (2006), the work of Chevaleyre et al. (2008), the extension of their work by Roos and Rothe (2010), and the other references cited in Table 1, which summarizes these results. Hence, the corresponding optimization problems are NP-hard in the sense that they cannot be solved in polynomial time, unless P = NP. Thus our goal is to study the (in)approximability of these problems. We give an overview of several known results and also provide some related new results for special instances of these problems.

Basic Notions of Approximation Theory

We assume the reader is familiar with complexity-theoretic notions such as reducibility, NP-completeness, etc. (see, e.g., the textbooks by Garey and Johnson (1979), Papadimitriou (1995), and Rothe (2005)). We now provide some formal definitions related to approximation algorithms and describe some basic techniques as the tools to prove inapproximability of optimization problems. We start with the definition of approximation algorithm.

Definition 2 (α-approximation algorithm) *Let $\mathscr{P}$ be an maximization problem and $\alpha < 1$. An α-*approximation algorithm *A for $\mathscr{P}$ is a polynomial-time algorithm that for every instance x of $\mathscr{P}$ produces a solution $A(x)$ whose value is at least $\alpha \cdot OPT(x)$, where $OPT(x)$ denotes the optimal value for x.*

We call α the *approximation factor* (or sometimes the *approximation ratio* or *performance guarantee*) of an α-approximation algorithm. Note that α may depend on the size of the given instance.

Definition 3 (PTAS) *A maximization problem $\mathscr{P}$ has a* polynomial-time approximation scheme *(*PTAS*) if for each ε, $0 < \varepsilon < 1$, there exists a $(1-\varepsilon)$-approximation algorithm A_ε for $\mathscr{P}$.*

Definition 4 (FPTAS) *A* PTAS *is said to be a* fully polynomial-time approximation scheme *(*FPTAS*) if the running time of A_ε is polynomial in both the input size and $1/\varepsilon$.*

For instance, if for each input of size n the algorithm A_ε runs in time $\mathcal{O}(n^{1/\varepsilon})$ then we have a PTAS, not an FPTAS. On the other hand, if the running time is $\mathcal{O}(n^{10}\varepsilon^{-100})$ then we have an FPTAS.

We now discuss some important techniques used to prove the inapproximability result in this paper. An easy way to prove that some optimization problem $\mathscr{P}$ is hard to approximate is to use a reduction from an NP-complete problem in order to design a *gap-introducing* reduction. We apply this technique to maximization problems only. In the following, *OPT* will always denote the function mapping an instance x to the value of an optimal solution with respect to the maximization problem at hand.

Definition 5 (α-gap-introducing reduction) *Let* $A \subseteq \Sigma^*$ *be an* NP*-complete problem,* $\mathscr{P}$ *be a maximization problem with input domain* Γ^**, and let* $\alpha : \mathbb{N} \to [0,1]$ *be a function of the input size. An* α-gap-introducing reduction from A to $\mathscr{P}$ *is given by two polynomial-time computable functions* $f : \Sigma^* \to \mathbb{R}$ *and* $g : \Sigma^* \to \Gamma^*$ *such that, for each* $x \in \Sigma^*$,

- *if* $x \in A$ *then* $OPT(g(x)) \geq f(x)$*, and*
- *if* $x \notin A$ *then* $OPT(g(x)) < \alpha(|x|) \cdot f(x)$.

Note that an α-approximation algorithm for a maximization problem $\mathscr{P}$ that has an α-gap-introducing reduction from an NP-complete problem A would allow us to distinguish between the yes- and no-instances of A in polynomial time. Hence, having such a reduction implies that there can be no α-approximation algorithm for $\mathscr{P}$, unless P $=$ NP.

Definition 6 (L-reduction) *Let* $\mathscr{P}_1$ *and* $\mathscr{P}_2$ *be some maximization problems. An* L-reduction from $\mathscr{P}_1$ to $\mathscr{P}_2$ *is given by two polynomial-time computable functions* f *and* g *and two parameters* α *and* β *such that for each instance* x *of* $\mathscr{P}_1$, $y = f(x)$ *is an instance of* $\mathscr{P}_2$ *and it holds that:*

1. $OPT(y) \leq \alpha \cdot OPT(x)$*, and*
2. *for each solution* s_2 *of value* v_2 *to* y, $s_1 = g(s_2)$ *is a solution of value* v_1 *to* x *such that*
$$OPT(x) - v_1 \leq \beta \cdot (OPT(y) - v_2).$$

The key point to note here is that when we have an L-reduction from $\mathscr{P}_1$ to $\mathscr{P}_2$ with parameters α and β and we have a $(1-\varepsilon)$-approximation algorithm for $\mathscr{P}_2$, where $\varepsilon > 0$ is a constant, then we obtain a $(1-\delta)$-approximation algorithm for $\mathscr{P}_1$ with $\delta = \alpha \cdot \beta \cdot \varepsilon$. In particular, if $\mathscr{P}_1$ L-reduces to $\mathscr{P}_2$ with parameters $\alpha = \beta = 1$ and we know that $\mathscr{P}_1$ cannot have an $(1-\varepsilon)$-approximation algorithm (unless, say, P $=$ NP), then $\mathscr{P}_2$ cannot have a $(1-\varepsilon)$-approximation algorithm either, unless P $=$ NP. For more background on approximation theory, see, e.g., the textbook by Vazirani (2003) and the survey by Arora and Lund (1996).

Approximability of Social Welfare Optimization for Utilities in the Bundle Form

After the overview of some basic notions of approximation theory given in the previous section, in this section we study the (in)approximability of maximizing social welfare under the bundle form. Regarding utilitarian social welfare, this problem has been studied as *Generalized Vickrey Auctioning* (GVA for short) in the field of combinatorial auctions. In this context, Lehmann, O'Callaghan, and Shoham (1999) showed that GVA is NP-hard to approximate[1] within $m^{\varepsilon-1/2}$.

[1]More precisely, they showed that this inapproximability result holds unless NP $=$ ZPP, where ZPP is the class of decision problems that admit a probabilistic algorithm that, on every input, runs in expected polynomial time and always returns the correct answer.

Their result holds even for so-called *single-minded agents* and is proven via a reduction from the well-known problem CLIQUE whose optimization version is known to be hard to approximate (Håstad 1999). Intuitively speaking, an agent a is said to be *single-minded* if for each bundle $R' \subseteq R$ of resources a is interested in, a has a specific utility v for the bundle R' as well as for all bundles containing the resources of R'. Table 2 summarizes this and other results mentioned in this section and some more.

Since utility functions represented in the bundle form can be exponentially in the number m of resources, we need to specify how an algorithm can access its input. One way to do this is to use a fixed natural "bidding language" (if one exists). Another approach uses different types of oracle to be queried. A *value oracle* returns the utility of a given bundle of resources for a given agent—this model is typical from the point of view of computer science. A stronger model is typical from an economics perspective: In the *demand oracle* model, a query is a vector $(v_1, \ldots, v_m)$ of particular utilities corresponding to the m resources $(r_1, \ldots, r_m)$, and for agent a_i the demand oracle returns a bundle $T \subseteq R$ such that the value $u_i(T) - \sum_{r_j \in T} v_j$ is maximal. Some results in Table 2 are given in either the demand or the value oracle model, some in the general case.

"Submodular" utilities have been studied intensely. A *submodular* utility function u fulfills the inequality

$$u(S \cup T) + u(S \cap T) \leq u(S) + u(T)$$

for every (not necessarily disjoint) pair of bundles S and T. In the context of combinatorial auctions, the focus is again on maximizing utilitarian social welfare. Lehmann, Lehmann, and Nisan (2001) achieved an approximation factor of $1/2$ for MAX-USW$_{\text{bundle}}$ in the value oracle model for submodular utilities. Vondrák (2008) improved their result to a $(1 - 1/e)$-approximation, where $e \approx 2,71828182$ is Euler's number, see also the previous work by Fleischer et al. (2006) and Calinescu et al. (2007). Khot et al. (2008) prove that this bound is tight: It is NP-hard to approximate MAX-USW$_{\text{bundle}}$ with a factor better than $1 - 1/e$ in the submodular setting and the value oracle model. Dobzinski and Schapira (2006) provide an approximation factor of $1 - 1/e$ for submodular utilities in the demand oracle model.

Similar to submodular utilities are "subadditive" utilities. A utility function u is said to be *subadditive* if for every pair $S, T \subseteq R$ of bundles of resources, we have that

$$u(S \cup T) \leq u(S) + u(T).$$

Typically, utilities are assumed to be nonnegative in this context. Feige (2009) proved an approximation factor of $1/2$ for MAX-USW$_{\text{bundle}}$ with subadditive utilities in the demand oracle model, and showed that it is tight in the value oracle model, i.e., it is NP-hard to approximate this problem within a factor of $1/2 + \varepsilon$. For fractionally subadditive utilities, he achieves even an approximation factor of $1 - 1/e$ by rounding to a relaxation of a linear program.

Recommendable literature about the problem of maximizing social welfare in the context of combinatorial auctions are the paper by Dobzinski, Nisan, and Schapira (2010) and

Problem (Restriction)	Approximability	Reference
Max-USW$_{\text{bundle}}$	$1/\sqrt{m}$ (DO)	Lehmann, O'Callaghan, and Shoham (1999)
	$\sqrt{\log m}/m$ (VO)	Holzman et al. (2002)
	NP-hard in factor $(\log m/m) + \varepsilon$ (VO)	Blumrosen and Nisan (2005)
	not approximable in polynomial time within factor $n^{\varepsilon-1}$, unless NP = ZPP	Chevaleyre et al. (2008), as noted by Roos and Rothe (2010)
Max-USW$_{\text{bundle}}$ (even for single-minded agents)	NP-hard in factor $m^{\varepsilon-1/2}$	Lehmann, O'Callaghan, and Shoham (1999)
Max-USW$_{\text{bundle}}$ (submodular utilities)	$1/2$ (VO)	Lehmann, Lehmann, and Nisan (2001)
	$1-1/e$ (DO)	Dobzinski and Schapira (2006)
	$1-1/e$ (VO)	Vondrák (2008)
	NP-hard in factor $1-1/e+\varepsilon$ (VO)	Khot et al. (2008)
Max-USW$_{\text{bundle}}$ (subadditive utilities)	$1/2$ (DO) NP-hard in factor $1/2+\varepsilon$ (VO)	Feige (2009)
Max-USW$_{\text{bundle}}$ (fractionally subadditive)	$1-1/e$ (DO)	
Max-ESW$_{\text{bundle}}$ (even with $u_i(r) \in \{0,1\}$) Max-NPSW$_{\text{bundle}}$ (even with $u_i(r) \in \{0,1\}$)	NP-hard in any factor NP-hard in any factor	Proposition 7

Table 2: Summary of (in)approximability of social welfare optimization problems for the bundle form. Key: DO means "demand oracle" and VO means "value oracle."

the bookchapter by Blumrosen and Nisan (2007). As mentioned above, typically utilitarian social welfare is studied in this context. In addition, we will give some inapproximability results for maximizing egalitarian and Nash-product social welfare later on (see Proposition 7).

Approximability of Social Welfare Optimization for *k*-Additive Utilities

Chevaleyre et al. (2008) showed that the problem $\mathbb{Q}$-Max-USW$_{\text{1-additive}}$ can be solved exactly in polynomial time. Indeed, it is easy to design a greedy algorithm for this problem according to the following rule: Every resource will be assigned to an agent who has maximum utility for it. Obviously, this algorithm runs in time $\mathcal{O}(n \cdot m)$ and returns the maximum utilitarian social welfare.

For $k = 2$, one immediately obtains an inapproximability result for $\mathbb{Q}$-Max-USW$_{k\text{-additive}}$ from the work of Chevaleyre et al. (2008) (see Proposition 8). They proved NP-completeness of the decision version of $\mathbb{Q}$-Max-USW$_{\text{2-additive}}$ by a reduction from the decision version of Maximum 2-Satisfiability (Max-2-Sat, for short). This optimization problem is defined as follows.

Maximum 2-Satisfiability	
Input:	A boolean formula φ in conjunctive normal form consisting of clauses having two literals each.
Output:	A truth assignment to the variables of φ that maximizes the number of satisfied clauses.

The best (unconditional[2]) inapproximability result currently known for Max-2-Sat is due to Håstad (2001), who shows that this problem is NP-hard to approximate within a factor of $21/22 \approx 0.9545$. Using this result, we obtain the corresponding inapproximability bound for $\mathbb{Q}^+$-Max-NPSW$_{\text{2-additive}}$ (see Proposition 12).

The problem $\mathbb{Q}$-Max-ESW$_{\text{1-additive}}$ is known as the "Santa Claus" problem[3] in the field of combinatorial auctions. studied by several authors. Recall from Table 1 that the decision version of $\mathbb{Q}$-Max-ESW$_{\text{1-additive}}$ is NP-complete (Roos and Rothe 2010). An inapproximability result for $\mathbb{Q}$-Max-ESW$_{\text{1-additive}}$ is due to Bezáková and Dani (2005): $\mathbb{Q}$-Max-ESW$_{\text{1-additive}}$ cannot be approximated in polynomial time within a factor of $\alpha > 1/2$, unless P = NP. This is also the best inapproximability result known for $\mathbb{Q}$-Max-ESW$_{\text{1-additive}}$. In the same paper, using techniques based on matching as well as rounding techniques for linear programming relaxation, Bezáková and Dani (2005) design two approximation algorithms for $\mathbb{Q}$-Max-ESW$_{\text{1-additive}}$, both having a performance guarantee of $1/(m-n+1)$.

Golovin (2005) studies a restricted version of $\mathbb{Q}$-Max-ESW$_{\text{1-additive}}$, which is also known as "*Big Goods/Small Goods*." In this restricted problem, the agents are allowed to choose among three values only (0, 1, and some $x > 1$) to express their utilities. For the *small* goods, each agent is allowed to assign utilities of zero and one, and for the *big* goods the allowed values are zero and $x > 1$. Golovin (2005) shows that this problem is approximable in polynomial time within a factor of $1/\sqrt{m}$, where m is the number of resources.

Golovin (2005) also studies another special variant of $\mathbb{Q}$-

[2] Khot et al. (2007) show that Max-2-Sat is NP-hard to approximate within a factor better than roughly 0.9439, provided that the so-called "Unique Games Conjecture" holds.

[3] Santa Claus gives his presents to the children, aiming at making the child worst off as happy as possible, i.e., aiming at maximizing egalitarian social welfare.

Problem (Restriction)	Approximability	Reference
MAX-USW$_{\text{1-additive}}$	P	Chevaleyre et al. (2008)
MAX-USW$_{k\text{-additive}}$ ($k \geq 2$, even for two agents)	NP-hard in any factor $\alpha > 21/22$	Proposition 8, based on a reduction due to Chevaleyre et al. (2008)
MAX-ESW$_{\text{1-additive}}$ ($u_i(r) = u_j(r)$ and $u_i(\emptyset) = 0$ for all i, j)	PTAS	Woeginger (1997)
MAX-ESW$_{\text{1-additive}}$ ($u_i(\emptyset) = 0$ for all i)	$1/(m-n+1)$ NP-hard in any factor $\alpha > 1/2$	Bezáková and Dani (2005)
MAX-ESW$_{\text{1-additive}}$ ($u_i(r) \in \{0,1,x\}$ and $u_i(\emptyset) = 0$ for all i) MAX-ESW$_{k\text{-additive}}$ ($k \geq 1$, $m = n$, and $u_i(\emptyset) = 0$ for all i)	$1/\sqrt{m}$ P	Golovin (2005)
MAX-ESW$_{\text{1-additive}}$ ($u_i(r_j) \in \{0,x_j\}$ and $u_i(\emptyset) = 0$ for all i, j)	$\mathcal{O}(\log\log\log n/\log\log n)$	Bansal and Sviridenko (2006)
MAX-ESW$_{\text{1-additive}}$ ($u_i(\emptyset) = 0$ for all i)	$\Omega(1/\sqrt{n}\log^3(n))$	Asadpour and Saberi (2007; 2010)
MAX-ESW$_{k\text{-additive}}$ ($k \geq 3$)	NP-hard in any factor	Theorem 9
MAX-NPSW$_{k\text{-additive}}$ ($k \geq 1$, $m = n$, and $u_i(\emptyset) = 0$ for all i)	P	Theorem 10
MAX-NPSW$_{\text{1-additive}}$ (two agents, $u_1(r) = u_2(r)$, $u_1(\emptyset) = 0$)	PTAS	Theorem 11
MAX-NPSW$_{\text{2-additive}}$	NP-hard in any factor $\alpha > 21/22$	Proposition 12
MAX-NPSW$_{k\text{-additive}}$ ($k \geq 3$)	NP-hard in any factor	Theorem 13

Table 3: Summary of (in)approximability of social welfare optimization problems for the k-additive form

MAX-ESW$_{\text{1-additive}}$, namely when there are as many agents as resources (i.e., $m = n$) and the empty bundle has zero utility, which is a reasonable assumption. It is then easy to see that each agent must get at least one resource to obtain an egalitarian social welfare distinct from zero. Hence, this problem can be solved in polynomial time (Golovin 2005), and the same applies to k-additive utilities for any $k \geq 1$. Analogous results are shown in Theorem 10 for the problem $\mathbb{Q}^+$-MAX-NPSW$_{k\text{-additive}}$ subject to the same restriction.

Bansal and Sviridenko (2006) investigate another restriction of the *Santa Claus* problem: If only two values are allowed for each single resource (i.e., r_j has either some value x_j or zero for each of the agents), there is an $\mathcal{O}(\log\log\log n/\log\log n)$-approximation. They left open a gap in the approximability for the general case, which could be reduced one year later by Asadpour and Saberi (2007; 2010), who improved the upper bound for the unrestricted problem by showing a performance guarantee of $\Omega(1/\sqrt{n}\log^3 n)$ for $\mathbb{Q}$-MAX-ESW$_{\text{1-additive}}$.

Building and improving on the work of Deuermeyer, Friesen, and Langston (1982) and Csirik, Kellerer, and Woeginger (1992), Woeginger (1997) studies the problem of maximizing the minimum completion time of jobs to be scheduled on parallel identical machines (which in some sense is dual to the problem of minimizing the *makespan*, i.e., the maximum completion time), and he designed a PTAS for it. Since this problem is NP-complete in the strong sense (Garey and Johnson 1979), there can be no FPTAS for it, unless P $=$ NP. Note that this problem is closely related to maximizing egalitarian social welfare and so in some sense corresponds to $\mathbb{Q}$-MAX-ESW$_{\text{1-additive}}$ (the *Santa Claus* problem). For $\mathbb{Q}^+$-MAX-NPSW$_{\text{1-additive}}$, restricted to two agents having the same utility functions and such that the empty bundle has zero utility, we design a PTAS in Theorem 11. Note also that this problem is closely related to the problem MINIMUM PARTITION (to be described right after the proof of Theorem 11), which is a special variant of the optimization version of SUBSET OF SUMS and has even an FPTAS (Kellerer, Pferschy, and Pisinger 2004).

Table 3 summarizes the results mentioned in this section.

Proving Some Special Results

In this section we present some new results for certain special cases referred to in the two previous sections, and their proofs. Recall from Table 2 that, as noted by Roos and Rothe (2010), it follows from a result of Chevaleyre et al. (2008) that MAX-USW$_{\text{bundle}}$ cannot be approximated within a factor of $n^{\varepsilon-1}$, unless NP $=$ ZPP. Can similar inapproximability results for $\mathbb{Q}$-MAX-ESW$_{\texttt{bundle}}$ and $\mathbb{Q}^+$-MAX-NPSW$_{\texttt{bundle}}$ be obtained? While this question still remains open, a gap-introducing reduction mentioned already in (Roos and Rothe 2010) and attributed to an anonymous reviewer of their paper provides a different kind of inapproximability result for these two problems. This gap-introducing reduction, which is presented in the proof of the following proposition, is from the NP-complete problem EXACT SET COVER (XSC, for short), which is defined as follows (see, e.g., the textbook by Garey and Johnson (1979)):

EXACT SET COVER	
Given:	A a finite set $B = \{b_1, \ldots, b_m\}$ and a family $\mathscr{S} = \{S_1, \ldots, S_n\}$ of subsets of B.
Question:	Is there a subset $I \subseteq \{1, \ldots, n\}$ such that $\{S_i \mid i \in I\}$ is a partition of B, i.e, $\bigcup_{i \in I} S_i = B$ and $S_i \cap S_j = \emptyset$ for all distinct $i, j \in I$?

Proposition 7 *The problems* $\mathbb{Q}$-MAX-ESW$_{\texttt{bundle}}$ *and* $\mathbb{Q}^+$-MAX-NPSW$_{\texttt{bundle}}$ *cannot be approximated in polynomial*

time within any factor, unless P = NP. *This result holds even when the utilities are restricted to the domain* $\{0,1\}$.

PROOF. Let $(B,\mathscr{S})$ with $B = \{b_1,\ldots,b_m\}$ and $\mathscr{S} = \{S_1,\ldots,S_n\}$ be an instance of XSC. Construct an instance $M = (A,R,U)$ of, respectively, $\mathbb{Q}$-MAX-ESW$_{\texttt{bundle}}$ and $\mathbb{Q}^+$-MAX-NPSW$_{\texttt{bundle}}$ as follows. Let $A = \{a_1,\ldots,a_n\}$ be our set of agents, and let $R = B$ be our set of m resources. For each i, $1 \le i \le n$, the utility function of agent a_i is defined as

$$u_i(T) = \begin{cases} 1 & \text{if } T = \emptyset \text{ or } T = S_j \text{ for some } j \\ 0 & \text{otherwise.} \end{cases}$$

If $(B,\mathscr{S})$ is a yes-instance of XSC then there exists some set $I \subseteq \{1,\ldots,n\}$ such that $\{S_i \mid i \in I\}$ is a partition of B, i.e, $\bigcup_{i\in I} S_i = B$ and $S_i \cap S_j = \emptyset$ for all distinct $i,j \in I$. Hence, by assigning the bundle S_i to agent a_i for each $i \in I$ and the empty bundle to all remaining agents, we have $\max_e(M) = \max_N(M) = 1$.

Conversely, suppose $(B,\mathscr{S})$ is a no-instance of XSC. As all resources need to be allocated, the optimal social welfare value (in both measures) is zero: $\max_e(M) = \max_N(M) = 0$.

If one could approximate either of these two problems within any factor in polynomial time, one could thus decide XSC in polynomial time, contradicting NP-hardness of XSC unless P = NP. ❑

Turning to k-additive utilities, we start with an inapproximability result for maximizing utilitarian social welfare that is based on the reduction of Chevaleyre et al. (2008) showing NP-hardness of the corresponding decision problem.

Proposition 8 *For each* $k \ge 2$, $\mathbb{Q}$-MAX-USW$_{k\text{-additive}}$ *cannot be approximated in polynomial time within any factor better than* $\alpha = 21/22$, *unless* P = NP. *This result holds even when there are only two agents.*

PROOF. The reduction Chevaleyre et al. (2008) used to show NP-hardness of the decision problem corresponding to $\mathbb{Q}$-MAX-USW$_{\text{2-additive}}$ from the decision version of MAX-2-SAT immediately yields an L-reduction from MAX-2-SAT to $\mathbb{Q}$-MAX-USW$_{\text{2-additive}}$. Let φ be an instance of MAX-2-SAT. Without loss of generality, we assume that φ does not contain any clause of the form $(x_i \vee x_i)$.

The $\mathbb{Q}$-MAX-USW$_{\text{2-additive}}$ instance $M = (A,R,U)$ constructed from φ has two agents (i.e., $A = \{a_1,a_2\}$), each resource in R corresponds to a propositional variable occurring in φ, and the agents' utilities are set in 2-additive form as follows. Agent a_1's coefficients are:

- $\alpha_1^{\emptyset} = 0$.
- For each $x_i \in R$, $\alpha_1^{\{x_i\}}$ is the number of times x_i occurs as a positive literal in a clause of φ.
- For all $x_i, x_j \in R$ with $i \ne j$, $\alpha_1^{\{x_i,x_j\}}$ is the negation of the number of times $(x_i \vee x_j)$ or $(x_i \vee x_j)$ occur in φ.

Agent a_2's coefficients are defined as $\alpha_2^T = 0$ for all bundles $T \subseteq R$, $\|T\| \le 2$.

Note that every assignment τ of truth values to the propositional variables of φ corresponds to a resource allocation X_τ for M, since agent a_1 receives resource x_i exactly if x_i is set to true under τ. It follows that $sw_u(X_\tau)$ equals the number of clauses in φ satisfied by τ. Thus $\max_u(M)$ equals the maximum number of satisfiable clauses in φ.

Since this is an L-reduction from MAX-2-SAT to $\mathbb{Q}$-MAX-USW$_{\text{2-additive}}$, the inapproximability bound of $21/22$ for MAX-2-SAT due to Håstad (2001) tranfers to $\mathbb{Q}$-MAX-USW$_{\text{2-additive}}$.

The same inapproximability result for the k-additive case for $k > 2$ follows immediately from the 2-additive case. ❑

Next, we consider egalitarian social welfare. The proof of Theorem 9 is similar to the proof of Proposition 7, but uses a reduction from the strongly NP-complete problem EXACT COVER BY 3-SETS (or X3C, for short), which is defined as follows (see, e.g., (Garey and Johnson 1979)):

EXACT COVER BY 3-SETS	
Given:	A finite set B with $\|B\| = 3n$ and a family $\mathscr{S} = \{S_1,\ldots,S_m\}$ of 3-element subsets of B.
Question:	Is there a subcollection $\mathscr{S}' \subseteq \mathscr{S}$ such that every element of B occurs in exactly one member of $\mathscr{S}'$?

Theorem 9 *For each* $k \ge 3$, $\mathbb{Q}$-MAX-ESW$_{k\text{-additive}}$ *cannot be approximated in polynomial time within any factor, unless* P = NP.

PROOF. It suffices to prove the theorem for the case of $k = 3$. Let $(B,\mathscr{S})$ be an instance of X3C, where $\|B\| = 3n$ and $\mathscr{S} = \{S_1,\ldots,S_m\}$. Construct an instance $M = (A,R,U)$ of $\mathbb{Q}$-MAX-ESW$_{\text{3-additive}}$ as follows. The set of n agents is $A = \{a_1,\ldots,a_n\}$ and the set of resources is $R = B$. For each agent $a_i \in A$, define the coefficients in the 3-additive representation of a_i's utility function as follows:

$$\alpha_i^T = \alpha^T = \begin{cases} 1 & \text{if } T = S_j \text{ for some } j \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $(B,\mathscr{S})$ is a yes-instance of X3C. Then there exist n pairwise disjoint subsets $S_1,\ldots,S_n$ of $\mathscr{S}$ such that $\bigcup_{1\le i\le n} S_i = B$. Hence, assigning the bundle S_i to agent a_i for each i, $1 \le i \le n$, we have $\max_e(M) = 1$.

Conversely, we show that if $(B,\mathscr{S})$ is a no-instance of X3C, then there is at least one agent who owns a bundle $T \subseteq B$ such that T does not contain any subset $S_i \in \mathscr{S}$. This implies that $sw_e(X) = 0$ for each allocation X, so $\max_e(M) = 0$. Indeed, assume that all agents are assigned bundles containing some $S_j \in \mathscr{S}$. Since the resources are indivisible and nonshareable, there must be n pairwise disjoint subsets in $\mathscr{S}$ that are an exact cover of B, a contradiction.

Therefore, if there were a polynomial-time approximation algorithm that approximates $\mathbb{Q}$-MAX-ESW$_{\text{3-additive}}$ within any factor, then it could distinguish yes- from no-instances of X3C. This contradicts the NP-hardness of X3C unless P = NP, and the theorem follows. ❑

Finally, turning to Nash product social welfare, we now present an exact polynomial-time algorithm for $\mathbb{Q}^+$-MAX-NPSW$_{\text{1-additive}}$ when restricted to $m = n$ and assuming that the empty bundle has always utility zero, by converting this problem to the problem of finding a maximum matching in some complete bipartite graph. We then extend this result to the general case of the k-additive form for any k. As mentioned in the previous section, an analogous result has also been established by Golovin (2005) for $\mathbb{Q}$-MAX-ESW$_{\text{1-additive}}$ with the same restriction and can also be applied to the k-additive form when $k \geq 2$.

Theorem 10 *For each $k \geq 1$, $\mathbb{Q}^+$-MAX-NPSW$_{k\text{-additive}}$ can be solved exactly in polynomial time when the number of agents and resources are the same and the empty bundle has always utility zero.*

PROOF. First suppose $k = 1$. Given a MARA setting $M = (A,R,U)$, we assume the utility functions $u_i \in U$ to be nonnegative and $u_i(\emptyset) = 0$ for all agents $a_i \in A$. Let $n = \|A\|$ and $m = \|R\|$. Note that by multiplying a large enough positive constant to each of these utility functions $u_i(r_j)$, $1 \leq i \leq n$ and $r_j \in R$, it suffices to consider the case when $u_i(r_j) \geq 1$ for all $u_i(r_j) \neq 0$. Construct an edge-weighted complete bipartite graph $G = (V,E)$ with vertex set $V = A \cup R$.

The edges $\{a_i, r_j\} \in E$ are weighted by some function $w : A \times R \to \mathbb{R}$, defined by

$$w(a_i, r_j) = \begin{cases} \log u_i(r_j) & \text{if } u_i(r_j) \neq 0 \\ -n \log \mu & \text{otherwise,} \end{cases}$$

where $\mu = \max\{u_i(r_j) \mid 1 \leq i \leq n \text{ and } r_j \in R\}$. Note that a perfect matching with maximal weight on the complete bipartite graph $G = (V,E)$ can be found in polynomial time, for example by the "Hungarian method" due to Kuhn (1955).

Now we prove that a maximum matching of the complete bipartite graph G corresponds to an allocation maximizing social welfare of M by the Nash-Product. Indeed, it is easy to see that each of the agents can get only one single resource.[4] Since the number of vertices of G is $2n$, there are $n!$ possible matchings and each of them has the form:

$$M_\pi = \{(a_1, r_{\pi(1)}), (a_2, r_{\pi(2)}), \ldots, (a_n, r_{\pi(n)})\}$$

where π is a permutation of $\{1,2,\ldots,n\}$. Define the weight of M_π by $W(M_\pi) = \sum_{i=1}^{n} w(a_i, r_{\pi(i)})$, and let $M_{\max}$ be a matching of maximum weight. Since $u_i(r_j) \leq \mu$ for all i, $1 \leq i \leq n$, and all $r_j \in R$, the weight of a matching is positive if and only if it does not contain any edge of negative weight. Thus, if $W(M_{\max}) < 0$ then the maximum Nash product of (A,R,U) is zero. We may thus suppose that $M_{\max}$ has no edge whose weight is negative, i.e., $u_i(r_{\pi(i)}) \geq 1$ for each i, $1 \leq i \leq n$, where π is the permutation corresponding to $M_{\max}$. Suppose there were another allocation (corresponding to a matching $M_{\pi'}$) with a larger Nash product than the allocation corresponding to $M_{\max} = M_\pi$:

$$\prod_{i=1}^{n} u_i(r_{\pi(i)}) < \prod_{i=1}^{n} u_i(r_{\pi'(i)}).$$

[4] Otherwise (i.e., if there were some agent getting more than one resource), since $n = m$, at least one other agent would get nothing at all, which means that the Nash product social welfare is zero.

This, however, would imply

$$\log\left(\prod_{i=1}^{n} u_i(r_{\pi(i)})\right) < \log\left(\prod_{i=1}^{n} u_i(r_{\pi'(i)})\right),$$

which in turn gives

$$\sum_{i=1}^{n} \log u_i(r_{\pi(i)}) < \sum_{i=1}^{n} \log u_i(r_{\pi'(i)}),$$

and hence we have

$$\sum_{i=1}^{n} w(a_i, r_{\pi(i)}) < \sum_{i=1}^{n} w(a_i, r_{\pi'(i)}),$$

a contradiction to the fact that $M_{\max}$ is a matching of maximum weight in G. Thus, $M_{\max}$ corresponds to an allocation maximizing social welfare of (A,R,U) by the Nash product.

The above argument can be easily extended to the case of $k \geq 2$, since each of the agents will get only one single resource, as otherwise the Nash product social welfare would be zero (see Footnote 4). ❑

Now consider the other special case of the problem $\mathbb{Q}^+$-NPSW$_{\text{1-additive}}$ mentioned in the previous section and suppose that there are only two agents having the same utility function. The problem thus restricted is still NP-hard, which follows from the construction given by Roos and Rothe (2010). We are interested in the approximability of the problem with this restriction. In particular, we will design an PTAS for $\mathbb{Q}^+$-NPSW$_{\text{1-additive}}$ under this restriction.

Theorem 11 *$\mathbb{Q}^+$-MAX-NPSW$_{1\text{-additive}}$ admits a* PTAS *when restricted to only two agents having the same utility function u with $u(\emptyset) = 0$.*

PROOF. Let $M = (A,R,U)$ be a given MARA setting with two agents (i.e., $A = \{a_1, a_2\}$) and nonnegative utilities $u_1(r) = u_2(r) \geq 0$ for all $r \in R$. Let u denote this utility function, i.e., $u = u_1 = u_2$. Consider the following greedy algorithm for this problem. Intuitively, this algorithm seeks to find disjoint subsets R' and R'' of R such that $R' \cup R'' = R$ and the product $u(R') \cdot u(R'')$ is maximized, where $u(T) = \sum_{r \in T} u(r)$. Without loss of generality, we can assume that R' is assigned to agent a_1 and R'' is assigned to agent a_2. Let ε be any fixed constant such that $0 < \varepsilon < 1$.

We now prove that Algorithm 1 is an $(1-\varepsilon)$-approximation algorithm for $\mathbb{Q}^+$-MAX-NPSW$_{\text{1-additive}}$ in our restricted setting. We need to show that the algorithm always returns in polynomial time two subsets R' and R'' such that $u(R') \cdot u(R'') \geq (1-\varepsilon)OPT$, where OPT is the optimal value of the problem, $\max_N(M)$.

Without loss of generality, we assume that $u(R') \geq u(R'')$ and that r_j is the last resource that was assigned to agent a_1. This implies that $u(R'') \geq u(R') - u(r_j)$. By addition of $u(R'')$ to the two sides of the inequality we get $2 \cdot u(R'') \geq u(R) - u(r_j)$, i.e., $u(R'') \geq (u(R) - u(r_j))/2$.

If $j \leq k$, it is easy to see that the obtained solution is indeed an optimal solution. Otherwise, we have $u(r_j) \leq u(r_i)$ for any $1 \leq i \leq k$, since the sequence $(r_1, r_2, \ldots, r_m)$ was sorted in nonincreasing order according to their utilities.

Algorithm 1 Greedy algorithm

```
1: Sort the resources in nonincreasing order of their utilities to get a sequence (r_1, r_2, ..., r_m) such that u(r_1) ≥ u(r_2) ≥ ··· ≥ u(r_m).
2: Set k := ⌈−1 + 1/√(1−ε)⌉.
3: Perform an exhaustive search for an optimal solution (R′, R″) over the k resources of (r_1, r_2, ..., r_k).
4: for i := k+1 to m do
5:     if u(R′) ≤ u(R″) then
6:         R′ := R′ ∪ {r_i}
7:     else
8:         R″ := R″ ∪ {r_i}
9:     end if
10: end for
11: return (R′, R″)
```

Therefore, one can easily check that $u(R) \geq (k+1)u(r_j)$. Furthermore, it is obvious that $OPT \leq {(u(R))^2}/4$ and, due to the "arithmetic mean – geometric mean" inequality (which says that $a \cdot b \leq {(a+b)^2}/4$ for all $a, b \geq 0$), we have

$$\begin{aligned} \frac{u(R') \cdot u(R'')}{OPT} &\geq \frac{(u(R''))^2}{\frac{(u(R))^2}{4}} \geq \frac{\left(\frac{u(R)}{2} - \frac{u(r_j)}{2}\right)^2}{\frac{(u(R))^2}{4}} \\ &= \left(1 - \frac{u(r_j)}{u(R)}\right)^2 \geq \left(1 - \frac{1}{k+1}\right)^2 \\ &\geq 1 - \varepsilon. \end{aligned}$$

The running time of Algorithm 1 is dominated by the work in lines 1 and 3, since the other steps have a smaller cost. At line 1 of the algorithm, we need to sort the sequence of m resources, which takes time $\mathcal{O}(m \log m)$. Regarding line 3, we have to search exhaustively on the set of subsets of $\{r_1, \ldots, r_k\}$, and we need time $\mathcal{O}(m^k)$ to do this. In total, our algorithm runs in polynomial time, more precisely in time $\mathcal{O}\left(m \log m + m^{\lceil -1 + 1/\sqrt{1-\varepsilon} \rceil}\right)$. ❑

As mentioned in the previous section, $\mathbb{Q}^+$-MAX-NPSW$_{\text{1-additive}}$ is closely related to the problem:

MINIMUM PARTITION	
Input:	A sequence $(c_1, c_2, \ldots, c_n)$ of n nonnegative integers.
Output:	A subset $I \subseteq \{1, 2, \ldots, n\}$ of indices that minimizes $\max\left(\sum_{i \in I} c_i,\ \sum_{i \notin I} c_i\right)$.

Note that $\max\left(\sum_{i \in I} c_i,\ \sum_{i \notin I} c_i\right)$ is minimal exactly when the two sums are equal. Similarly, assigning a bundle of equal value to both agents maximizes social welfare by the Nash product. As mentioned in the previous section, the problem MINIMUM MAKESPAN SCHEDULING (see (Vazirani 2003) for the definition) is also closely related to $\mathbb{Q}$-MAX-ESW$_{\text{1-additive}}$. Since MINIMUM MAKESPAN SCHEDULING is strongly NP-complete, it is impossible to have an FPTAS for this problem, assuming P $\neq$ NP. However, there does exist a PTAS (Hochbaum and Shmoys 1987). This problem even has an FPTAS when the number of machines is fixed (Horowitz and Sahni 1976)). Despite the similarities between the four problems mentioned above, since they have different objective functions, it is not clear whether approximability results for one problem transfer to one of the other three. In particular, we conjecture that when the number of agents is fixed, $\mathbb{Q}^+$-MAX-NPSW$_{\text{1-additive}}$ has an FPTAS as well.

Finally, we have the following results on the inapproximability of $\mathbb{Q}^+$-MAX-NPSW$_{k\text{-additive}}$ for every fixed $k \geq 2$. We start with the case of $k = 2$.

Proposition 12 *$\mathbb{Q}^+$-MAX-NPSW$_{\text{2-additive}}$ cannot be approximated in polynomial time within any factor better than $\alpha = {21}/{22}$, unless* P = NP. *This result holds even when there are only two agents.*

PROOF. Modify the reduction from the proof of Proposition 8: For agent a_2, the empty bundle has utility one and all other bundles with at most two resources have utility zero. Thus the number of clauses satisfied is exactly the maximum Nash product social welfare. Everything else remains unchanged. Using the inapproximability bound of $\alpha = {21}/{22}$ for MAX-2-SAT due to Håstad (2001), the result follows. ❑

For the case of $k \geq 3$, an even stronger inapproximability result can be shown: $\mathbb{Q}^+$-MAX-NPSW$_{k\text{-additive}}$ is NP-hard to approximate to within any factor.

Theorem 13 *For each $k \geq 3$, $\mathbb{Q}^+$-MAX-NPSW$_{k\text{-additive}}$ cannot be approximated in polynomial time within any factor, unless* P = NP.

PROOF. The proof is similar to the proof of Theorem 9. ❑

Conclusions and Open Questions

Social welfare optimization is one of the most important and vibrant research fields in multiagent resource allocation. We have surveyed some known and presented some new approximability and inapproximability results for the problems of maximizing utilitarian, egalitarian, or Nash product social welfare, where the agents' utilities are represented either in the bundle or the k-additive form. These results are summarized in Tables 2 and 3 and complement the results known for the corresponding decision problems stated in Table 1.

Still, many problems remain open, especially for the unrestricted problems considered in this survey. While for the bundle form mainly results on *utilitarian* social welfare optimization are known (see Table 2), most results known for the k-additive form regard *egalitarian* social welfare optimization (see Table 3). We propose to study all three types of social welfare for both representation forms, and have started by focusing in particular on the previously somewhat neglected Nash product social welfare in this paper.

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