Extending Tournament Solutions

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Abstract

An important subclass of social choice functions, so-called *C1 functions*, only take into account the pairwise majority relation between alternatives. In the absence of majority ties—e.g., when there is an odd number of agents with linear preferences—the majority relation is antisymmetric and complete and can thus conveniently be represented by a tournament. Tournaments have a rich mathematical theory and many formal results for C1 functions assume that the majority relation constitutes a tournament. Moreover, most C1 functions have only been *defined* for tournaments and allow for a variety of generalizations to unrestricted preference profiles, none of which can be seen as the unequivocal extension of the original function. In this paper, we argue that restricting attention to tournaments is justified by the existence of a *conservative extension*, which inherits most of the commonly considered properties from its underlying tournament solution.

1 Introduction

Perhaps one of the most natural ways to aggregate binary preferences from individual agents to a group of agents is *simple majority rule*, which prescribes that one alternative is socially preferred to another whenever a majority of agents prefers the former to the latter. Majority rule has an intuitive appeal to democratic principles, is easy to understand and—most importantly—satisfies some attractive formal properties [20]. Moreover, almost all common voting rules coincide with majority rule in the two-alternative case. It would therefore seem that the existence of a majority of individuals preferring alternative a to alternative b signifies something fundamental and generic about the group's preferences over a and b.

A *C1 function* is a social choice function that is based on the majority relation only. When dealing with C1 functions, it is often assumed that there are no majority ties. For example, this can be guaranteed by insisting on an odd number of voters with linear preferences. Under this assumption, a preference profile gives rise to a *tournament* and a C1 function is equivalent to a *tournament solution*, i.e., a function that associates with every complete and antisymmetric directed graph a subset of the vertices of the graph. Examples of well-studied tournament solutions are the Copeland set, the top cycle, the uncovered set, or the Slater set (see [19]). Recent years have witnessed an increasing interest in tournament solutions both in terms of their axiomatic as well as algorithmic properties by the multiagent systems community [7, 17, 10, 11, 9, 3] and the theoretical computer science community [23, 2, 4].

It is natural to ask how a given C1 function can be generalized to the class of preference profiles that may admit majority ties. Mathematically speaking, we are looking for ways to apply a tournament solution to a complete, but not necessarily antisymmetric, directed graph—a so-called *weak tournament*. For many tournament solutions, generalizations or extensions to weak tournaments have been proposed. Often, it turns out that there are several sensible ways to generalize a tournament solution and it is unclear whether there exists a unique "correct" generalization. Even for something as elementary as the Copeland set or the top cycle, there is a variety of extensions that are regularly considered in the literature. A natural criterion for evaluating the different proposals is whether the extension satisfies appropriate generalizations of the axiomatic properties that the original tournament solution satisfied.

In this paper, we propose a generic way to extend any tournament solution to the class of weak tournaments. The *conservative extension* of a tournament solution S returns all alternatives that are chosen by S in *some* orientation of the weak tournament at hand. We show that many of the most common axiomatic properties of tournament solution are "inherited" from S to its conservative extension. We argue that these results provide a justification for restricting attention to tournaments when studying C1 functions.

The conservative extension also leads to interesting *computational* problems that are strongly related to the possible winner problem [18]. In fact, computing the conservative extension of a tournament solution is equivalent to solving its possible winner problem when pairwise comparisons are only partially specified. Of course, there is an exponential number of orientations of a weak tournament in general. Early results, however, indicate that for many well-known tournament solutions, the corresponding conservative extensions can be computed efficiently by exploiting individual peculiarities of these concepts.

2 Preliminaries

Let U be a universe of alternatives. For notational convenience we assume that $\mathbb{N} \subseteq U$. Every nonempty finite subset of U is called a *feasible set*. For a relation $R \subseteq U \times U$ and $(a, b) \in U \times U$, we usually write $a\ R\ b$ instead of the more cumbersome $(a, b) \in R$. A *weak tournament* is a pair $W = (A, \succsim)$, where A is a feasible set and $\succsim$ is a complete relation on U, i.e., for all $a, b \in U$, we have $a \succsim b$ or $b \succsim a$ (or both).[1] Intuitively, $a \succsim b$ signifies that alternative a is (weakly) preferable to b. Note that completeness implies reflexivity, i.e., $a \succsim a$ for all $a \in U$. We write $a \succ b$ if $a \succsim b$ and not $b \succsim a$, and $a \sim b$ if both $a \succsim b$ and $b \succsim a$. We denote the class of all weak tournaments by $\mathcal{W}$.

The relation $\succsim$ is often referred to as the *dominance relation*. One of the best-known concepts defined in terms of the dominance relation is that of a Condorcet winner. Alternative a is a *Condorcet winner* in a weak tournament $W = (A, \succsim)$ if $a \succ b$ for all alternatives $b \in A \setminus \{a\}$.

A *tournament* is a weak tournament whose dominance relation $\succsim$ is also antisymmetric, i.e., for all *distinct* $a, b \in U$, we have either $a \succsim b$ or $b \succsim a$ (but not both).[2] For a tournament $T = (A, \succsim)$ and distinct alternatives $a, b \in A$, $a \succsim b$ if and only if $a \succ b$. We therefore often write $T = (A, \succ)$ instead of $T = (A, \succsim)$. We denote the class of all tournaments by $\mathcal{T}$. Obviously, $\mathcal{T} \subseteq \mathcal{W}$.

For a pair of weak tournaments $W = (A, \succsim)$ and $W' = (A', \succsim')$, we say that W is *contained* in W', and write $W \subseteq W'$, if $A = A'$ and $a \succsim b$ implies $a \succsim' b$ for all $a, b \in U$. We will often deal with the set of all tournaments that are contained in a given weak tournament W.

Definition 1. *For a weak tournament $W \in \mathcal{W}$, the set of* orientations *of W is given by $[W] = \{T \in \mathcal{T} : T \subseteq W\}$.*

Every orientation of a weak tournament $W = (A, \succsim)$ can be obtained from W by omitting either $a \succsim b$ or $a \succsim b$ for every pair in $\{(a, b) \in U \times U : a \sim b\}$.

The relation $\succsim$ can be raised to sets of alternatives and we write $A \succsim B$ to signify that $a \succsim b$ for all $a \in A$ and all $b \in B$. For a weak tournament $W = (A, \succsim)$ and a feasible set $B \subseteq A$, we will sometimes consider the *restriction* $W|_B = (B, \succsim)$ of W to B.

A *tournament solution* is a function S that maps each tournament $T = (A, \succ)$ to a nonempty subset $S(T)$ of its alternatives A called the choice set. It is generally assumed that tournament solutions cannot distinguish between isomorphic tournaments, i.e., if $h\colon A \to A'$ is an isomorphism between two tournaments $(A, \succ)$ and $(A', \succ')$, then $S(A', \succ') = \{h(a) : a \in S(A, \succ)\}$.

Two examples of well-known tournament solutions are the top cycle and the Copeland set. The *top cycle* $TC(T)$ of a tournament $T = (A, \succ)$ is defined as the smallest set $B \subseteq A$ such that $B \succ A \setminus B$. The *Copeland set* $CO(T)$ consists of all alternatives whose dominion is of maximal size, i.e., $CO(T) = \arg\max_{a \in A} |\{b \in A \setminus \{a\} : a \succ b\}|$.

A very basic requirement for any tournament solution is that a Condorcet winner should be uniquely selected whenever there is one.

Definition 2. *A tournament solution S is* Condorcet-consistent *if $S(T) = \{a\}$ for all tournaments T such that a is a Condorcet winner in T.*

Both TC and UC are Condorcet-consistent.

3 The Conservative Extension

In order to make tournament solutions applicable to general preference profiles, we need to generalize them to weak tournaments. A *generalized tournament solution* is a function S that maps each weak tournament $W = (A, \succsim)$ to a nonempty subset $S(W)$ of its alternatives A. A generalized tournament solution S is called an *extension* of tournament solution S' if $S(W) = S'(W)$ whenever W is a tournament. For several tournament solutions, extensions have been proposed in the literature (e.g., [16, 22]). Of course, there are many ways to extend any given tournament solution, and there is no definite way of judging whether one proposal is better than another one.

In this paper, we are interested in *generic* ways to extend any tournament solution to the class of weak tournaments. In particular, our goal is to extend tournament solutions in such a way that common axiomatic properties are "inherited" from a tournament solution to its extension. This task is not trivial, as even the arguably most cautious approach has its problems: let the *trivial extension* of a tournament solution S be defined as the generalized tournament solution that always selects the whole feasible set A whenever the weak tournament $W = (A, \succsim)$ is *not* a tournament. It is easy to see that the trivial extension does not satisfy Condorcet-consistency.[3][4] Indeed, for the weak tournament $(\{a, b, c\}, \succsim)$ with $a \succ \{b, c\}$ and $b \sim c$, the trivial extension of any tournament solution selects $\{a, b, c\}$.

We therefore propose to extend tournament solutions in a slightly more sophisticated way. The *conservative extension* of a tournament solution S returns all alternatives that are chosen by S in *some* orientation of the weak tournament at hand.

Definition 3. *Let S be a tournament solution. The conservative extension $[S]$ of S is the generalized tournament solution that maps a weak tournament $W \in \mathcal{W}$ to*

$$[S](W) = \bigcup_{T \in [W]} S(T).$$

This definition is reminiscent of the parallel-universes tie-breaking approach in social choice theory [14, 12] and corresponds to selecting the set of all possible winners of W when ties are interpreted as missing edges [3].

[1] This definition slightly diverges from the common graph-theoretic definition where $\succsim$ is defined on A rather than on U. However, it facilitates the sound definition of tournament solutions and their properties.

[2] Defining tournaments with a reflexive dominance relation is non-standard. The reason we define tournaments in such a way is to ensure that every tournament is a weak tournament. Whether the dominance relation of a tournament is reflexive or not does not make a difference for any of our results.

[3] Definition 2 directly applies to *generalized* tournament solutions.

[4] The trivial extension also fails to inherit composition-consistency.

The example weak tournament depicted in Figure 1 has four orientations. It can be checked that $\{CO(T) : T \in [W]\} = \{\{a\}, \{a,b\}, \{a,c\}\}$ and $\{TC(T) : T \in [W]\} = \{\{a\}, \{a,b,c,d\}\}$. Therefore, $[CO](W) = \{a,b,c\}$ and $[TC](W) = \{a,b,c,d\}$.

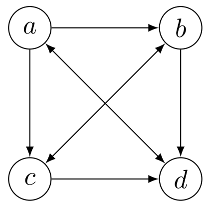

Figure 1: Weak tournament $W = (\{a,b,c,d\}, \succsim)$ with $[CO](W) = \{a,b,c\}$ and $[TC](W) = \{a,b,c,d\}$. An edge from vertex x to vertex y represents $x \succsim y$.

4 Inheritance of Properties

The literature on (generalized) tournament solutions has identified a number of desirable properties for these concepts. In this section, we study which properties are inherited when a tournament solution is generalized via the conservative extension. After stating a useful lemma, we consider three classes of properties: *dominance-based properties* (Section 4.2) that deal with changes in the dominance relation, *choice-theoretic properties* that deal with varying feasible sets (Section 4.3), and *composition-consistency* (Section 4.4). Due to the space constraint, proofs are omitted.

4.1 A General Lemma

Many properties express the invariance of alternatives being chosen (or alternatives not being chosen) under certain type of transformations of the weak tournament. That is, they have the form that if an alternative a is chosen (not chosen) from some weak tournament W, then a is also chosen (not chosen) from $f(W)$, where f is an operation that transforms weak tournaments in a particular way.

Formally, a *tournament operation* is a mapping f from the class of all weak tournaments to itself. A tournament operation f is *orientation-consistent* if applying the operation to any orientation of a weak tournament W results in a tournament that is an orientation of $f(W)$.

Definition 4. *A tournament operation f is* orientation-consistent *if for all weak tournaments W and all $T \in [W]$,*

$$f([W]) = [f(W)],$$

where $f([W]) = \{f(T) : T \in [W]\}$. Furthermore, a class F of tournament operations is orientation-consistent *if each operation in F is orientation-consistent.*

The commutative diagram in Figure 2 illustrates this definition. Observe that a necessary condition for f to be orientation-consistent is that $f(\mathcal{T}) \subseteq \mathcal{T}$.

Let F be a class of tournament operations and $\mathcal{C}$ a subclass of weak tournaments. We then say that a generalized

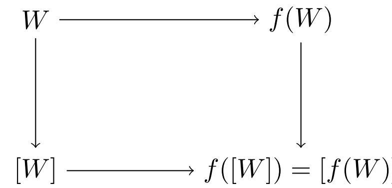

Figure 2: Orientation-consistency

tournament solution S is *inclusion-invariant under F on $\mathcal{C}$* if, for all weak tournaments W in $\mathcal{C}$,

$$a \in S(W) \quad \text{implies} \quad a \in S(f(W)) \text{ for all } f \in F.$$

Similarly, we say that S is *exclusion-invariant under F on $\mathcal{W}$* if

$$a \notin S(W) \quad \text{implies} \quad a \notin S(f(W)) \text{ for all } f \in F.$$

We are now in a position to formulate a very useful lemma.

Lemma 1. *Let F be an orientation-consistent class of tournament operations and S a tournament solution. Then,*

(i) *if S is inclusion-invariant under F on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$,*

(ii) *if S is exclusion-invariant under F on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

4.2 Dominance-Based Properties

We first look at three properties that deal with changes in the dominance relation, namely monotonicity (MON), set-monotonicity (SMON), and independence of unchosen alternatives (IUA).

A tournament solution is *monotonic* if a chosen alternative remains in the choice set when it is strengthened against some other alternative, while leaving everything else unchanged. Here, *strengthening* a versus b refers to replacing $b \succ a$ with $a \succ b$. In weak tournaments, we can also strengthen a against b by replacing $a \sim b$ with $a \succ b$.[5] In order to formalize monotonicity, let $W = (A, \succsim)$ be a weak tournament and define $W_{a \succ b} = (A, \succsim')$, where $\succsim' = \succsim \setminus \{(b,a)\} \cup \{(a,b)\}$.

Definition 5. *A generalized tournament solution S is* monotonic *if for all $W = (A, \succsim)$ and $b \in U$,*

$$a \in S(W) \quad \textit{implies} \quad a \in S(W_{a \succ b}).$$

It is easy to see that monotonicity can be phrased as an inclusion-invariance condition. Let $f_{a \succ b}$ be the tournament operation that maps a weak tournament W to $W_{a \succ b}$ and define $F_a^{MON}(W) = \{f_{a \succ b} : b \in U \setminus \{a\}\}$.

Lemma 2. *A generalized tournament solution is monotonic if and only if it is inclusion-invariant under F_a^{MON}.*

Since F_a^{MON} is clearly orientation-consistent, we can apply Lemma 1 and get our first inheritance result.

[5] A more subtle way to strengthen a against b consists in replacing $b \succ a$ with $a \sim b$. Although this case is not covered by our definition of monotonicity, it can be shown that $[S]$ satisfies this additional property as long as S is monotonic.

Corollary 1. *If a tournament solution S is monotonic on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

Independence of unchosen alternatives (IUA) prescribes that the choice set is invariant under any changes in the dominance relation among unchosen alternatives.

Definition 6. *A generalized tournament solution S is* independent of unchosen alternatives (IUA) *if for all $W = (A, \succsim)$ and $a, b \in U \setminus S(A)$,*

$$S(W) = S(W_{a \succ b}).$$

In order to apply Lemma 1, define $F^{IUA}(W) = \{f_{a \succ b} : a, b \in U \setminus S(A)\}$ and observe that $F^{IUA}(W)$ is orientation-consistent.

Lemma 3. *A generalized tournament solution satisfies IUA if and only if it is both inclusion-invariant under F^{IUA} and exclusion-invariant under F^{IUA}.*

Corollary 2. *If a tournament solution S is independent of unchosen alternatives on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

Set-monotonicity is a strengthening of both monotonicity and IUA and features prominently in a characterization of group-strategyproof social choice functions [6]. A tournament solution is *set-monotonic* if the choice set remains the same whenever some alternative is strengthened against some unchosen alternative.

Definition 7. *A generalized tournament solution S is* set-monotonic (SMON) *if for all $W = (A, \succsim)$ and $a, b \in U$ with $b \notin S(A)$,*

$$S(W) = S(W_{a \succ b}).$$

Similar to the case of IUA, we can define an orientation-consistent class of tournament operations that allows us to phrase set-monotonicity in terms of the conjunction of inclusion-invariance and exclusion-invariance. Define $F^{SMON}(W) = \{f_{a \succ b} : a \in U, b \in U \setminus S(A)\}$.

Lemma 4. *A generalized tournament solution is set-monotonic if and only if it is both inclusion-invariant under F^{SMON} and exclusion-invariant under F^{SMON}.*

Corollary 3. *If a tournament solution S is set-monotonic on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

4.3 Choice-Theoretic Properties

We now turn to a class of properties that relate choices from different feasible sets to each other. For all of these properties, the dominance relation $\succsim$ is fixed. We can therefore simplify notation and write $S(A)$ as a shorthand for $S((A, \succsim))$.

The central property in this section is stability [8], which requires that a set is chosen from two different sets of alternatives if and only if it is chosen from the union of these sets.

Definition 8. *A generalized tournament solution S is* stable *if for all feasible sets A, B and $X \subseteq A \cap B$,*

$$X = S(A) = S(B) \quad \text{if and only if} \quad X = S(A \cup B).$$

Stability can be factorized into conditions $\widehat{\alpha}$ and $\widehat{\gamma}$ by considering each implication in the above equivalence separately.

Definition 9. *Let A, B be feasible sets and $X \subseteq A \cap B$. A generalized tournament solution S satisfies*

(i) $\widehat{\alpha}$ *if $X = S(A \cup B)$ implies $X = S(A) = S(B)$, and*

(ii) $\widehat{\gamma}$ *if $X = S(A) = S(B)$ implies $X = S(A \cup B)$.*

Property $\widehat{\alpha}$ is also known as Chernoff's *postulate* 5^* [13], the *strong superset property* [5], or *outcast* [1] (see [21] for a more thorough discussion of the origins of this condition).

For a finer analysis, we introduce a new perspective on these two properties by splitting them up further.

Definition 10. *Let A, B be feasible sets and $X \subseteq A \cap B$. A generalized tournament solution S satisfies*

(i) $\widehat{\alpha}_{\subseteq}$ *if $S(A \cup B) = X$ implies $S(A) \subseteq X \wedge S(B) \subseteq X$,*

(ii) $\widehat{\alpha}_{\supseteq}$ *if $S(A \cup B) = X$ implies $S(A) \supseteq X \wedge S(B) \supseteq X$,*

(iii) $\widehat{\gamma}_{\subseteq}$ *if $X = S(A) = S(B)$ implies $X \subseteq S(A \cup B)$, and*

(iv) $\widehat{\gamma}_{\supseteq}$ *if $X = S(A) = S(B)$ implies $X \supseteq S(A \cup B)$.*

Property $\widehat{\alpha}_{\subseteq}$ has been called the *weak superset property* or the *Aïzerman property* before.

Obviously, we have $\widehat{\alpha} \Leftrightarrow \widehat{\alpha}_{\subseteq} \wedge \widehat{\alpha}_{\supseteq}$ and $\widehat{\gamma} \Leftrightarrow \widehat{\gamma}_{\subseteq} \wedge \widehat{\gamma}_{\supseteq}$. Furthermore note that $\widehat{\alpha}_{\subseteq}$ implies *idempotency*, i.e., $S(S(W)) = S(W)$ for all weak tournaments W.

Lemma 1 is not directly applicable to these choice-theoretic properties. However, some of the properties can be formulated in a more accessible way. Brandt and Harrenstein [8] have shown that a generalized tournament solution S satisfies $\widehat{\alpha}$ if and only if for all feasible sets A, B, $S(A) \subseteq B \subseteq A$ implies $S(A) = S(B)$. Similar characterizations exist for $\widehat{\alpha}_{\subseteq}$ and $\widehat{\alpha}_{\supseteq}$.

Lemma 5. *Let S be a generalized tournament solution and A, B feasible sets.*

(i) S satisfies $\widehat{\alpha}_{\subseteq}$ if and only if

$$S(A) \subseteq B \subseteq A \text{ implies } S(A) \subseteq S(B).$$

(ii) S satisfies $\widehat{\alpha}_{\supseteq}$ if and only if

$$S(A) \subseteq B \subseteq A \text{ implies } S(A) \supseteq S(B).$$

For a feasible set B, we let f_B denote the tournament operation that maps a weak tournament $W = (A, \succsim)$ with $B \subseteq A$ to its restriction to B, i.e., $f_B(W) = (B, \succsim)$. We can now reformulate $\widehat{\alpha}$, $\widehat{\alpha}_{\subseteq}$, and $\widehat{\alpha}_{\supseteq}$ as invariance properties by letting $F^{\widehat{\alpha}} = \{f_B : S(A) \subseteq B \subseteq A\}$.

Lemma 6. *Let S be a generalized tournament solution.*

(i) S satisfies $\widehat{\alpha}_{\subseteq}$ if and only if S is inclusion-invariant under $F^{\widehat{\alpha}}$.

(ii) S satisfies $\widehat{\alpha}_{\supseteq}$ if and only if S is exclusion-invariant under $F^{\widehat{\alpha}}$.

(iii) S satisfies $\widehat{\alpha}$ if and only if S is both inclusion-invariant under $F^{\widehat{\alpha}}$ and exclusion-invariant under $F^{\widehat{\alpha}}$.

Since $F^{\widehat{\alpha}}$ is orientation-consistent, we can apply Lemma 1 to get the following inheritance statements.

Corollary 4. *Let $\phi \in \{\widehat{\alpha}, \widehat{\alpha}_{\subseteq}, \widehat{\alpha}_{\supseteq}\}$. If a tournament solution S satisfies property ϕ on $\mathcal{T}$, so does $[S]$ on $\mathcal{W}$.*

For $\widehat{\gamma}$ and its descendants $\widehat{\gamma}_{\subseteq}$ and $\widehat{\gamma}_{\supseteq}$, no characterization is known that is similar in spirit to Lemma 5. In fact, we were not able to prove that any of these three properties is inherited from a tournament solution S to its conservative extension $[S]$. However, all three properties are inherited if S satisfies $\widehat{\alpha}$.

Proposition 1. *Let S be a tournament solution that satisfies $\widehat{\alpha}$ and let $\phi \in \{\widehat{\gamma}, \widehat{\gamma}_{\subseteq}, \widehat{\gamma}_{\supseteq}\}$. If S satisfies property ϕ on $\mathcal{T}$, so does $[S]$ on $\mathcal{W}$.*

Since stability is equivalent to the conjuction of $\widehat{\alpha}$ and $\widehat{\gamma}$, the following statement follows immediately from Corollary 4 and Proposition 1.

Corollary 5. *If S is stable on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

Interestingly, assuming $\widehat{\alpha}$ in order to ensure that $\widehat{\gamma}$ is inherited is less restrictive than it seems because all common tournament solution satisfy $\widehat{\alpha}$ *if and only if* they satisfy $\widehat{\gamma}$. In general, however, it is the case that $\widehat{\alpha}$ and $\widehat{\gamma}$ are independent from each other, even though this requires the construction of rather artificial tournament solutions.

Proposition 2. *There exists a tournament solution that satisfies $\widehat{\alpha}$, but not $\widehat{\gamma}$.*

Proposition 3. *There exists a tournament solution that satisfies $\widehat{\gamma}$, but not $\widehat{\alpha}$.*

4.4 Composition-Consistency

We finally consider a structural property that deals with sets of similar alternatives. A component of a tournament is a subset of alternatives that bear the same dominance relationship to all alternatives not in the set. A decomposition is a partition of the alternatives into components. A decomposition induces a summary tournament with the components as alternatives. A tournament solution is then said to be composition-consistent if it selects the best alternatives from the components it selects from the summary tournament.

In order to extend the definition of composition-consistency to weak tournaments, we first redefine the concept of a component. By a *component* of a weak tournament $W = (A, \succsim)$ we understand feasible set $X \subseteq A$ such that X is a singleton or for all $y \in A \setminus X$ either $X \succ y$ or $y \succ X$. We have the following lemma.

Lemma 7. *Let $W = (A, \succsim)$ be a weak tournament and $X \subseteq A$. Then, X is a component of W if and only if X is a component of every orientation $T \in [W]$.*

A *decomposition* of a weak tournament $W = (A, \succsim)$ we define as a partition $\{X_1, \ldots, X_k\}$ of A such that each X_i is a component of W. Moreover, let $W_1 = (B_1, \succsim_1), \ldots, W_k = (B_k, \succsim_k)$, and $\tilde{W} = (\{1, \ldots, k\}, \tilde{\succsim})$ be weak tournaments with $B_1, \ldots, B_k$ pairwise disjoint. Then, define the *product* $\prod(\tilde{W}, W_1, \ldots, W_k)$ of $W_1, \ldots, W_k$ with respect to $\tilde{W}$ as the weak tournament $(A, \succsim')$ such that $A = \bigcup_{i=1}^{k} B_i$ and, for all $b_1 \in B_i, b_2 \in B_j$,

$$b_1 \succsim' b_2 \quad \text{if and only if} \quad i = j \text{ and } b_1 \succsim_i b_2, \text{ or } i \neq j \text{ and } i \mathrel{\tilde{\succsim}} j.$$

We are now in a position to define composition-consistency for weak tournaments.

Definition 11. *A generalized tournament solution S is* weakly composition-consistent *if for all weak tournaments W, decompositions $\{X_1, \ldots, X_k\}$ of W, and $\tilde{W} = (\{1, \ldots, k\}, \tilde{\succsim})$ such that $W = \prod(\tilde{W}, W|_{X_1}, \ldots, W|_{X_k})$,*

$$S(T) \quad = \bigcup_{i \in S(\tilde{W})} S(W|_{X_i}).$$

Let $W = \prod(\tilde{W}, W_1, \ldots, W_k)$ where $W_i = (B_i, \succsim_i)$ for all i with $1 \leq i \leq k$. Observe that then $\{B_1, \ldots, B_k\}$ is a decomposition of W whenever $\tilde{W}$ is a tournament. If, moreover, no B_i is a singleton, the implication also holds in the opposite direction. We find that, if a tournament solution S is composition-consistent on the class of tournaments, so is its conservative extension $[S]$ on the class of weak tournaments.

Proposition 4. *If a tournament solution S is composition-consistent on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

5 Computational Complexity

When a tournament solution S is generalized via the conservative extension to $[S]$, it is natural to ask whether the choice set of $[S]$ can be computed efficiently. Since the number of orientations of a weak tournament can be exponential in the size of the weak tournament, tractability of the winner determination problem of S is a necessary, but not a sufficient, condition for the tractability of $[S]$.

Proposition 5. *There is a tournament solution S such that the winner determination problem is in P for S, and NP-complete for $[S]$.*

In light of Proposition 5, it is interesting to check for each tractable tournament solution S, whether the choice set of $[S]$ can be computed efficiently. This question is mathematically equivalent to the problem of computing the set of *possible winners for a partially specified tournament*. The latter problem has been studied for the Copeland set CO, the top cycle TC, and the uncovered set UC.

Proposition 6 ([15])**.** *Computing $[CO]$ is in P.*

Proposition 7 ([18])**.** *Computing $[TC]$ is in P.*

Proposition 8 ([3])**.** *Computing $[UC]$ is in P.*

While the proof of Proposition 6 consists in a polynomial-time reduction to maximum network flow, $[TC]$ and $[UC]$ can be computed by greedy algorithms. It is a very interesting open problem whether the conservative extensions of more elaborate tournament solutions such as the minimal covering set or the bipartisan set can be computed efficiently.

If computing winners is NP-complete for a tournament solution, the same is true for its conservative extension.

Lemma 8. *If winner determination for S is NP-complete, then winner determination for $[S]$ is NP-complete.*

Since the winner determination problem is NP-complete for the Banks set BA [23], we have an immediate corollary.

Corollary 6. *Computing $[BA]$ is NP-complete.*

6 Conclusion

We have shown that the conservative extension inherits many desirable properties from its underlying tournament solution. In general, the conservative extension $[S]$ of tournament solution S is rather large and there might be finer extensions of S that still satisfy its characterizing properties. However, the conservative extension may serve as "proof of concept" to show that generalizing a tournament solution in a meaningful way is possible. Whether there are finer solutions that are equally attractive is a different issue that needs to be settled for each tournament solution at hand.

Besides its axiomatic properties, the conservative extension is also interesting from a computational point of view. Particularly intriguing is the question whether the conservative extension of the minimal covering set or the bipartisan set can be computed efficiently.

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References

[1] M. Aizerman and F. Aleskerov. *Theory of Choice*, volume 38 of *Studies in Mathematical and Managerial Economics*. North-Holland, 1995.

[2] N. Alon. Ranking tournaments. *SIAM Journal on Discrete Mathematics*, 20(1):137–142, 2006.

[3] H. Aziz, M. Brill, F. Fischer, P. Harrenstein, J. Lang, and H. G. Seedig. Possible and necessary winners of partial tournaments. In V. Conitzer and M. Winikoff, editors, *Proceedings of the 11th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*. IFAAMAS, 2012.

[4] D. Baumeister, F. Brandt, F. Fischer, J. Hoffmann, and J. Rothe. The complexity of computing minimal unidirectional covering sets. *Theory of Computing Systems*, 53(3):467–502, 2013.

[5] G. Bordes. Some more results on consistency, rationality and collective choice. In J. J. Laffont, editor, *Aggregation and Revelation of Preferences*, chapter 10, pages 175–197. 1979.

[6] F. Brandt. Group-strategyproof irresolute social choice functions. In T. Walsh, editor, *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*, pages 79–84. AAAI Press, 2011.

[7] F. Brandt and F. Fischer. Computing the minimal covering set. *Mathematical Social Sciences*, 56(2):254–268, 2008.

[8] F. Brandt and P. Harrenstein. Set-rationalizable choice and self-stability. *Journal of Economic Theory*, 146(4): 1721–1731, 2011.

[9] F. Brandt, M. Brill, F. Fischer, and P. Harrenstein. Minimal retentive sets in tournaments. In W. van der Hoek, G. A. Kaminka, Y. Lespérance, and M. Luck, editors, *Proceedings of the 9th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 47–54. IFAAMAS, 2010.

[10] F. Brandt, F. Fischer, P. Harrenstein, and M. Mair. A computational analysis of the tournament equilibrium set. *Social Choice and Welfare*, 34(4):597–609, 2010.

[11] F. Brandt, M. Brill, and H. G. Seedig. On the fixed-parameter tractability of composition-consistent tournament solutions. In T. Walsh, editor, *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*, pages 85–90. AAAI Press, 2011.

[12] M. Brill and F. Fischer. The price of neutrality for the ranked pairs method. In J. Hoffmann and B. Selman, editors, *Proceedings of the 26th AAAI Conference on Artificial Intelligence (AAAI)*, pages 1299–1305. AAAI Press, 2012.

[13] H. Chernoff. Rational selection of decision functions. *Econometrica*, 22:422–443, 1954.

[14] V. Conitzer, M. Rognlie, and L. Xia. Preference functions that score rankings and maximum likelihood estimation. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence (IJCAI)*, pages 109–115, 2009.

[15] W. J. Cook, W. H. Cunningham, W. R. Pulleyblank, and A. Schrijver. *Combinatorial Optimization*. Wiley and Sons, 1998.

[16] B. Dutta and J.-F. Laslier. Comparison functions and choice correspondences. *Social Choice and Welfare*, 16(4):513–532, 1999.

[17] P. Faliszewski, E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Llull and Copeland voting computationally resist bribery and constructive control. *Journal of Artificial Intelligence Research*, 35:275–341, 2009.

[18] J. Lang, M. S. Pini, F. Rossi, D. Salvagnin, K. B. Venable, and T. Walsh. Winner determination in voting trees with incomplete preferences and weighted votes. *Journal of Autonomous Agents and Multi-Agent Systems*, 25(1):130–157, 2012.

[19] J.-F. Laslier. *Tournament Solutions and Majority Voting*. Springer-Verlag, 1997.

[20] K. May. A set of independent, necessary and sufficient conditions for simple majority decisions. *Econometrica*, 20:680–684, 1952.

[21] B. Monjardet. Statement of precedence and a comment on IIA terminology. *Games and Economic Behavior*, 62:736–738, 2008.

[22] J. E. Peris and B. Subiza. Condorcet choice correspondences for weak tournaments. *Social Choice and Welfare*, 16(2):217–231, 1999.

[23] G. J. Woeginger. Banks winners in tournaments are difficult to recognize. *Social Choice and Welfare*, 20: 523–528, 2003.