Hardness Results for Approximate Pure Horn CNF Formulae Minimization*

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Abstract

We show that for a pure Horn Boolean function on n variables, unless P $=$ NP, it is not possible to approximate in polynomial time (in n) the minimum numbers of clauses and literals to within factors of $2^{O(\log^{1-o(1)} n)}$ even when the inputs are restricted to 3-CNFs with $O(n^{1+\varepsilon})$ clauses, for some small $\varepsilon > 0$. Furthermore, we show that unless the ETH is false, it is not possible to obtain constant factor approximations for these problems even having sub-exponential time (in n).

1 Introduction

We focus on the hardness of approximating the minimum numbers of clauses and literals of a pure Horn Boolean functions specified through pure Horn CNFs and 3-CNFs.

The first partial result in this realm was provided in (Bhattacharya et al. 2010). Specifically, it was shown that unless NP $\subseteq$ QP, that is, every problem in NP can be solved in super-polynomial time (in the size of the input), the minimum number of clauses of a pure Horn function on n variables specified through a pure Horn CNF formula cannot be approximated in polynomial time (in n) to within a factor of $2^{O(\log^{1-\varepsilon} n)}$, for any small constant $\varepsilon > 0$. Their result is based on a gap-preserving reduction from a fairly well known network design problem, namely, MinRep (Kortsarz 2001) and has two main components: a gadget that associates to every MinRep instance M a pure Horn CNF formula h and links the size of an optimal solution to M to the size of a clause minimum pure Horn CNF representation of h, and a gap amplification device that provides the referred gap. Despite being both necessary to accomplish the result, each component works in an rather independent way.

We strengthen (Bhattacharya et al. 2010)'s result in the following ways. We start from a related but slightly different problem, called Label-Cover, and present a new gap-preserving reduction to the problem of determining the minimum number of clauses in a pure Horn CNF representation of a pure Horn function. In arguing about the correctness of our reduction, we show that our gadget forms an exclusive component of the function in question and hence can be minimized separately (Boros et al. 2010) from the gap amplification device (and it is then clear that the same principle underlies Bhattacharya et al.'s result). Our hardness of approximation factor is slightly larger and stands upon a weaker complexity hypothesis.

The explicit independence of both parts and the fact that our gadget is somewhat more complicated (for MinRep is a "simplification" of Label-Cover) allows us to slightly modify our reduction and obtain a similar result for the case where the representation is restricted to clauses of size at most 3 (i.e., 3-CNFs). We can then derive in a straightforward fashion a hardness result on the minimum number of literals of pure Horn functions. We note these results cannot be obtained from Bhattacharya et al.'s gadget. Furthermore, as our reductions have the Label-Cover problem as starting point, using a stronger complexity hypothesis, we can show that even in the case where sub-exponential time (in the number of variables) is allowed for computation of the minimum number of clauses and literals of the function, it is not possible to obtain a constant factor approximation of such values without refuting the new hypothesis.

More specifically, for a pure Horn Boolean function h on n variables, we show that unless P $=$ NP, the minimum number of clauses in a CNF or 3-CNF representation of h and the minimum number of literals in a 3-CNF representation of h cannot be approximated in polynomial time (in n) to within factors of $2^{O(\log^{1-o(1)} n)}$ even when the inputs are restricted to CNFs and 3-CNFs with $O(n^{1+\varepsilon})$ clauses, for some small $\varepsilon > 0$; it is worth mentioning that $o(1) \approx (\log\log n)^{-c}$ for any constant $c \in (0, 1/2)$ in this case. After that, we show that unless the Exponential Time Hypothesis (Impagliazzo and Paturi 2001) is false, it is not possible to approximate the minimum number of clauses and the minimum number of literals in time $\exp(n^{\delta})$ for some $\delta \in (0, 1)$, to within factors of $O(\log^{\beta} n)$ for some $\beta \in (0, 1)$, even when the inputs are restricted to 3-CNFs with $O(n^{1+\varepsilon})$ clauses, for some small constant $\varepsilon > 0$. For these same problems, we also obtain a hardness of approximation factor of $O(\log n)$ under what we call hyper-polynomial time.

The remaining of the text is organized as follows. We introduce some basic concepts and present the problem on

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which our reduction is based in § 2. The reduction for CNFs and its proof of correctness are shown in § 3. In § 4, we extend that result for 3-CNF formulae and address the case of minimizing the number of literals. In § 5 we show that sub-exponential time availability gives better but still super-constant hardness of approximation factors. We then offer some conclusions in § 6.

2 Preliminaries and LABEL-COVER

Below we succinctly define the Boolean concepts we need. For a more thorough exposition consult the book (Crama and Hammer 2011). A mapping $h : \{0,1\}^n \to \{0,1\}$ is called a *Boolean function* on n propositional variables. We denote its set of variables by $V_h = \{v_1, \ldots, v_n\}$. A *literal* is a propositional variable or its negation. An elementary disjunction of literals $C = \bigvee_{i \in I} \bar{v}_i \vee \bigvee_{j \in J} v_j$, with $I, J \subseteq V_h$ and $I \cap J = \emptyset$ is called a *clause*; its *degree* is given by $\mathsf{deg}(C) = |I \cup J|$ and the set of variables it depends upon by $\mathsf{Vars}(C) = I \cup J$. A clause C is *pure Horn* if $|J| = 1$; the literal in J is the *head* of C and $\bigvee_{i \in I} \bar{v}_i$ is its *subgoal*. A conjunction of pure Horn clauses is a *pure Horn CNF*. A Boolean function h is *pure Horn* if there is a pure Horn CNF formula $\Phi \equiv h$, that is, if $\Phi(v) = h(v)$ for all $v \in \{0,1\}^n$. A clause C is an *implicate* of a Boolean function h if for all $v \in \{0,1\}^n$ it holds that $h(v) = 1$ implies $C(v) = 1$. An implicate is *prime* if it is inclusion-wise minimal w.r.t. its set of literals. It is known that prime implicates of pure Horn functions are pure Horn clauses. We denote the set of prime implicates of h by $\mathcal{I}^p(h)$.

For a pure Horn CNF $\Phi = \bigvee_{i=1}^n C_i$ representing a pure Horn function h, we denote by $|\Phi|_c := n$ and $|\Phi|_l := \sum_{i=1}^n \mathsf{deg}(C)$ the numbers of clauses and literals of Φ, respectively. Φ is a clause (literal) minimum representation of h if $|\Phi|_c \leq |\Psi|_c$ ($|\Phi|_l \leq |\Psi|_l$) for every other pure Horn CNF representation Ψ of h. A pure Horn CNF Φ representing h is *prime* if its clauses are prime and is *irredundant* if the CNF obtained after removing any of its clauses does not represent h anymore. It is known that every clause (literal) minimum representation of h is prime and irredundant.

Let $C_1 = S_1 \vee v_1$ and $C_2 = S_2 \vee v_2$ be pure Horn clauses such that $\bar{v}_1 \in S_2$ and S_1 and S_2 have no other complemented literals. The *resolvent* of C_1 and C_2 is the pure Horn clause $R(C_1, C_2) = S_1 \vee (S_2 \setminus \{\bar{v}_1\}) \vee v_2$ and C_1 and C_2 are said to be *resolvable*. If C_1 and C_2 are implicates of a pure Horn function h, then so is $R(C_1, C_2)$. A set of clauses $\mathcal{C}$ is *closed under resolution* if for all $C_1, C_2 \in \mathcal{C}$, $R(C_1, C_2) \in \mathcal{C}$. The *resolution closure* of $\mathcal{C}$, $R(\mathcal{C})$, is the smallest set $\mathcal{X} \supseteq \mathcal{C}$ closed under resolution. For a Boolean function h, let $\mathcal{I}(h) := R(\mathcal{I}^p(h))$.

Let Φ be a pure Horn CNF representing a pure Horn function h and let $Q \subseteq V_h$. The *forward chaining* of Q on Φ, $F_\Phi(Q)$, is defined by the following algorithm. Initially, $F_\Phi(Q) = Q$. As long as there is a pure Horn clause $S \vee v$ in Φ such that $S \subseteq F_\Phi(Q)$ and $v \notin F_\Phi(Q)$, add v to $F_\Phi(Q)$. Two distinct pure Horn CNFs Φ and Ψ represent the same pure Horn function h if and only if $F_\Phi(U) = F_\Psi(U)$ for all $U \subseteq V_h$. This implies we can do forward chaining on h, denoted by $F_h(U)$, using any of its representations.

Definition 1 ((Boros et al. 2010))**.** *Let h be a Boolean function and $\mathcal{X} \subseteq \mathcal{I}(h)$ be a set of clauses. $\mathcal{X}$ is an* exclusive set of clauses of *h if for all resolvable clauses $C_1, C_2 \in \mathcal{I}(h)$ it holds that: $R(C_1, C_2) \in \mathcal{X}$ implies $C_1 \in \mathcal{X}$ and $C_2 \in \mathcal{X}$.*

Definition 2 ((Boros et al. 2010))**.** *Let $\mathcal{X} \subseteq \mathcal{I}(h)$ be an exclusive set of clauses for a Boolean function h and let $\mathcal{C} \subseteq \mathcal{I}(h)$ be such that $\mathcal{C} \equiv h$. The Boolean function $h_\mathcal{X} = \mathcal{C} \cap \mathcal{X}$ is called the $\mathcal{X}$-component of h.*

Lemma 3 ((Boros et al. 2010))**.** *Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{I}(h)$, $C_1 \neq C_2$ such that $\mathcal{C}_1 \equiv \mathcal{C}_2 \equiv h$ and let $\mathcal{X} \subseteq \mathcal{I}(h)$ be an exclusive set of clauses. Then $\mathcal{C}_1 \cap \mathcal{X} \equiv \mathcal{C}_2 \cap \mathcal{X}$ and in particular $(\mathcal{C}_1 \setminus \mathcal{X}) \cup (\mathcal{C}_2 \cap \mathcal{X})$ also represents h.*

Lemma 4 ((Boros, Čepek, and Kučera 2011))**.** *Let Φ be a pure Horn CNF representing the function h, let $W \subseteq V_h$ be such that $F_\Phi(W) = W$, and define $\mathcal{X}(W) = \{C \in \mathcal{I}(h) : \mathsf{Vars(C)} \subseteq W\}$. Then $\mathcal{X}(W)$ is an exclusive family for h.*

The LABEL-COVER problem is a graph labeling promise problem formally introduced in (Arora et al. 1997) as a combinatorial abstraction of interactive proof systems (Arora et al. 1998; Arora and Safra 1998). It comes in maximization and minimization flavors (linked by a 'weak duality' relation) and it is probably the most popular starting point for hardness of approximation reductions. We introduce here a minimization version.

Definition 5. *A* LABEL-COVER instance *is a quadruple $\mathcal{L}_0 = (G, L_0, L'_0, \Pi_0)$, where $G = (X, Y, E)$ is a bipartite graph, L_0 and L'_0 are disjoint sets of labels for the vertices in X and Y respectively, and $\Pi_0 = (\Pi^0_e)_{e \in E}$ is a set of constraints with each $\Pi^0_e \subseteq L_0 \times L'_0$ being a non-empty relation of admissible pairs of labels for the edge e. The* size *of $\mathcal{L}_0$ is a function of the number of vertices in Y.*

Definition 6. *A* labeling *for $\mathcal{L}_0$ is any function $f_0 : X \to 2^{L_0}, Y \to 2^{L'_0} \setminus \{\emptyset\}$ assigning subsets of labels to vertices. A labeling f_0* covers *an edge (x, y) if for every label $\ell'_0 \in f_0(y)$ there is a label $\ell_0 \in f_0(x)$ such that $(\ell_0, \ell'_0) \in \Pi^0_{(x,y)}$. A* total-cover *for $\mathcal{L}_0$ is a labeling that covers every edge in E. $\mathcal{L}_0$ is said to be* feasible *if it admits a total-cover.*

It is well known (Arora and Lund 1996) that non-feasible LABEL-COVER instances can be easily transformed into feasible ones preserving the same structural relations.

Definition 7. *For a total-cover f_0 of $\mathcal{L}_0$, let $f_0(Z) = \sum_{z \in Z} |f_0(z)|$ with $Z \subseteq X \cup Y$. The* cost *of f_0 is given by $\kappa(f_0) = f_0(X)/|X|$ and f_0 is said to be* optimal *if $\kappa(f_0)$ is minimum among the costs of all total-covers for $\mathcal{L}_0$.*

It follows that $1 \leq \kappa(f_0) \leq |L_0|$ for any total-cover f_0 and without loss of generality, we can assume G has no isolated vertices (for they do not influence a labeling's cost).

Definition 8. *A total-cover f_0 is* tight *if $f_0(Y) = |Y|$, i.e., if for every $y \in Y$ it holds that $|f_0(y)| = 1$.*

Lemma 9. *Every* LABEL-COVER *instance $\mathcal{L}_0$ admits a tight optimal total-cover $f_0 : X \to 2^{L_0}, Y \to L'_0$.*

Proof. Suppose f_0 as in Definition 6 is a minimally non-tight optimal total-cover for $\mathcal{L}_0$. Hence, there is a $y \in Y$ such that $|f_0(y)| > 1$. Let $\ell'_0 \in f_0(y)$ and define a new labeling g where $g(z) = f_0(z)$ for all $z \in X \cup (Y \setminus \{y\})$ and $g(y) = f_0(y) \setminus \{\ell'_0\}$. Note that $g(y) \neq \emptyset$ and that every edge (x, y) for $x \in N(y) := \{z \in X : (z, y) \in E\}$ is covered (for f_0 is a total-cover). Moreover, clearly $\kappa(f_0) = \kappa(g)$. Hence, g is an optimal total-cover for $\mathcal{L}_0$ in which $g(Y) < f_0(Y)$, contradicting the assumption made. The result thus follows. □

Notation 10. *For $\mathcal{L}_0$ being a* LABEL-COVER *instance as in Definition 5, define $r = |X|$, $s = |Y|$, $m = |E|$, $\lambda = |L_0|$, $\lambda' = |L'_0|$, $\pi_e = |\Pi^0_e|$ for $e \in E$, and set $\pi = \sum_{e \in E} \pi_e$.*

Problem 11. *For any $\rho > 0$, a* LABEL-COVER *instance $\mathcal{L}_0$ has promise ρ if it falls in one of two cases: either there is a tight optimal total-cover for $\mathcal{L}_0$ of cost 1, or every tight optimal total-cover for $\mathcal{L}_0$ has cost at least ρ. The* LABEL-COVER$_\rho$ *problem is a promise problem which receives a* LABEL-COVER *instance with promise ρ as input and correctly classify it in one of those two cases.*

Notice the behavior of LABEL-COVER$_\rho$ is left unspecified for non-promise instances and so any answer is acceptable in that case. Due to this behavior, LABEL-COVER$_\rho$ is also referred to as a *gap-problem* with gap ρ in the literature. The following result holds for it.

Theorem 12 ((Dinur and Safra 2004))**.** *Let c be a constant in $(0, 1/2)$, $\rho_c(s) = 2^{(\log s)^{1-1/\delta_c(s)}}$ with $\delta_c(s) = (\log \log s)^c$, and let $\mathcal{L}_0$ be a* LABEL-COVER *instance with promise $\rho_c(s)$. Then it is NP-hard to distinguish between the cases in which $\kappa(\mathcal{L}_0)$ is equal to 1 or at least $\rho_c(s)$, where $\kappa(\mathcal{L}_0)$ is the cost of a tight optimal total-cover for $\mathcal{L}_0$.*

Remark 13. *Every* LABEL-COVER *instance produced by Dinur and Safra's reduction is feasible, has promise $\rho_c(s)$, and the following relations hold: $r = s\lfloor \delta_c(s) \rfloor$, $\lambda = \Theta(\rho_c(s))$, $\lambda' = \Theta(\rho_c(s)^{\delta_c(s)}) = o(s)$, $s\lfloor \delta_c(s) \rfloor \leq m \leq s^2 \lfloor \delta_c(s) \rfloor$, and $\pi \leq m\lambda\lambda' = O(s^2 \delta_c(s) \rho_c(s)^{\delta_c(s)+1}) = o(s^3)$, for s and c as specified in Notation 10 and Theorem 12, respectively.*

Without any loss of generality, it is possible to modify the LABEL-COVER definitions so that the vertices in X and Y have their own copies of the label-sets L_0 and L'_0, respectively.

Definition 14. *Let $\mathcal{L}_0 = (G, L_0, L'_0, \Pi_0)$ be a feasible* LABEL-COVER *instance and consider the sets $L_x := \{(x, \ell_0) : \ell_0 \in L_0\}$ for each vertex $x \in X$, and $L'_y := \{(y, \ell'_0) : \ell'_0 \in L'_0\}$ for each vertex $y \in Y$. Also, define the sets $L := \cup_{x \in X} L_x$, $L' := \cup_{y \in Y} L'_y$, and $\Pi := \cup_{(x,y) \in E} \Pi_{(x,y)}$, with*

$$\Pi_{(x,y)} := \left\{ \big((x, \ell_0), (y, \ell'_0)\big) : (\ell_0, \ell'_0) \in \Pi^0_{(x,y)} \right\}.$$

The quadruple $\mathcal{L} = (G, L, L', \Pi)$ is an extension *of $\mathcal{L}_0$.*

It is clear that $|L_x| = \lambda$ for each vertex $x \in X$, $|L_y| = \lambda'$ for each vertex $y \in Y$, $|\Pi_e| = \pi_e$ for each edge $e \in E$, $|\Pi| = \pi$, and that a labeling for $\mathcal{L}$ is a mapping f such that $x \mapsto f(x) \subseteq L_x$ for each vertex $x \in X$, and $y \mapsto f(y) \subseteq L'_y$, $f(y) \neq \emptyset$ for each vertex $y \in Y$. Furthermore, the remaining definitions and concepts extend in a straight forward fashion.

Lemma 15. *There is a one-to-one cost preserving correspondence between solutions to the* LABEL-COVER$_\rho$ *problem and to the* EXTENDED LABEL-COVER$_\rho$ *problem, for any $\rho > 0$ and the sizes of both instances are linearly related.*

Proof. It is easy to see that f_0 is a (tight) total-cover for the LABEL-COVER$_\rho$ problem if and only if f is a (tight) total-cover for the EXTENDED LABEL-COVER$_\rho$ problem, where $f(x) = \{(x, \ell_0) : \ell_0 \in f_0(x)\}$ for every $x \in X$, and $f(y) = \{(y, \ell'_0) \in L'_y : \ell'_0 \in f_0(y)\}$ for every $y \in Y$ (or $f(y) = (y, f_0(y))$ if the total-covers are tight). Furthermore it is clear that $\kappa(f_0) = \kappa(f)$, in any case. The linearity relation on the instances' sizes follows from Definition 14. □

3 Reduction to CNFs

Let $\mathcal{L} = (G = (X, Y, E), L, L', \Pi)$ be a LABEL-COVER extension, and d and t be positive integers to be specified later. For nonnegative integers n, define $[n] := \{1, \ldots, n\}$.

Associate propositional variables $u(\ell)$ with every label $\ell \in L \cup L'$, $e(x, y, i)$ and $e(x, y, \ell', i)$ with every edge $(x, y) \in E$, every label $\ell' \in L'_y$ and every index $i \in [d]$. Let $v(j)$ for indices $j \in [t]$ be extra variables and consider the following families of clauses:

(a) $u(\ell) \wedge u(\ell') \longrightarrow e(x, y, \ell', i)$
$$\forall\, (x, y) \in E,\ (\ell, \ell') \in \Pi_{(x,y)},\ i \in [d];$$

(b) $\bigwedge_{z \in N(y)} e(z, y, \ell', i) \longrightarrow e(x, y, i)$
$$\forall\, (x, y) \in E,\ \ell' \in L'_y,\ i \in [d];$$

(c) $e(x, y, i) \longrightarrow e(x, y, \ell', i)$
$$\forall\, (x, y) \in E,\ \ell' \in L'_y,\ i \in [d];$$

(d) $\bigwedge_{i \in [d]} \bigwedge_{(x,y) \in E} e(x, y, i) \longrightarrow u(\ell) \qquad \forall\, \ell \in L \cup L';$

(e) $v(j) \longrightarrow u(\ell) \qquad \forall\, j \in [t],\ \ell \in L \cup L',$

where as before $N(y) := \{x \in X : (x, y) \in E\}$ for each $y \in Y$.

Let us call Ψ and Φ the *canonical* pure Horn CNF formulae defined, respectively, by the families of clauses (a) through (d) and by all the families of clauses above, and let g and h, in that order, be the pure Horn functions they represent.

Lemma 16. *For d and t as above and r, s, m, λ, λ', and π as in Notation 10, it holds that the number of clauses and variables in Φ are $|\Phi|_c = (t+1)(r\lambda + s\lambda') + d(\pi + 2m\lambda')$ and $|\Phi|_v = t + dm(\lambda' + 1) + r\lambda + s\lambda'$, respectively.*

The driving idea behind our reduction is that of tying the cost of tight optimal solutions of $\mathcal{L}$ to the size of clause minimum pure Horn CNF representations of h. Once this is done, we achieve the desired dichotomy using a gap amplification device (the parameter t, on which clauses of type (e) depend upon). In more details, the families of clauses (a) through (d) form the core of our gadget as they locally reproduce the LABEL-COVER structural properties: clauses of type (a) correspond to the constraints on the pairs of labels that can be assigned to each edge; clauses of type (b) represent the condition for an edge to be considered covered; clauses of type (c) state that if an edge can be covered in a certain way, then it can be covered in all legal ways (implying it is not necessary to keep track of more than one covering possibility for each edge in clause minimum representations); clauses of type (d) translate the total-cover requirement and reintroduces all the labels available ensuring if a total-cover is achieved, so are all the others (and this is the basis of our gadget's global behavior). The second part of our gadget is composed by the family of clauses (e), which locally plays the role of introducing an initial collection of labels. On a global level, as long as this initial collection of labels forms a total-cover, the remaining clauses of type (e) can be removed from the CNF representation of h without any loss (for the clauses of type (d) reintroduces all the labels). However, at first, there is no guarantee a subset of the clauses of type (e) belongs to a clause minimum pure Horn CNF representation of h as it could be advantageous having some prime implicates with $v(j)$, for $j \in [t]$, on their subgoals and variables other than $u(\ell)$ as heads, for $\ell \in L \cup L'$. It could even be the case only a constant times t number of such prime implicates are needed in a clause minimum pure Horn CNF representation h, rendering the gap amplification device innocuous. To avoid this undesirable scenario, we introduce another amplification device (the parameter d, on which clauses (a)–(d) depend upon) that helps to shape the prime implicates involving the variables $v(j)$ into an appropriate form.

Let V_g be the set of variables occurring in Ψ. By definition, these are the variables the function g depends on. Clearly, V_g is closed under Forward Chaining on Φ as no clause in Ψ has head outside V_g and no clause in $\Phi \setminus \Psi$ has subgoals contained in V_g. Hence, by the fact that Φ represents h and by Lemma 4, it follows that $\mathcal{X}(V_g)$, as defined there, is an exclusive family for h. It then follows.

Corollary 17. *The function g is an exclusive component of the function h. Consequently, g can be minimized separately.*

With some effort, it is possible to prove that Ψ is a clause minimum pure Horn CNF representation of g. For our proofs however, a weaker result suffices.

Lemma 18. *Let Θ be a clause minimum pure Horn CNF representation of g. We have $|\Psi|_c/(\lambda + \lambda') \leq |\Theta|_c \leq |\Psi|_c$.*

Proof. The upper bound is obvious. For the lower bound, observe that each variable of g appears no more than $\lambda + \lambda'$ times as a head. As in any clause minimum representation of g they must appear as head at least once, the claim follows. □

Lemma 19. *For all $j \in [t]$, it holds that $F_h(\{v(j)\}) = V_g \cup \{v(j)\}$.*

Proof. It is enough to show that $\{e(x, y, i) : (x, y) \in E, i \in [d]\} \subseteq F_\Phi(\{v(j)\})$, for a fixed $j \in [t]$. The inclusion is false if there is a label $\ell'' \in L \cup L'$ such that $u(\ell'') \notin F_\Phi(\{v(j)\})$. As for every label $\ell \in L \cup L'$, $v(j) \longrightarrow u(\ell)$ is a clause in Φ, the inclusion holds and the claim follows. □

Lemma 20. *A variable $v(j)$, for some index $j \in [t]$, is never the head of an implicate of h. Moreover, every prime implicate of h involving $v(j)$ is quadratic.*

Proof. The first claim is straight forward as all implicates of h can be derived from Φ by resolution, and $v(j)$ is not the head of any clause of Φ. By Lemma 19, $v(j) \longrightarrow z$ is an implicate of h for all $z \in V_g$. Since h is a pure Horn function, the claim follows. □

Lemma 21. *For $d > r\lambda + s\lambda'$, the prime implicates involving variables $v(j)$, for indices $j \in [t]$, in any clause minimum pure Horn CNF representation of h have the form $v(j) \longrightarrow u(\ell)$ with $\ell \in L \cup L'$.*

Proof. By Lemma 20, there are only three types of prime implicates involving the variables $v(j)$, with $j \in [t]$. Let $\Upsilon = \Theta \wedge \Gamma$ be a clause minimum pure Horn CNF representation of h, with Θ being a clause minimum pure Horn CNF representation of g. Let $j \in [t]$ and for $i \in [d]$, consider the sets

$$\Gamma_u^j = \Gamma \cap \{v(j) \longrightarrow u(\ell) : \ell \in L \cup L'\},$$
$$\Gamma_i^j = \Gamma \cap \{v(j) \longrightarrow e(x, y, i),\ v(j) \longrightarrow e(x, y, \ell', i) : (x, y) \in E, \ell' \in L'_y\}.$$

Suppose $\sum_{j \in [t], i \in [d]} |\Gamma_i^j| > 0$ for otherwise, there is nothing to show. Fix some $j \in [t]$, let $i_1, i_2 \in [d]$, $i_1 \neq i_2$, and adjust the notation such that $|\Gamma_{i_1}^j| \leq |\Gamma_{i_2}^j|$. If the equality does not hold, as the conclusions drawn in h for each $i \in [d]$ must be the same it follows that $\Upsilon' = (\Upsilon \setminus \Gamma_{i_2}^j) \cup \Delta_{i_1 \to i_2}^j$ is also a representation of h, where $\Delta_{i_1 \to i_2}^j$ is equal to

$$\{v(j) \longrightarrow e(x, y, i_2) : v(j) \longrightarrow e(x, y, i_1) \in \Gamma_{i_1}^j\} \cup \{v(j) \longrightarrow e(x, y, \ell', i_2) : v(j) \longrightarrow e(x, y, \ell', i_1) \in \Gamma_{i_1}^j\}.$$

Since $|\Upsilon'| < |\Upsilon|$, we get a contradiction with the clause minimality of Υ. Thus, $|\Gamma_i^j| = \gamma_j > 0$ for all $i \in [d]$.

This implies $\sum_{i \in [d]} |\Gamma_i^j| = d\gamma_j \geq d$. If $d > r\lambda + s\lambda' = |L \cup L'|$, we have that the pure Horn CNF

$$\Delta_j = (\Upsilon \setminus \cup_{i \in d} \Gamma_i^j) \cup \{v(j) \longrightarrow u(\ell) : \ell \in L \cup L'\}$$

has less clauses than Υ. As Δ_j represents h as well, this contradicts the clause minimality of Υ and the result follows. □

Definition 22. *Let Υ be a clause minimum pure Horn CNF representation of h. For each $j \in [t]$, consider the set $S_j = \{\ell \in L \cup L' : v(j) \longrightarrow u(\ell) \in \Upsilon\}$ and define the function $f_j : X \to L, Y \to L'$ given by $f_j(x) = S_j \cap L_x$ for vertices $x \in X$ and $f_j(y) = S_j \cap L'_y$ for vertices $y \in Y$.*

Lemma 23. *For all indices $j \in [t]$ and vertices $y \in Y$, it holds that $|f_j(y)| = 1$.*

Proof. Assume the opposite and for some $j \in [t]$ and vertex $y \in Y$ such that $|f_j(y)| > 1$, let $\ell' \in S_j(y)$. Let $\Upsilon' = \Upsilon \setminus \{v(j) \longrightarrow u(\ell')\}$ (cf. Lemma 21). If $F_{\Upsilon'}(\{v(j)\}) = V_g \cup \{v(j)\}$ then Υ is not clause minimum and we are done.

Therefore, consider that is not the case and fix an index $i \in [d]$ arbitrarily.

As $u(\ell') \notin F_{\Upsilon'}(\{v(j)\})$, there must be an edge $(x, y) \in E$ such that $e(x, y, i) \notin F_{\Upsilon'}(\{v(j)\})$ as well and so, for every label $\ell'' \in L'_y$ there is a vertex $z \in N(y)$ for which $e(z, y, \ell'', i) \notin F_{\Upsilon'}(\{v(j)\})$. That is the case when the clauses $u(\ell) \wedge u(\ell'') \longrightarrow e(z, y, \ell'', i)$ are not satisfied for every label $\ell \in L_z$ with $(\ell, \ell'') \in \Pi_{(z,y)}$. Now, since $u(\ell'') \in F_{\Upsilon'}(\{v(j)\})$ for all labels $\ell'' \in f_j(y) \setminus \{\ell'\}$, it holds that $u(\ell) \notin F_{\Upsilon'}(\{v(j)\})$, for all labels $\ell \in L_z$ such that $(\ell, \ell'') \in \Pi_{(z,y)}$ and $\ell'' \in f_j(y) \setminus \{\ell'\}$. This gives $\Upsilon' \equiv \Upsilon' \setminus \Delta$, with $\Delta = \{v(j) \longrightarrow u(\ell'') : \ell'' \in (f_j(y) \setminus \{\ell'\})\}$, and implies $\Upsilon \equiv \Upsilon \setminus \Delta$, contradicting the clause minimality of Υ. The claim thus follows. □

Lemma 24. *For each index $j \in [t]$, the function f_j is a tight total-cover for $\mathcal{L}$.*

Proof. Fix an index $j \in [t]$. By construction, f_j is a labeling for $\mathcal{L}$ and Lemma 23 implies f_j is tight independently of being a total-cover or not. Suppose f_j is not a total-cover for $\mathcal{L}$. So, there exists an edge $(x, y) \in E$ and a label $\ell' \in f_j(y)$ such that for all labels $\ell \in f_j(x)$ it holds that $(\ell, \ell') \notin \Pi_{(x,y)}$. As Υ is clause minimum, no variable $e(x, y, \ell', i)$ belongs to $F_{\Upsilon(\{v(j)\})}$ for every index $i \in [d]$. That however, contradicts the fact that Υ represents h (cf. Lemma 19). □

In order to relate the size of a clause minimum representation of h to the cost of an optimal solution to $\mathcal{L}$, we need a comparison object. Let f be a tight total-cover for $\mathcal{L}$ and consider the following subfamily of clauses:

(e') $v(j) \longrightarrow u(\ell)$

$$\forall\, j \in [t],\ x \in X,\ y \in Y,\ \ell \in f(x) \cup f(y),$$

with $f(x) \subseteq L_x$ and $f(y) \subseteq L'_y$. Let Φ_f be the *refined canonical* (w.r.t. f) pure Horn CNF formula resulting from the conjunction of Ψ with the implications (e').

Lemma 25. *Φ_f represents h.*

Proof. Suppose the opposite. As g is also an exclusive component of Φ_f, that means $u(\ell'') \notin F_{\Phi_f}(\{v(j)\})$ for some label $\ell'' \in L \cup L'$ and index $j \in [t]$. For that to happen, for any index $i \in [d]$ there must be an edge $(x, y) \in E$ such that $e(x, y, i) \notin F_{\Phi_f}(\{v(j)\})$ and for every $\ell' \in L'_y$ there is a vertex $z \in N(y)$ such that $e(z, y, \ell', i) \notin F_{\Phi_f}(\{v(j)\})$ as well. But since f is a tight total-cover, there is a pair of labels $(\ell_z, \ell'_y) \in \Pi_{(z,y)}$ satisfying the clause $u(\ell_z) \wedge u(\ell'_y) \longrightarrow e(z, y, \ell'_y, i)$ as both $u(\ell_z)$ and $u(\ell'_y)$ belong to $F_{\Phi_f}(\{v(j)\})$, contradicting the assumption. Therefore, $F_{\Phi_f}(\{v(j)\}) = V_g \cup \{v(j)\}$ and the claim follows (cf. Lemma 19). □

Lemma 26. *For each index $j \in [t]$, f_j is a tight minimum cost total-cover for $\mathcal{L}$.*

Proof. Fix an index $j \in [t]$. Suppose the tight total-cover f_j for $\mathcal{L}$ (cf. Lemma 24) is not optimal, and let f^* be a minimum cost one. Consider the refined canonical formula Φ_{f^*} and define $\Upsilon_{f^*} = (\Phi_{f^*} \setminus \Psi) \cup \Theta$, where $\Theta \subset \Upsilon$ represents g. Lemma 25 and Corollary 17 guarantee Υ_{f^*} represents h as well. As $|\Upsilon|_c = |\Theta|_c + |\Gamma|_c$, $|\Upsilon_{f^*}|_c = |\Theta|_c + |\Phi_{f^*} \setminus \Psi|_c$, and $|\Phi_{f^*} \setminus \Psi|_c = t(\kappa(f^*)r + s) < t(\kappa(f_j)r + s) = |\Gamma|_c$, it follows that $|\Upsilon_{f^*}|_c < |\Upsilon|_c$, a contradiction. Therefore, f_j is a tight total-cover of minimum cost. □

Remark 27. *The tight optimal total-covers f_j and f_k, for $j, k \in [t]$ and $j \neq k$, might be different. As they have the same optimal cost, any one of them can be exhibited as solution to $\mathcal{L}$.*

Corollary 28. $|\Psi|_c/(\lambda+\lambda') \leq |\Upsilon|_c - t(\kappa(f)r+s) \leq |\Psi|_c$, *where $\kappa(f)r + s$ is the total number of labels in a tight optimal total-cover f for $\mathcal{L}$.*

We can now prove the main result of this section.

Theorem 29. *Unless $P = NP$, the minimum number of clauses of a pure Horn function on n variables cannot be approximated in polynomial time (in n) to within a factor of*

$$\rho_c(n^\varepsilon) \geq 2^{\varepsilon(\log n)^{1-1/\delta_c(n)}} = 2^{O(\log^{1-o(1)} n)},$$

where $\delta_c(n) = (\log\log n)^c$ and c is any fixed constant in $(0, 1/2)$, even when the input is restricted to CNFs with $O(n^{1+2\varepsilon})$ clauses, for some $\varepsilon \in (0, 1/4]$.

Proof. Let $\mathcal{L}_0$ be one of the LABEL-COVER promise instances produced by Dinur and Safra's reduction (cf. Section 2) and let $\mathcal{L}$ be its extension.

Choose $d = r\lambda + s\lambda' + 1$ (cf. Lemma 21), let Φ be the canonical formula constructed from $\mathcal{L}$, and let h be the pure Horn function it defines. Let Υ be a clause minimum pure Horn CNF representation of h obtained by some minimization algorithm when Φ is given as input.

Let c be fixed arbitrarily close to $1/2$ (for the larger the value of c, the larger the hardness of approximation factor). For convenience, let $\delta = \delta_c(s)$ and $\rho = \rho_c(s)$ (cf. Theorem 12). Using Notation 10 and the parametrization given by Remark 13, Lemma 16 and Corollary 28 give

$$\begin{aligned}
|\Upsilon|_c &\geq t(\kappa(f)r + s) + \frac{d(\pi + 2m\lambda') + r\lambda + s\lambda'}{\lambda + \lambda'} \\
&\geq st(\kappa(f)(\delta - 1) + 1) + \Omega(s^2\delta\rho^\delta) \\
&\geq st(\kappa(f)(\delta - 1) + 1) + \omega(s^2),
\end{aligned}$$

and

$$\begin{aligned}
|\Upsilon|_c &\leq t(\kappa(f)r + s) + d(\pi + 2m\lambda') + r\lambda + s\lambda' \\
&\leq st(\kappa(f)\delta + 1) + O(s^3\delta^2\rho^{2\delta}) \\
&\leq st(\kappa(f)\delta + 1) + o(s^4).
\end{aligned}$$

Moreover,

$$\begin{aligned}
t \leq |\Upsilon|_v &= t + dm(\lambda' + 1) + r\lambda + s\lambda' \\
&\leq t + O(s^3\delta\rho^{2\delta}) \leq t + o(s^4).
\end{aligned}$$

Choosing $\varepsilon > 0$ such that $t = s^{1/\varepsilon} = \Omega(s^4)$, gives $|\Upsilon|_c \longrightarrow s^{(1+1/\varepsilon)}\delta_c(s)(\kappa(f) + o(1))$ and $|\Upsilon|_v \longrightarrow s^{1/\varepsilon}$ as $s \longrightarrow \infty$, implying that the construction of Φ can be performed in polynomial time in the number of variables of h.

It also gives the following dichotomy

$$\kappa(\mathcal{L}) = 1 \Longrightarrow |\Upsilon|_c \leq s^{(1+1/\varepsilon)}\delta_c(s)(1 + o(1)),$$

$$\kappa(\mathcal{L}) \geq \rho_c(s) \Longrightarrow |\Upsilon|_c \geq s^{(1+1/\varepsilon)}\delta_c(s)(\rho_c(s) + o(1)).$$

Letting $n = |\Upsilon|_v$ and relating $|\Upsilon|_c$ to the number of variables of h, the above dichotomy reads as

$$\kappa(\mathcal{L}) = 1 \Longrightarrow |\Upsilon|_c \leq n^{(1+\varepsilon)}\delta_c(n^\varepsilon)(1 + o(1)),$$
$$\kappa(\mathcal{L}) \geq \rho_c(s) \Longrightarrow |\Upsilon|_c \geq n^{(1+\varepsilon)}\delta_c(n^\varepsilon)(\rho_c(n^\varepsilon) + o(1)),$$

giving a hardness of approximation factor of $\rho_c(n^\varepsilon)$ for the pure Horn CNF minimization problem (cf. Theorem 12).

To conclude the proof, notice that the gap

$$\rho_c(n^\varepsilon) = 2^{(\varepsilon \log n)^{1-1/\delta_c(n^\varepsilon)}} \geq 2^{\varepsilon(\log n)^{1-1/\delta_c(n)}}$$

and that the number of clauses $n^{(1+\varepsilon)}\delta_c(n^\varepsilon) \leq n^{1+2\varepsilon}$. □

4 3-CNFs and Number of Literals

Clauses of type (b) and (d) might have arbitrarily long subgoals in the reduction we presented (cf. Section 3). It is possible to strengthen our result by modifying the gadget so that every clause is either quadratic or cubic and proving that even in this case, the pure Horn minimization problem is still hard to approximate.

Specifically, we replace the clauses of type (b) in a similar way to what is done on the reduction from SAT to 3-SAT (Garey and Johnson 1979), that is, in a linked list fashion. The same technique does not work with the clauses of type (d), though. The reason being it would introduce clauses of the form $w_1 \wedge w_2 \longrightarrow u(\ell)$, for $\ell \in L \cup L'$ and propositional variables w_1 and w_2, implying that it could be advantageous to have $2t'$ prime implicates of the form $v(j) \longrightarrow w_i$, with $i \in [2]$ and $j \in [t']$ (where t' is a positive integer to be fixed in the same way as t was), in a clause minimum representation of h' (the pure Horn function after introducing the new variables), thus rendering the gap amplification device innocuous once again. We circumvent such possibility replacing the clauses of type (d) as follows. Associate new propositional variables $e(k,i)$ with all indices $k \in [m-1]$ and $i \in [d]$, where as before, $m = |E|$ and $d = r\lambda + s\lambda' + 1$. Let $\langle e_1, e_2, \ldots, e_m\rangle$ be an arbitrary but fixed ordering of the edges in E. Let $e(k,i)$ with indices $k \in \{m, m+1, \ldots, 2m-1\}$ and $i \in [d]$ be aliases for the original propositional variables $e(x,y,i)$ in such a way that for each edge $(x,y) \in E$, $(x,y) = e_{k-m+1}$ is associated to $e(k,i)$. Also fix an arbitrary ordering for the labels in $L \cup L'$. Create a complete binary tree for each index $i \in [d]$ through the clauses

(d$_1$) $e(2k,i) \wedge e(2k+1,i) \longrightarrow e(k,i)$ $\quad \forall\, k \in \left[\lfloor \frac{2m-1}{2} \rfloor\right],\ i \in [d]$;

(d$_2$) $e(2m-1,i) \longrightarrow e\left((2m-1)/2, i\right)$ $\quad \forall\, i \in [d]$, when $2m-1$ is even;

(d$_3$) $e(1,i) \wedge e(1,i+1) \longrightarrow u(\ell_i)$ $\quad \forall\, i \in [d-1]$;

in which the variables $e(x,y,i)$, for edges $(x,y) \in E$, are the leaves. The clauses of type (d$_3$) associates each label from $L \cup L'$ with the roots of two aforementioned trees in an orderly fashion. It helps to think about the roots as nodes and the labels as links of a path of size d. In order for the path to be well defined (all labels being reintroduced), all nodes must be present (every variable $e(1,i)$, $i \in [d]$, must be reached). This implies that if for some index $j \in [t']$ a pure Horn 3-CNF formula representing h' has a prime implicate of the form $v(j) \longrightarrow e(k,i)$, for some $k \in [2m-1]$ and $i \in [d]$, this prime implicate is either superfluous or it is accompanied by at least $d-1$ other prime implicates with head $v(j)$ and having the same form. As d works as a second amplification device (in a similar way to the CNF case), it is clearly not advantageous for clause minimum pure Horn 3-CNF formulae representing h' to have prime implicates involving $v(j)$ of forms other than $v(j) \longrightarrow u(\ell)$, with $\ell \in L \cup L'$. Therefore, this construction suits our purposes.

Lemma 30. *For Φ' as described, it holds that $d(\pi + 2m\lambda' + 2m) - 1 \leq |\Phi'|_c - t(r\lambda + s\lambda') \leq d(\pi + m^2\lambda' + 4m - 2)$ and that $t \leq |\Phi'|_v \leq t + dm(\lambda' + 2) + m^2\lambda'$, with d and t' as above and r, s, m, λ, λ', and π as in Notation 10.*

The same arguments used in Corollary 17 apply in this new setting as the differences between the two gadgets are local. Indeed, it is the case that Lemmas 18–20, Lemmas 23–26 and Corollary 28 also transfer in a semi-verbatim fashion: sometimes, small changes on the objects and arguments used are needed, but the overall ideas and the structure of the proofs are maintained. A small note on the proof of the analogue of Lemma 21 is in order. With the introduction of the new variables in the 3-CNF construction, the sets Γ_i^j for $j \in [t']$ and $i \in [d]$ need to be accordingly redefined to include every possible prime implicate involving variables $v(j)$ not having the form of those in Γ_u^j, that is, not having the form $v(j) \longrightarrow u(\ell)$ for $\ell \in L \cup L'$. The remaining of the proof transfer in a verbatim fashion. We then have the following.

Theorem 31. *Unless P = NP, the minimum number of clauses of a pure Horn function on n variables cannot be approximated in polynomial time (in n) to within a factor of*

$$\rho_c(n^\varepsilon) \geq 2^{\varepsilon(\log n)^{1-1/\delta_c(n)}} = 2^{O(\log^{1-o(1)} n)},$$

where $\delta_c(n) = (\log\log n)^c$ and c is any fixed constant in $(0, 1/2)$, even when the input is restricted to 3-CNFs with $O(n^{1+2\varepsilon})$ clauses, for some $\varepsilon \in (0, 1/6]$.

Now notice that with exception of the variables $v(j)$, with $j \in [t']$, that only appear as subgoals in quadratic prime implicates, every other variable appears in subgoals and heads of mostly cubic pure Horn clauses. Also, since the functions we are dealing with are pure Horn, we have no unit clauses. Therefore, it holds that $2|\Phi'|_c \leq |\Phi'|_l \leq 3|\Phi'|_c$ or in other words, that $|\Phi'|_l = \Theta(|\Phi'|_c)$, where Φ' is a pure Horn 3-CNF formula obtained from our 3-CNF construction above. We then have the following result.

Corollary 32. *Unless P = NP, the minimum number of literals of a pure Horn function on n variables cannot be approximated in polynomial time (in n) to within a factor of $2^{O(\log^{1-o(1)} n)}$, even when the input is restricted to 3-CNFs with $O(n^{1+\varepsilon})$ clauses, for some small constant $\varepsilon > 0$.*

5 Hardness for Sub-exponential and Hyper-polynomial Times

The hardness of approximation results shown on Sections 3 and 4 concern approximations obtained in polynomial time on the number of variables of the pure Horn functions in question. In this section, we show that under another well known complexity hypothesis, allowing sub-exponential time in the number of variables is still not enough to obtain a constant factor approximations.

The following conjecture concerning the time solvability of the k-SAT problem (i.e., the problem of determining if a k-CNF has a satisfying solution) was introduced in (Impagliazzo and Paturi 2001).

Conjecture 33. *For $k \geq 3$, define s_k to be the infimum of*

$$\{\delta : \text{there exists an } O(2^{\delta n} poly(n)) \text{ time algorithm for solving the } k\text{-SAT problem}\},$$

for n being the number of variables of the k-SAT instance. The Exponential Time Hypothesis (ETH) *states that $s_k > 0$ for $k \geq 3$.*

In other words, if true, the ETH implies that there is no sub-exponential time algorithm for k-SAT with $k \geq 3$ and hence, that P $\neq$ NP (the converse however, does not hold). The ETH has many implications beyond search problems, for instance, in fields as communication complexity and structural complexity and is widely believed to be true.

In a recent breakthrough (Moshkovitz and Raz 2010) introduced a new two-query projection test PCP system with sub-constant error and quasi-linear size. One way of interpreting their result is the following. The maximization flavor of the LABEL-COVER$_\varepsilon$ promise problem receives a LABEL-COVER instance in which each vertex can receive at most one label and the goal is to decide whether there is a total-cover for the instance in question or if every labeling covers at most an ε fraction of the edges. Let ϕ be any 3-SAT instance of size (number of clauses) equal to σ and let $\varepsilon = \varepsilon(\sigma) \geq 1/(\log^\beta \sigma)$, for some positive and sufficiently small β. There is a reduction from ϕ to a LABEL-COVER instance of maximization flavor and promise ε whose graph has size at most $\sigma^{1+o(1)} poly(1/\varepsilon)$ and such that $\log |L| \leq poly(1/\varepsilon)$ and $\log |L'| \leq O(\log(1/\varepsilon))$, where as before L and L' are the label sets. For ε as above, this reduction is quasi-linear sized in σ. It also gives that it is NP-hard to solve the maximization flavor of LABEL-COVER$_\varepsilon$. While the soundness error ε is not as small as that of (Dinur and Safra 2004), its quasi-linear size implies that assuming the ETH (i.e., 3-SAT takes $\exp(\Omega(\sigma))$ time to solve), the maximization version of the LABEL-COVER$_\varepsilon$ promise problem cannot be solved in less than $\exp(\sigma^{1-o(1)})$ time, thus ruling out better approximations in sub-exponential time.

A simpler proof was presented in (Dinur and Harsha 2009) who also observed that the reduction can be carried out with ε taken as small as $2^{-O(\log^\beta \sigma)}$ for some $\beta \in (0, 1)$ at the cost of the label set L having size sub-exponential in σ. Using the 'weak duality' between the maximization and minimization flavors, we can read Dinur and Harsha's result as follows.

Theorem 34 ((Dinur and Harsha 2009))**.** *There exists constants $c > 0$ and $\beta \in (0, 1)$, such that for every function $1 < \rho(\sigma) \leq 2^{O(\log^\beta \sigma)}$, there exists alphabets L and L' of sizes $O(2^{\rho(\sigma)^c})$ and $O(\rho(\sigma)^c)$, respectively, such that even under nearly linear length preserving reductions it is NP-hard to distinguish between the case a* LABEL-COVER *instance with promise $\rho(\sigma)$ has a total-cover of cost 1 from the case where every total-cover for it has cost at least $\rho(\sigma)$.*

Remark 35. *We then have a new parametrization, slightly different from the one in Remark 13: $s = \sigma^{1+o(1)}$ with $o(1) \approx (\log\log\sigma)^{-\Omega(1)}$, $r = \sigma 2^{O(\log^\beta \sigma)}$, $m = r\rho(\sigma)^c$, $\lambda = O(2^{\rho(\sigma)^c})$, and $\lambda' = O(\rho(\sigma)^c)$.*

We shall deal with polynomial time reductions first. Let $\vartheta \leq \beta$ such that $c\vartheta \leq 1$ and take $\rho(\sigma) = \log^\vartheta \sigma$. Then, $\lambda = O(2^{(\log^\vartheta \sigma)^c}) = O(\sigma)$ and $\lambda' = O((\log^\vartheta \sigma)^c) \leq O(\log \sigma)$, and the size of the instance in question is nearly linear in σ. Moreover, as $m \leq \pi \leq m\lambda\lambda'$, it follows that

$$2^{O(\log^\beta \sigma)} \sigma \log^{c\vartheta} \sigma \leq \pi \leq O(2^{O(\log^\beta \sigma)} \sigma^2 \log^2 \sigma),$$

and that d, one of the amplification devices we use in our reductions (cf. Lemma 21), is equal to

$$d = r\lambda + s\lambda' + 1 = \Theta(2^{O(\log^\beta \sigma)}\sigma^2 + \sigma^{1+o(1)} \log^{c\vartheta} \sigma). \quad (1)$$

It is then possible to show hardness results for the sub-exponential time case in a similar way.

Theorem 36. *Assuming the ETH, the minimum numbers of clauses and literals of pure Horn functions on n variables cannot be approximated in $\exp(n^\delta)$ time, for some $\delta \in (0, 1)$, to within factors of $O(\log^\vartheta n)$, for some $\vartheta \in (0, 1)$, even when the input is restricted to 3-CNFs with $O(n^{1+2\varepsilon})$ clauses, for some small constant $\varepsilon > 0$.*

Proof. Let $\mathcal{L}_0$ be one of the Dinur and Harsha's LABEL-COVER promise instances and let $\mathcal{L}$ be its extension. Let d be as in Equation (1), Φ' be the canonical pure Horn 3-CNF formula constructed from $\mathcal{L}$, and h' be the pure Horn function it defines. Let Υ' be a clause minimum pure Horn 3-CNF representation of h' obtained by some minimization algorithm when Φ' is given as input.

For convenience, let $\rho = \rho(\sigma)$ and $\zeta = \zeta(\sigma) = r/s = 2^{O(\log^\beta \sigma)}/\sigma^{o(1)}$. Using Notation 10 and the new parametrization above, Lemma 30 and an analogue of Corollary 28 give

$$\begin{aligned} |\Upsilon'|_c &\geq t(\kappa(f)r + s) + \frac{d(\pi + 2m\lambda' + 2m) - 1}{\lambda + \lambda'} \\ &\geq st(\kappa(f)r/s + 1) + \Omega(\sigma^{1+o(1)} \log^{2c\vartheta} \sigma 2^{\log^\tau \sigma}) \\ &\geq st(\kappa(f)\zeta + 1) + \omega(s), \end{aligned}$$

and

$$\begin{aligned} |\Upsilon'|_c &\leq t(\kappa(f)r + s) + d(\pi + m^2\lambda' + 4m - 2) \\ &\leq st(\kappa(f)r/s + 1) + O(\sigma^4 \log^3 \sigma 2^{\log^\tau \sigma}) \\ &\leq st(\kappa(f)\zeta + 1) + o(s^5), \end{aligned}$$

where $\tau > 0$ is some constant. Moreover,

$$\begin{aligned} t \leq |\Upsilon'|_v &\leq t + dm(\lambda' + 2) + m^2\lambda' \\ &\leq t + O(\sigma^2 \log^2 \sigma 2^{\log^{2\tau} \sigma}) \\ &\leq t + o(s^3). \end{aligned}$$

Choosing $\varepsilon' > 0$ such that $t = s^{1/\varepsilon'} = \Omega(s^4)$, gives $|\Upsilon'|_c \longrightarrow s^{(1+1/\varepsilon')}\zeta(\sigma)(\kappa(f) + o(1))$ and $|\Upsilon'|_v \longrightarrow s^{1/\varepsilon'}$ as $s \longrightarrow \infty$, implying the construction of Φ' can be performed in polynomial time in the number of variables of h'.

It also gives the following dichotomy

$$\kappa(\mathcal{L}) = 1 \Longrightarrow |\Upsilon'|_c \leq s^{(1+1/\varepsilon')}\zeta(\sigma)(1 + o(1)),$$
$$\kappa(\mathcal{L}) \geq \rho(\sigma) \Longrightarrow |\Upsilon'|_c \geq s^{(1+1/\varepsilon')}\zeta(\sigma)(\rho(\sigma) + o(1)),$$

with $\rho(\sigma) = \log^{\vartheta} \sigma$ in this case. Let $n = |\Upsilon'|_v$ and let $\varepsilon = \varepsilon'/(1 + o(1))$. Relating the number of clauses of Υ' to the number of variables of h, the above dichotomy reads as

$$\kappa(\mathcal{L}) = 1 \Longrightarrow |\Upsilon'|_c \leq n^{(1+\varepsilon')}\zeta(n^{\varepsilon})(1 + o(1)),$$
$$\kappa(\mathcal{L}) \geq \rho(\sigma) \Longrightarrow |\Upsilon'|_c \geq n^{(1+\varepsilon')}\zeta(n^{\varepsilon})(\rho(n^{\varepsilon}) + o(1)),$$

giving a hardness of approximation factor of $\rho(n^{\varepsilon})$ for the pure Horn 3-CNF clause minimization problem (cf. Theorem 34). As ε^{ϑ} is a constant, it follows that the gap

$$\rho(n^{\varepsilon}) = \log^{\vartheta} n^{\epsilon} = O(\log^{\vartheta} n).$$

Also, the number of clauses $n^{(1+\varepsilon')}\zeta(n^{\varepsilon}) \leq n^{1+2\varepsilon'}$.

To conclude the clause minimization part, observe that sub-exponential time in σ, namely, $2^{o(\sigma)}$ is equivalent to $2^{o(n^{1/4-o(1)})}$ time in n and since $|\Upsilon'|_c = O(n^{1+2\varepsilon'})$, the time bound follows for $\delta < (1 - 2\varepsilon')/4$.

Regarding literal minimization, the structure of Φ' implies its numbers of clauses and literals differ by only a constant (cf. Section 4) and therefore, similar results hold in this case as well. □

A natural next step is trying to push the hardness of approximation factor further by allowing, for instance, super-polynomially sized LABEL-COVER instances. More specifically, let $b > 1$ be such that $\alpha = bc > 1$ and take $\rho(\sigma) = \log^{\alpha} \sigma = o(2^{O(\log^{\beta} \sigma)})$ for some $\beta \in (0, 1)$. It then follows that $\lambda = O(2^{\log^{\alpha} \sigma}) = O(\sigma^{\log^{\alpha-1} \sigma})$, $\lambda' = O(\log^{\alpha} \sigma)$, and that $\pi \leq O(2^{\log^{\alpha} \sigma + O(\log^{\beta} \sigma)}\sigma \log^{\alpha} \sigma)$, implying the size of the LABEL-COVER instance is now super-polynomial in σ. Also, it follows that the value we chose for d is

$$d = \Theta(\sigma 2^{\log^{\alpha} \sigma + O(\log^{\beta} \sigma)}).$$

Following the steps of Theorem 36's proof, we obtain that

$$\omega(s^2) \leq |\Upsilon'|_c - t(\kappa(f)r + s) \leq o(\sigma^{4+\log^{\alpha-1} \sigma})$$

and that

$$t \leq |\Upsilon'|_v \leq t + o(\sigma^{3+\log^{\alpha-1} \sigma}).$$

Now, choosing $t = (8\sigma)^{\log^{\alpha-1} 8\sigma}$ gives a hardness of approximation factor for pure Horn clause minimization equal to $\rho(\sigma)$. Writing σ as a function of n, we obtain that $\sigma = (2^{\log^{1/\alpha} n})/8$ and hence, a hardness of approximation factor

$$\rho(n) = \log^{\alpha} \left(\frac{2^{\sqrt[\alpha]{\log n}}}{8} \right) = \left(\sqrt[\alpha]{\log n} - 3 \right)^{\alpha} = O(\log n).$$

As $|\Upsilon'|_c = O(n^2)$, sub-exponential time $2^{o(\sigma)}$ means

$$\mu(n) := 2^{o\left(\left(2^{\sqrt[\alpha]{\log n}}/8\right)^{1/2}\right)} = o\left(2^{n^{1/(\log\log n)^C}}\right)$$

time, for huge constant $C > 0$. We call $\mu(n)$ a *hyper-polynomial* time function as it sits between super-polynomial and sub-exponential ones. We then just proved the following theorem.

Theorem 37. *Assuming the ETH, the minimum numbers of clauses and literals of pure Horn functions on n variables cannot be approximated in $\mu(n)$ time to within factors of $O(\log n)$, for all $\alpha > 1$, even when the input is restricted to 3-CNFs with $O(n^2)$ clauses.*

Now, if we take $\rho = 2^{O(\log^{\beta} \sigma)}$ for some $\beta \in (0, 1)$ we obtain a sub-exponential sized LABEL-COVER instance. After similar calculations nevertheless, we obtain a similar $O(\log n)$ hardness factor under more stringent time constraints, thus rendering such result obsolete in light of Theorem 37.

6 Conclusions

In the last three sections, we have shown improved hardness of approximation results for the Horn CNF formulae minimization (regarding the number of clauses and the number of literals). A natural question that comes to mind concerns the tightness of these results. It was shown in (Hammer and Kogan 1993) that for a pure Horn function on n variables, it is possible to approximate the minimum number of clauses and the minimum number of literals of a pure Horn CNF formula representing it to within a factor of $n-1$ and $\binom{n}{2}$, respectively. This leaves a considerable gap on the polynomial time case and an even larger gap on the sub-exponential time case. Whether these gaps can be improved remain yet to be seen. Improvements on two-query sub-constant error PCP systems, leading to larger gaps on the LABEL-COVER problem, would immediately imply (as long as the instances are polynomially sized) larger hardness of approximation results for the problems we discussed. Such improvement might allow one to obtain hardness results for super-polynomial time and close the gap between $\exp(n^{\delta})$ and $\exp(o(n))$ sub-exponential times, for some $\delta \in (0, 1)$, both of which we left open. A second possibility is the use of a different promise problem (with a larger gap than that of LABEL-COVER) as a starting point and construction of a new reduction.

On the opposite direction, it might be possible to design new approximation algorithms with lower factors of approximation.

We conjecture that it is possible to improve the hardness of approximation factor for clause minimization in polynomial time, of a Horn function in n variables to at least n^{ε} for some small $\varepsilon > 0$ (perhaps even $n^{1/2-\varepsilon}$) and that a similar factor holds in the case of literal minimization (up to a constant).

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