Bi-tone and k-tone Decompositions of Boolean Functions

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Abstract

For a Boolean Matrix A a binary vector v is called t-frequent if Av has at least t entries of value $\operatorname{supp}(v)$. Given two parameters $t_1 < t_2$ the t_1-frequent but t_2-infrequent vectors of a matrix represent a Boolean function that has two domains of (opposite) monotonicity. These functions were studied for the purpose of data analysis and abstract concept discovery in (Eisenschmidt et al. 2010). In this paper we introduce as a generalization very structured Boolean functions partitioning their domain into k alternating regions of monotonicity. We show how given two sets W_0 and W_1 prescribing values 0 and 1 such a function with minimal k can be constructed. We believe that in data analysis or data fitting tasks such well-structured functions are advantageous, especially for small values of k.

1 Bitone Boolean functions

A *Boolean* function is a function f mapping from $\{0,1\}^n$ into $\{0,1\}$. A Boolean function is called *(up-)monotone*, if for $x, y \in \{0,1\}^n$ with $x \leq y$ also $f(x) \leq f(y)$ holds. A Boolean function is monotone if and only if it is representable as conjunction and disjunction of binary variables (i.e., without any negations). We will denote by $|f|$ the number of maximal false points of f. For an exhaustive overview see (Crama and Hammer 2011).

Let $f_1, f_2 \colon \{0,1\}^n \to \{0,1\}$ be monotone Boolean functions such that $f_1(x) \leq f_2(x)$ for all $x \in \{0,1\}^n$. Let $g := \neg f_1 \wedge f_2$. Then g is a Boolean function that is in general not monotone. We will call a function arising in this way a *bitone* Boolean function.

The term *bitonicity* can be explained as follows: Let $x_1, \ldots, x_s \in \{0,1\}^n$ be the maximal false points (MFPs) of f_1, i.e. $f_1(x_i) = 0$ for all $i = 1, \ldots, s$. Let $y_1, \ldots, y_t$ be MFPs of f_2. Then the relation $f_1 \leq f_2$ implies that for every y_i there exists x_j such that $y_i \leq x_j$. Now we partition the set of Boolean vectors into three sets:

$$\begin{aligned} M_0 &:= \{x \in \{0,1\}^n \colon \exists i \in \{1,\ldots,t\} \text{ such that } x \leq y_i\} \\ M_2 &:= \{x \in \{0,1\}^n \colon \exists j \in \{1,\ldots,s\} \text{ such that } x_j < x\} \\ M_1 &:= \{0,1\}^n \setminus (M_0 \cup M_2). \end{aligned} \tag{1}$$

Figure 1 illustrates the partition induced by a bitone function. The set M_1 forms a band between the sets M_0 and M_2. The function g evaluates to 1 for all elements of M_1. It evaluates to 0 for all elements of M_0 and M_2. The function g is *up-monotone* w.r.t. M_0 and M_1, it is *down-monotone* w.r.t. M_1 and M_2.

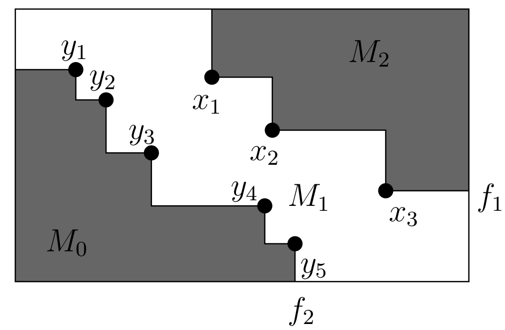

Figure 1: Partition colored by value of bitone Boolean function.

In the following, we will show that the concept of bitone Boolean functions is closely related to the concept of FMI-sets as defined in (Eisenschmidt et al. 2010).

Lemma 1. Let $0 \leq t_1 < t_2 \leq m$ and let $A \in \{0,1\}^{m \times n}$ be a binary matrix. The t_1-frequent and t_2-infrequent sets of matrix A correspond to the Boolean vectors yielding a value of 1 of a bitone Boolean function $g \colon \{0,1\}^n \to \{0,1\}$.

Proof. Taking the t_1-infrequent sets in matrix A as minimal true points we obtain an (up-)monotone Boolean function f_1. Analogously, the t_2-infrequent sets in matrix A induce another monotone Boolean function $f_2 \colon \{0,1\}^n \to \{0,1\}$. As $t_1 < t_2$, we obtain that $f_1 \leq f_2$ holds. Therefore, the function $g := \neg f_1 \wedge f_2$ is a bitone Boolean function. Furthermore, $g(x) = 1$ if and only if $\operatorname{supp}(x)$ is t_1-frequent and t_2-infrequent w.r.t. matrix A. ☐

Now we will show that for each bitone function $g \colon \{0,1\}^n \to \{0,1\}$ we can construct a matrix A such that the 1-frequent and l-infrequent sets correspond to those Boolean vectors for which $g(x) = 1$ holds.

Lemma 2. Let $g := \neg f_1 \wedge f_2 \colon \{0,1\}^n \to \{0,1\}$ be a bitone function. There exists a binary matrix $A \in \{0,1\}^{m \times n}$ with at most $|f_1| + (|f_1|+1)|f_2|$ rows and a parameter $l \leq m$ such that the elements $x \in \{0,1\}^n$ with $g(x) = 1$ correspond to 1-frequent and l-infrequent sets of the matrix A.

Proof. Let $x_1, \dots, x_s \in \{0,1\}^n$ be the maximal vectors such that $f_1(x_i) = 0$ for $i = 1, \dots, s$. Analogously, let $y_1, \dots, y_t \in \{0,1\}^n$ be the maximal binary vectors such that $f_2(y_j) = 0$ for all $j = 1, \dots, t$.

The first step of the construction works as follows: Every element $x \in \{x_1, \dots, x_s\}$ gives rise to a row of the matrix A. Having put up a preliminary matrix $\tilde{A}$ this way, we determine the maximal integer z such that there is a column index $i \in \{1, \dots, n\}$ with $\{i\}$ being a z-frequent set w.r.t. $\tilde{A}$. For every y_j, $j = 1, \dots, t$, let t_j denote the frequency of $\mathrm{supp}\,(y_j)$ w.r.t. $\tilde{A}$. For every y_j, we add $(z+1) - t_j$ copies of the vector y_j to the matrix $\tilde{A}$. The matrix we obtain in this way is the matrix A we were looking for.

We set $l := z + 1$. Let $J \subseteq \{1, \dots, n\}$ be a column index set that is 1-frequent and l-infrequent. We have to show that $g(x_J) = 1$. This is equivalent to showing that $\neg f_1(x_J) = 1$ and $f_2(x_J) = 1$ holds. As J is 1-frequent it is contained in a maximally 1-frequent set $\bar{J}$. Consider the set of rows of matrix A that are all one on the set of columns $\bar{J}$. Assume that this set of rows contains rows that are added to the matrix A in the second construction step. Then $\bar{J}$ would be contained in a l-frequent set and it would be l-frequent itself. This is a contradiction. Therefore, the rows chosen by $\bar{J}$ are added in the first construction step. As $\bar{J}$ is maximally chosen it is clear that there is an index $i \in \{1, \dots, s\}$ such that $\bar{J} = \mathrm{supp}\,(x_i)$. Therefore, $f_1(x_J) \leq f_1(x_{\bar{J}}) = f_1(x_i) = 0$. It remains to show that $f_2(x_J) = 1$. As J is l-infrequent it is not contained in any of the support sets $\mathrm{supp}\,(y_j)$, $j = 1, \dots, t$. Therefore, there is no vector y_j with $x_J \leq y_j$, $j = 1, \dots, t$. As the vectors y_j were maximally chosen such that $f_2(y_j) = 0$, we obtain that $f_2(x_J) = 1$. □

We note that in general such a representation will not be compact: It is known that a (degenerate) bitone Boolean function that is simply monotone may require exponentially many rows in any matrix representing it as t-infrequent sets (Sloan, Takata, and Turán 1998).

Corollary 1 (cf. (Sloan, Takata, and Turán 1998)). Almost all n-variable bitone Boolean functions require $\Omega(\frac{2^{n/2}}{n})$ rows to be represented by a matrix with any threshold.

Proof. We will show first that there are at least $3^{\frac{2^n}{n^{1/2}}}$ bitone functions on n variables. Let f be a monotone Boolean function with all its minimal true points contained in the set of Boolean vectors of support $\binom{n}{\lfloor n/2 \rfloor}$. For such a Boolean function f, let X^f denote its set of minimal true points. Let g be a monotone Boolean with all its minimal true points contained in

$$\{x \in \{0,1\}^n \colon \mathrm{supp}(x) = \lfloor n/2 \rfloor\} \setminus X^f.$$

Then clearly, $f \leq g$. It thus follows that there are at least

$$\sum_{k=0}^{\binom{n}{\lfloor n/2 \rfloor}} \binom{\binom{n}{\lfloor n/2 \rfloor}}{k} 2^{\binom{n}{\lfloor n/2 \rfloor} - k} = 3^{\binom{n}{\lfloor n/2 \rfloor}} \geq 3^{2^{\frac{n}{2}}}.$$

On the other hand, there are at most $m^2 2^{mn}$ binary-matrices with at most m and corresponding thresholds (Sloan, Takata, and Turán 1998). Comparing the two expressions yields our claim. □

2 From bitone to k-tone Boolean functions

We have seen in the previous section that bitone Boolean function can be characterized as those functions $g : \{0,1\}^n \to \{0,1\}$ partitioning the set of Boolean vectors $\{0,1\}^n$ into three disjoint sets M_0, M_1 and M_2 such that

$$\begin{aligned} &\forall\, x \in M_2 \quad \nexists\, y \in M_1 \cup M_0 \text{ with } y \geq x \\ &\forall\, x \in M_1 \quad \nexists\, y \in M_0 \text{ with } y \geq x \end{aligned} \tag{2}$$

and g evaluates to 0 on M_0 and M_2 and evaluates to 1 on M_1. Now, we generalize this concept to k-tone functions:

Definition 1. A Boolean function $g \colon \{0,1\}^n \to \{0,1\}$ is called k-tone, if it partitions the set of Boolean vectors $\{0,1\}^n$ into $k+1$ disjoint sets such that

$$\forall\, x \in M_i, \forall j < i \quad \nexists\, y \in M_j \text{ with } y \geq x \quad i = 1, \dots, k \tag{3}$$

and $g|_{M_i} = 1 - g|_{M_{i+1}}$ for $i = 0, \dots, k-1$.

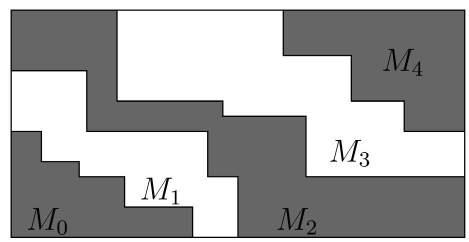

Figure 2: Partition corresponding to a 4-tone Boolean function colored by its values.

Remark 1. For convenience, we will standardize on the case that $g|_{M_0} \equiv 0$ for every k-tone function.

Lemma 3. A k-tone Boolean function can be represented as the disjuction of $\frac{k}{2}$ bitone Boolean functions if k is even and of $\lfloor \frac{k}{2} \rfloor - 1$ bitone functions and one monotone Boolean function if k is odd.

Proof. We know from the definition that every k-tone Boolean function partitions the set of all Boolean vectors $\{0,1\}^n$ into $k+1$ sets $M_0, \dots, M_k$. For every set $\bigcup_{t=0}^{i} M_t$ ($i = 0, \dots, k-1$) determine the maximal (w.r.t. $\leq$) vectors $x_{1,i}, \dots, x_{j(i),i}$. We define monotone Boolean functions $f_i \colon \{0,1\}^n \to \{0,1\}$ with the help of these maximal elements:

$$f_i(x) = \begin{cases} 0, & \text{if } \exists\, l \in \{1, \dots, j(i)\} \text{ such that } x \leq x_{l,i} \\ 1, & \text{otherwise.} \end{cases} \tag{4}$$

This implies immediately that $f_j \leq f_i$ for $i \leq j$ holds. Let $j \in \{1, \dots, k-1\}$ be an odd index. Then $g|_{M_j} \equiv 1$. We define a bitone Boolean function $h_j \colon \{0,1\}^n \to \{0,1\}$ as $h_j := \neg f_j \wedge f_{j-1}$. h_j evaluates to 1 on M_j and to

0 otherwise. To see this, let $x \in M_j$. Then there exists $x_{i,j} \in M_j$ (a maximal element) such that $x \leq x_{i,j}$. Therefore, $f_j(x) = 0$. As $x \in M_j$ the partition property (3) implies that there is no element y in M_{j-1} such that $y \geq x$. Therefore, $f_{j-1}(x) = 1$. Together this yields $h_j(x) = 1$. Now let $x \in \{0,1\}^n \setminus M_j$. We want to show that $h_j(x) = 0$ then. There exists an index i such that $x \in M_i$. Let us assume first that $i < j$. With the same argument as before we find that $f_{j-1}(x) = 0$ and thus $h(x) = 0$. Therefore, let us assume that $i > j$. From Definition (1) we obtain that there is no element in $x_{k,j} \in \{x_{1,j}, \ldots, x_{j(j),j}\}$ such that $x_{k,j} \leq x$. Therefore $f_j(x) = 1$ and thus $h_j(x) = 0$.

Then the k-tone Boolean function g may be expressed as follows:

$$g = \begin{cases} \bigvee_{\substack{j=1 \\ j \text{ odd}}}^{k-1} h_j, & \text{if } k \text{ is even,} \\ \bigvee_{\substack{j=1 \\ j \text{ odd}}}^{k-2} h_j \vee f_{k-1}, & \text{if } k \text{ is odd.} \end{cases} \tag{5}$$

If k is even, then $g|_{M_k} \equiv 0$ and the disjuction of the bitone functions h_j for odd $j \in \{1, \ldots, k-1\}$ equals the k-tone function h. If k is odd, then $g|_{M_k} \equiv 1$ and we have to add the monotone Boolean function f_{k-1} to the disjuction of the bitone functions h_j for odd $j \in \{1, \ldots, k-2\}$ to represent the k-tone Boolean function g. □

Note that by construction we always have

Observation 1. For a k-tone Boolean function $f = \bigvee_{i \in \{0,\ldots,\lfloor \frac{k}{2} \rfloor\}} (\neg f_i \wedge f_{i+1}) \vee h_k$ (where $h_k = 0$ if k is even) it is always true that for $i \neq j$:

$$\operatorname{supp}(\neg f_i \wedge f_{i+1}) \cap \operatorname{supp}(\neg f_j \wedge f_{j+1}) = \emptyset.$$

Proof. Assume $i < j$ and take any x with $g_j(x) = 1$. Then $f_j(x) = 0$, hence $f_i(x) = f_{i+1} = 0$, hence $g_i(x) = 0$. □

3 Application to data analysis

In data fitting applications where measurements only partially determine a Boolean function we may want to require k-tonicity of the function determined. This may be because the alternating structure is natural to the application since the data naturally clusters in 'stripes', or because the direct decomposition into monotone Boolean functions is an advantage to understanding the system described by it. Obviously the smaller k for such an application, the better.

We hence consider the following problem: let $W_0 \subseteq \{0,1\}^n$ and $W_1 \subseteq \{0,1\}^n$ such that $W_0 \cap W_1 = \emptyset$. We want to determine the minimal integer k such that there exists a k-tone Boolean function g that evaluates to 0 on W_0 and to 1 on W_1. A first upper bound for k integer is easily determined:

Lemma 4. Let $W_0 \subseteq \{0,1\}^n$ and $W_1 \subseteq \{0,1\}^n$ with $W_0 \cap W_1 = \emptyset$. Then there is a $(2n+1)$-tone Boolean function $g\colon \{0,1\}^n \to \{0,1\}$ evaluating to 0 on W_0 and to 1 on W_1.

Proof. We partition the set of Boolean vectors into $2n+2$ sets according to the total support and the set W_1:

$$K_i := \left\{ x \in W_1 \colon \sum_{j=1}^{n} x_j = i \right\} \tag{6}$$

$$S_i := \left\{ x \in \{0,1\}^n \colon \sum_{j=1}^{n} x_j = i \right\} \setminus K_i \tag{7}$$

We order these sets as follows: $S_0, K_0, S_1, K_1, \ldots, S_n, K_n$. It is immediately clear that the sequence of sets partitions $\{0,1\}^n$ and satisfies property (3). A Boolean function alternating on these sets and evaluating to 0 on S_0 is $2n+1$-tone. □

In a second step we show that the parameter k is bounded from below by the length of the longest chain of elements alternating between W_0 and W_1. (Note that the length of this chain is bounded itself by the maximal Hamming-distance of a pair of points in $W_0 \times W_1$.) Afterwards, we will present an algorithm yielding a partition with a length realizing this lower bound.

Lemma 5. Let $x_1 \leq \ldots \leq x_k$ be a chain of elements with $x_{2i} \in W_0$ ($x_{2i+1} \in W_0$) and $x_{2i+1} \in W_1$ ($x_{2i} \in W_1$) for all $i = 0, \ldots, \lfloor \frac{k}{2} \rfloor$. Let f be an l-tone Boolean function taking the value of 0 on the set W_0 and the value 1 on the set W_1. Then $l \geq k-1$.

Proof. Let $M_0, \ldots, M_l$ be a partition of $\{0,1\}^n$ with partition property (3) and such that $M_{2j} \cap W_0 = \emptyset$ and $M_{2j+1} \cap W_1 = \emptyset$ or such that $M_{2j} \cap W_1 = \emptyset$ and $M_{2j+1} \cap W_0 = \emptyset$ ($j = 0, \ldots, \lfloor \frac{l}{2} \rfloor$). As the elements x_i alternate between W_0 and W_1 and as the partition $M_0, \ldots, M_l$ has property (3), it is clear that there is no M_j containing more than one of the x_i. □

Observation 2. A longest chain as in Lemma 5 for explicitly given sets W_0 and W_1 can be computed in polynomial time. For sets implicitly given by a membership oracle the problem is $\mathcal{NP}$-hard.

Proof. If we are allowed to list the elements of W_0 and W_1, then we can build the following digraph: $G = (W_0 \cup W_1, A)$ where $(x_0, x_1) \in A$ if and only if $x_1 \leq x_2$ and ($x_i \in W_i$ or $x_i \in W_{1-i}$). The graph is acyclic, and a longest path from a node in W_0 to a node in W_1 can hence be computed in polynomial time.

Without listing W_0 and W_1 the problem is as hard as determining the largest independent set in a graph: For a given graph G let W_0 be the incidence vectors of all independent sets with even, W_1 the independent sets with odd cardinality. Then a longest chain as for Lemma 5 determins the largest independent set, assuming it is odd (otherwise reverse the roles of W_0 and W_1). □

We now investigate the complexity of determining the minimal k necessary for a k-tone representation of a given function:

Lemma 6. The decision problem "Given a Boolean function h represented as a k-tone Boolean function, does there exist a representation as an k'-tone Boolean function with $k' < k$" is co-$\mathcal{N}P$-complete.

Proof. We will assume that $h(0) = 0$ as usual, and show that deciding whether a representation for $k = 2$ is minimal is already co-$\mathcal{N}P$-complete.

Let $h = \neg f \wedge g$ for monotone Boolean functions $f \leq g$. We can assume that $f \neq 0$ since that is easy to test. We will show that testing minimality of the representation of h amounts to checking whether h is (up-)monotone, and that checking equivalency of monotone Boolean functions, which is known to be co-$\mathcal{N}P$-complete (Reith 2003), can be reduced to this problem.

Clearly, if the 2-tone representation $h = \neg f \wedge g$ is not minimal, then the smaller 1-tone representation will have to be monotone. Conversely, if h is in fact monotone, then it can serve as a 1-tone representation, and hence the two-tone representation is not minimal.

Hence the decision problem is in co-$\mathcal{N}P$: In the NO-case (i.e. there exists no $k' < k$ to represent h as a k'-tone Boolean function), a certifcate for non-monotonicity of h is given by a point x^* such that $0 = h(0) < h(x) > h(1)$.

It was shown in (Reith 2003) that checking equivalency $f = g$ of monotone Boolean formulas f, g is co-$\mathcal{N}P$-complete. This is still true if we assume $f \leq g$: An oracle for deciding equivalency under the assumption $f \leq g$ need only be called twice, with $f \wedge g = g$ and $f \wedge g = f$ to decide $f = g$ for arbitrary monotone Boolean formulas.

Now, let an oracle deciding monotonicity of $h = \neg f \wedge g$ for two monotone Boolean functions f, g with $f \leq g$ be given. It can decide equivalency of f and g as follows: h can only be monotone if $g = 1$ (which is impossible because $h(0) = 0$), if $\neg f = 1$ (which we excluded), or if $f = g$, since if there exists a point x such that $f(x) < g(x)$ we will have $0 = h(0) < h(x) > h(1) = 0$. Here $h(1) = 0$ because $f(1) = 1$, since $f \neq 0$. □

In the following we will present an algorithm that – given two sets $W_0, W_1 \subseteq \{0,1\}^n$ and an oracle returning the maximal elements in a set – returns a sequence of sets of minimal cardinality yielding a partition with the property (3).

Lemma 7. Algorithm 3.1 returns a sequence of sets with property (3).

Proof. We will show that the sequence of sets $M_0, \ldots, M_k$ has the following property:

$$\forall x \in M_i \quad \nexists y \in M_j \text{ with } j > i \text{ and } x \leq y.$$

Apart from different indexing, this is property (3). Assume not. Then there is an $x \in M_i$ and a $y \in M_j$, $j > i$, with $y \geq x$. If this was true, then the element x would not have been added to the set M_i in the i-th step (as it is not maximal w.r.t. y). This proves that the sequence of sets returned by Algorithm 3.1 has property (3). □

Lemma 8. Let $M_0, \ldots, M_i$ be the sequence of sets returned by Algorithm 3.1. Let $x \in M_j \cap W_l$ for $j \in \{0, \ldots, i\}$ and $l \in \{0, 1\}$. If $j \geq 2$, there is $y \in M_{j-1} \cap W_{1-l}$ with $y \geq x$.

Algorithm 3.1 Algorithm determining the minimal parameter for a k-tone Boolean function

Input: $W_0, W_1 \subseteq \{0,1\}^n$; an oracle returning the maximal elements of a subset of $\{0,1\}^n$.
Output: Sets $M_0, \ldots, M_k$ yielding a partition with property (3) with minimal $k \in \mathbb{Z}_+$

- $i := -1$; $V_0 := W_0$ and $V_1 := W_1$
- **while** $V_0 \cup V_1 \neq \emptyset$ **do**
  - $i := i + 1$, $j := i \bmod 2$ and $\bar{M}_i := \emptyset$
  - $\mathcal{M} :=$ maximal elements in $V_0 \cup V_1$
  - **while** $\mathcal{M} \cap V_j \neq \emptyset$ **do**
    - $\bar{M}_i := \bar{M}_i \cup \mathcal{M}$
    - $V_j := V_j \setminus \bar{M}_i$
    - $\mathcal{M} :=$ maximal elements in $V_0 \cup V_1$
  - **end while**
  - $X := \bar{M}_i \setminus V_j$
  - **for all** $x \in X$ **do**
    - Determine $S_x := \{l \in \{1, \ldots, n\} \colon x + e_l \in \{0,1\}^n \setminus W_{1-j}\}$
  - **end for**
  - $M_i := \bar{M}_i \setminus X \cup \bigcup_{x \in X} \bigcup_{l \in S_x} x + e_l$
- **end while**
- Repeat the algorithm with interchanged W_0 and W_1, return the sequence of sets of lower cardinality.
- **return:** $M_0, \ldots, M_i$.

Proof. x is added to the set M_j in the j-th step of the algorithm. W.l.o.g. we assume that $x \in W_0$. Let us consider

$$\mathcal{M} = \text{maximal elements in } V_0 \cap V_1$$

at the beginning of the $(j-1)$-th step of the algorithm. $j \geq 2$ implies that $\mathcal{M} \subseteq W_1$. The maximality of the elements yields the existence of $y \in \mathcal{M}$ with $y \geq x$. The element y is added to the set M_{j-1} in the course of the $(j-1)$-th step of the algorithm. □

It remains to show that Algorithm 3.1 returns indeed a sequence of sets of minimal cardinality. We will show this by showing that the cardinality of the sequence of sets equals the cardinality of the largest chain of elements with elements alternating between W_0 and W_1.

Lemma 9. Let $x_0 \leq x_1 \leq \ldots \leq x_t$ be the largest chain of elements with elements alternating between W_0 and W_1. Let $M_0, \ldots, M_i$ be the sequence of sets returned by Algorithm 3.1. Then $i = t$ and $x_{t-j} \in M_j$ for all $j = 0, \ldots, t$.

Proof. In a first step, we assume that $x_t \in W_0$ and $M_i \cap W_1 = \emptyset$. The other case, i.e., $x_t \in W_0$ and $M_i \cap W_0 = \emptyset$ will be excluded in the end.

We show that $x_t \in M_0$. Assume not. Then there is an element $z \in M_0$ with $z \geq x_t$. Thus there exists a vector $y \in W_1$ with $z \geq y \geq x_t$ yielding that the chain $x_1 \leq \ldots \leq x_t$ is not the largest chain with elements alternating between W_0 and W_1. The chain $x_1 \leq \ldots \leq x_t \leq y \leq z$ is of larger cardinality contradicting our assumption.

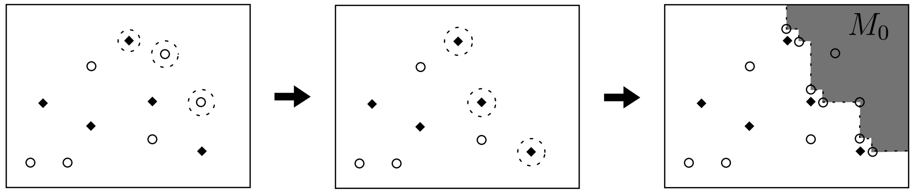

Figure 3: The first step of the algorithm. Elements of W_0 are marked with circles, elements of W_1 are marked with squares.

Assume the claim is true for all $j < j*$, i.e., $x_{t-j} \in M_j$ for all $j < j^*$. Now consider the j^*-th step of Algorithm 3.1. Assume that x_{t-j^*} is not added to M_{j^*} in this step. This implies that there is $z \in (W_0 \cup W_1) \setminus (M_0 \cup \ldots \cup M_{j^*-1})$ with $z \geq x_{t-j^*}$. There are several cases possible:

1. $z \in W_0$.
   (a) $\exists y \in W_1 \setminus (M_0 \cup \ldots \cup M_{j^*-1})$ with $x_{t-j^*} \leq y \leq z$. With Lemma 8 we can construct a larger chain of alternating elements. This contradicts our assumption.
   (b) $\nexists y \in W_1 \setminus (M_0 \cup \ldots \cup M_{j^*-1})$ with $x_{t-j^*} \leq y \leq z$. But then x_{t-j^*} would have been added to $\bar{M}_{j^*}$ and thus to M_{j^*}.
2. $z \in W_1$.
   As in the beginning of the j^*-th step all maximal elements in $(W_0 \cup W_1) \setminus (M_0 \cup \ldots \cup M_{j^*-1})$ are contained in W_0, there exists $x \in M_{j^*}$ with $x \geq z \geq x_k$. With Lemma 8 it is then possible to construct a larger chain of alternating elements. This contradicts our assumption.

All in all, we obtain that x_{t^*-j} is indeed added to the set M_{j^*}.

It remains to consider the case where the element $x_t \in W_1$. Then the interchange of W_0 and W_1 and the repetition of Algorithm 3.1 with these interchanged sets yields the claim. □

Algorithm 3.1 returns a sequence of sets with property 3. It remains to give the partition of $\{0,1\}^n$ with property (3) that we obtain from this sequence of sets.

$$\begin{aligned} X_0 :=& \{x \in \{0,1\}^n \colon \exists y \in M_0 \text{ with } y \leq x\}, \\ X_j :=& \{x \in \{0,1\}^n \colon \exists y \in M_j \text{ with } y \leq x \\ & \text{and } x \notin X_l \text{ for any } l < j\}, \qquad 0 < j < i, \\ X_i :=& \{x \in \{0,1\}^n \colon \exists y \in X_{i-1} \text{ with } y \geq x \text{ and } x \notin X_1\}. \end{aligned} \tag{8}$$

Finally, we set $\bar{X}_0 := \{0,1\}^n \setminus \bigcup_{t=1}^{i} X_i$. The partition we were looking for is $\bar{X}_0, X_1, \ldots, X_i$.

Lemma 10. The sequence of sets $\bar{X}_0, X_1, \ldots, X_i$ is a partition of $\{0,1\}^n$ with property (3). For each $j \in \{0, \ldots i\}$ either $X_j \cap W_0 = \emptyset$ or $X_j \cap W_1 = \emptyset$. Additionally for $b \in \{0,1\}$: $X_j \cap W_b = \emptyset \Leftrightarrow X_{j+1} \cap W_{1-b} = \emptyset$. $X_0, \ldots, X_i$ is a sequence of minimal cardinality with this property.

Proof. The claim follows directly from the construction of $\bar{X}_0, \ldots, X_i$ and from the previous arguments. □

We note that if W_0 and W_1 are given by membership oracles and there are only polynomially many elements in $W_0 \cup W_1$ the algorithm has polynomial running time.

4 k-tone Boolean functions and FMI-sets

In the previous section, we showed that there is a connection between bitone functions and FMI sets for one pair of parameters (t_1, t_2). Now we will show that k-tone functions are in correspondence with $\lfloor \frac{k}{2} \rfloor$ sets of (t_1^i, t_2^i)-FMI sets $(i = 1, \ldots, \lfloor \frac{k}{2} \rfloor)$ where $t_2^i < t_1^{i+1}$.

Let $A \in \{0,1\}^{m \times n}$ be a binary matrix, let $1 \leq t_1^1 < t_2^1 < t_1^2 < \ldots < t_k^1 < t_k^2 \leq m$ be an integer sequence with

$$S_i := \{x \in \{0,1\}^n \colon \operatorname{supp}(x) \text{ is } t_1^i\text{-frequent and } t_2^i\text{-infrequent}\}. \tag{9}$$

and for $i = 1, \ldots, k-1$ let

$$T_i := \{x \in \{0,1\}^n \colon \operatorname{supp}(x) \text{ is } t_2^i\text{-frequent and } t_1^{i+1}\text{-infrequent}\}. \tag{10}$$

Let $T_0 := \{x \in \{0,1\}^n \colon \operatorname{supp}(x) \text{ is not } t_1^1\text{-frequent}\}$ and $T_k := \{x \in \{0,1\}^n \colon \operatorname{supp}(x) \text{ is } t_k^2\text{-frequent}\}$. We order these sets as follows: $T_0, S_1, T_1, \ldots, S_k, T_k$. The sequence of these sets forms a partition satisfying property (3). This follows from the fact that t-frequency is down-monotone for every $t \in \{0, \ldots, m\}$. Therefore, there is a $2k$-tone Boolean function that evaluates to 1 on S_i and to 0 on the sets T_i.

Now the converse:

Lemma 11. Let $f \colon \{0,1\}^n \to \{0,1\}$ be a k-tone Boolean function and let $l := \lfloor \frac{k}{2} \rfloor$. Then there exists a matrix $A \in \{0,1\}^{m \times n}$ and an integer sequence $1 \leq t_1^1 < t_2^1 < \ldots < t_1^l < t_2^l \leq m$ such that the subsets of $\{0,1\}^n$ where f evaluates to 1 correspond to t_1^i-frequent and t_2^i-infrequent sets.

Proof. Definition 1 yields that the k-tone function f induces a partition $M_0, \ldots, M_k$ satisfying property (3). Recall that we restricted ourselves to k-tone Boolean functions evaluating to 0 on M_0. Assume w.l.o.g. that k is even. We know from Lemma 3 that f may be expressed as a disjunction of l bitone Boolean functions:

$$f = (\neg f_1 \wedge f_2) \vee (\neg f_3 \wedge f_4) \vee \ldots \vee (\neg f_{k-1} \wedge f_k)$$

where the f_i are monotone Boolean functions and we have $f_1 \leq \ldots \leq f_k$. For each of the monotone Boolean functions f_i let $x_{1,i}, \ldots, x_{j(i),i}$ denote the maximal vectors such that $f_i(x_{k,i}) = 0$.

Let $i \in \{1, \ldots, k\}$ be an odd index. For every maximal element $x_{k,i}$, $k = 1, \ldots, j(i)$, we introduce one row to the binary matrix, namely the vector $x_{k,i}$ itself. The resulting preliminary matrix is denoted by $\tilde{A}^i$. Let s_i denote the maximal frequency of a single column $\{c\}$ ($c \in \{1, \ldots, n\}$) with respect to the matrix $\tilde{A}^i$.

Now we turn to index $i + 1$: for f_{i+1} consider the maximal elements $x_{1,i+1}, \ldots, x_{j(i+1),i+1}$. For each of these elements, we determine the frequency of its support w.r.t. $\tilde{A}^i$. For $x_{k,i+1}$ we denote by $t_{k,i+1}$ this parameter. For each maximal element $x_{k,i+1}$ we introduce $s_i - t_{k,i+1}$ copies of the vector to the matrix $\tilde{A}^i$. The resulting matrix is denoted by A^i. The 1-frequent and s_i-infrequent sets in A^i correspond to the Boolean vectors with $\neg f_i \wedge f_{i+1}$ evaluating to 1. Glueing the matrices A^i together yields the matrix A.

Now we will show the following:

The Boolean vectors $x \in \{0, 1\}^n$ for which $(\neg f_i \wedge f_{i+1})(x) = 1$ holds correspond to index sets $J = \text{supp}\,(x)$ that are

$$1 + \sum_{\substack{j<i,\\ j \text{ odd}}} (1 + s_j) - \text{frequent and } \sum_{\substack{j\leq i,\\ j \text{ odd}}} (1 + s_j) - \text{infrequent.}$$

with respect to the matrix A.

Let $J \subseteq \{1, \ldots, n\}$ be a set that is $1 + \sum_{\substack{j<i,\\ j \text{ odd}}} (1 + s_j)$-frequent and $\sum_{\substack{j\leq i,\\ j \text{ odd}}} (1 + s_j)$-infrequent.

Consider the corresponding incidence vector x_J and determine the maximal parameter $j \in \{1, \ldots, k\}$ such that $f_j(x_J) = 0$. Clearly, $f_l(x_J) = 0$ holds for all $l \leq j$, as $f_1 \leq \ldots \leq f_k$. Therefore, for every $l \leq j$ there exists a vector x_l (maximal such that $f_l(x_l) = 0$) such that $x_l \geq x_J$.

Therefore, the set J is

$$\begin{aligned} 1 + \sum_{\substack{l<j,\\ l \text{ odd}}} (1 + s_l) - \text{frequent}, &\quad \text{if } j \text{ odd},\\ \sum_{\substack{l\leq j,\\ l \text{ odd}}} (1 + s_l) - \text{frequent}, &\quad \text{if } j \text{ even}. \end{aligned}$$

We conclude that $j = i$ and thus $0 = f_i(x_J)$. It remains to prove that $f_{i+1}(x_J) = 1$. Assume not, i.e., $f_{i+1}(x_J) = 0$. Then the same argumentation as above yields that J is $\sum_{\substack{j\leq i,\\ j \text{ odd}}} (1 + s_j)$-frequent contradicting our assumption. Therefore, $(\neg f_i \wedge f_{i+1})(x_J) = 1$ holds and the first direction of our claim is proved.

The construction of the matrix A and the property that $f_1 \leq \ldots \leq f_k$ yields that the reverse direction holds as well. □

We note that all of the above can be generalized from the binary case to the case of order-preserving functions mapping into an arbitrary Boolean algebra (see e.g. (Dwinger 1961)), since monotonicity and k-tonicity can be similarly defined.

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