Toward the Complexity of the Existence of Wonderfully Stable Partitions and Strictly Core Stable Coalition Structures in Hedonic Games*

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Abstract

We study the computational complexity of the existence and the verification problem for wonderfully stable partitions (WSPE and WSPV) and of the existence problem for strictly core stable coalition structures (SCSCS) in hedonic games. We show that WSPV is NP-complete and both WSPE and SCSCS are DP-hard, where DP is the second level of the boolean hierarchy, and we discuss an approach for classifying the latter two problems in terms of their complexity.

1 Introduction

Hedonic games are an interesting model combining the central ideas of, on the one hand, cooperative game theory (see, e.g., the textbooks by Peleg and Sudhölter (2003) and Chalkiadakis, Elkind, and Wooldridge (2011)) where players form coalitions in order to manage certain tasks as a team and, on the other hand, voting scenarios (see, e.g., the bookchapters by Brams and Fishburn (2002) and Brandt, Conitzer, and Endriss (2013)) where players give their preferences over several alternatives in order to elect mutually desirable alternatives by aggregating their preferences. In a hedonic game, the alternatives are groups (coalitions) of players and players "vote" on coalitions they want to join by expressing their preferences. In particular, hedonic games have been studied from a computational perspective, for example by Dimitrov et al. (2006), Sung and Dimitrov (2006), Aziz, Brandt, and Seedig (2013), and Woeginger (2013b). In his survey, Woeginger (2013a) gives an overview of several core stability concepts in hedonic games and their analysis.

Here, we in particular focus on the concepts of wonderfully stable partitions and strictly core stable coalition structures that have been considered in this survey. A partition of the vertices of an undirected graph is called *wonderfully stable* if each vertex is assigned to a clique of largest size that contains the vertex. In the context of hedonic games, this notion can be interpreted to express the following scenario. If the players are represented by the vertices in a graph and there is an undirected edge between two vertices if and only if the two related players like each other, then—under so-called enemy-oriented preferences (Dimitrov et al. 2006)—a largest clique containing a vertex corresponds to the coalition that is most preferred by the player this vertex represents. A wonderfully stable partition for this graph thus corresponds to a coalition structure of the game where each player ends up in her most preferred coalition. In the same domain, we also consider another stability concept. Intuitively, a coalition structure is *(strictly) core stable* if no group of players have an incentive to form a different coalition, thus breaking away from the given coalition structure.

It is known that, under enemy-oriented preferences, there always exists a core stable coalition structure in a given game (Dimitrov et al. 2006), and deciding whether a given coalition structure is core stable or strictly core stable is strongly NP-complete (Sung and Dimitrov 2006; Woeginger 2013a). Let WSPE be the problem of deciding whether there exists a wonderfully stable partition in a given graph, and let SCSCS be the problem of deciding whether there exists a strictly core stable coalition structure in a given enemy-oriented hedonic game. The exact complexity of these problems is unknown so far. Woeginger (2013a) points out that these interesting open problems might be difficult to solve. The best known upper bounds are Θ_2^p for WSPE and Σ_2^p for SCSCS (where Θ_2^p and Σ_2^p are levels of the polynomial hierarchy), and Woeginger (2013a) conjectures that they are complete for these classes. We provide DP-hardness lower bounds for both problems. This is a first step toward classifying these two problems in terms of their complexity. Moreover, we show that proving coDP-hardness for them would already suffice to establish their Θ_2^p-hardness.

2 Preliminaries

Hedonic Games

A *hedonic game* consists of a finite set $N = \{1,\dots,n\}$ of *players* and a profile $\succeq = (\succeq_1,\dots,\succeq_n)$ of *preference relations*, where $\succeq_i$ denotes player i's preference relation. Each such preference relation $\succeq_i$ defines a weak preference order over all *coalitions* (i.e., subsets of N) that contain player i. Let A and B be coalitions containing i. We say that *i weakly prefers A to B* if $A \succeq_i B$, and we say *i prefers A to B* (denoted by $A \succ_i B$) if $A \succeq_i B$, but not $B \succeq_i A$.

Since the number of coalitions in a player's preference order is exponential in the number of players, it is reasonable to consider compactly represented hedonic games; see the survey of Woeginger (2013a) for an overview of vari-

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ous possible encodings. Here, we consider so-called enemy-oriented preferences as introduced by Dimitrov et al. (2006). In their setting, every player $i \in N$ has a set of friends $F_i \subseteq N$ including i itself and a set of enemies $E_i = N \setminus F_i$. We assume that these sets are symmetric, that is, a player $j \in N$ is a player i's friend if and only if i is j's friend.[1]

Definition 2.1. *1. A player $i \in N$* prefers *a coalition A to a coalition B (where $i \in A \cap B$) if $|A \cap E_i| < |B \cap E_i|$, or if $|A \cap E_i| = |B \cap E_i|$ and $|A \cap F_i| > |B \cap F_i|$.*

2. A player $i \in N$ weakly prefers *a coalition A to a coalition B (where $i \in A \cap B$) if $|A \cap E_i| < |B \cap E_i|$, or if $|A \cap E_i| = |B \cap E_i|$ and $|A \cap F_i| \geq |B \cap F_i|$.*

An enemy-oriented hedonic game can be modeled by an undirected graph $G = (V,E)$, where the set $N = \{1,\dots,n\}$ of players corresponds to the set $V = \{v_1,\dots,v_n\}$ of vertices, and there is an edge $e = \{v_i, v_j\}$, $i \neq j$, in E if and only if i and j are friends. We use the following graph-theoretic notions. A *clique* in an undirected graph $G = (V,E)$ is a set $C \subseteq V$ such that each two distinct vertices are connected by an edge. For each $v \in V$, let $\omega_G(v)$ denote the *clique number of v in G*, which is the size of a largest clique in G that contains v. A player always prefers a coalition represented by a clique to one represented by an incomplete subgraph of the same size, and a larger clique results in a better coalition.

In a game, a *coalition structure* $\Gamma = \{C_1,\dots,C_k\}$ is a partition of the players into $k \geq 1$ disjoint coalitions $C_1,\dots,C_k \subseteq N$ such that $\bigcup_{i=1}^{k} C_i = N$. For a coalition structure Γ, we denote the coalition C_j that contains player i by $\Gamma(i)$. In the associated graph $G = (V,E)$, a coalition structure corresponds to a *partition* $\Pi = \{P_1,\dots,P_k\}$ of the vertices of G into disjoint sets of vertices $P_1,\dots,P_k \subseteq V$ such that $\bigcup_{i=1}^{k} P_i = V$, and we denote the set P_j that contains a vertex v by $\Pi(v)$.

Stablility Concepts

We now define some stability concepts for hedonic games.

Definition 2.2. *1. A coalition $C \subseteq N$* blocks *a coalition structure Γ if each player $i \in C$ prefers C to $\Gamma(i)$.*

2. A coalition structure Γ is core stable *if there is no nonempty coalition $C \subseteq N$ that blocks Γ.*

3. A coalition $C \subseteq N$ weakly blocks *a coalition structure Γ if for each player $i \in C$, we have $C \succeq_i \Gamma(i)$, and for at least one player $j \in C$, we have $C \succ_j \Gamma(j)$.*

4. A coalition structure Γ is strictly core stable *if there is no nonempty coalition $C \subseteq N$ that weakly blocks Γ.*

Note that in a hedonic game with enemy-oriented preferences, a core stable coalition structure corresponds always to a partition into cliques in the associated graph. Recall from Section 1 that the concept of wonderfully stable partition in hedonic games has a purely graph-theoretic interpretation:

Definition 2.3. *Given a graph $G = (V,E)$, a partition $\Pi = \{P_1,\dots,P_k\}$ of the vertex set of G is called* wonderfully stable *if P_i is a clique for each i, $1 \leq i \leq k$, and $|\Pi(v)| = \omega_G(v)$ for each vertex $v \in V$.*

[1] As Woeginger (2013a) points out, in the context of stability only symmetric friendship relations matter in the enemy-oriented scenario, which justifies our assumption of symmetry and why we consider undirected graphs.

Adopting the notation from core stability in hedonic games, we say that a clique $P \subseteq V$ *blocks* a partition Π into cliques if there exists a vertex $v \in P$ with $\omega_G(v) > |\Pi(v)|$. (Note that $\omega_G(v) \geq |\Pi(v)|$, for each vertex $v \in V$, since $\Pi(v)$ is a clique that contains v.) Furthermore, note that the problem of whether there exists a partition into a limited number of cliques in a graph, is NP-hard (see, e.g., (Garey and Johnson 1979)). If, however, the number of cliques is not limited, a partition into cliques can easily be found.

The following lemma provides a relation between strictly core stable coalition structures and wonderfully stable partitions. The easy proof is omitted due to space constraints.

Lemma 2.4. *Let $G = (V,E)$ be the graph representing an enemy-oriented hedonic game $\mathcal{G}$. Let Π be a partition of V and let Γ be the corresponding coalition structure in $\mathcal{G}$.*

1. *If Π is a wonderfully stable partition for G, then Γ is a strictly core stable coalition structure for $\mathcal{G}$.*
2. *If there is an integer $c \in \mathbb{N}$ such that $\omega_G(v) = c$ for all vertices $v \in V$ and Γ is a strictly core stable coalition structure for $\mathcal{G}$, then Π is a wonderfully stable partition for G.*

Note the following useful property that holds for graphs consisting of several independent components.

Property 2.5. *Let G be the graph representing an enemy-oriented hedonic game $\mathcal{G}$ and let G consist of k independent components G_i, $1 \leq i \leq k$, corresponding to games $\mathcal{G}_i$. There exists a wonderfully stable partition Π for G (respectively, a strictly core stable coalition structure Γ for $\mathcal{G}$) if and only if there exist wonderfully stable partitions Π_i for all components G_i of G (respectively, strictly core stable coalition structures Γ_i for all games $\mathcal{G}_i$), $1 \leq i \leq k$.*

For problems in general, a similar property has been defined by Chang and Kadin (1995): A problem A *has* AND$_\omega$ *functions* if there exists a polynomial-time computable f such that for all $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n$, it holds that $x_i \in A$ for all $i \in \{1,2,\dots,n\}$ if and only if $f(x_1,x_2,\dots,x_n) \in A$.

We will analyze the following decision problems.

STRICTLY CORE STABLE COALITION STRUCTURE (SCSCS)	
Given:	A hedonic game $\mathcal{G} = (N, \succeq)$ with enemy-oriented preferences.
Question:	Does there exist a strictly core stable coalition structure in $\mathcal{G}$?

WONDERFULLY STABLE PARTITION EXISTENCE (WSPE)	
Given:	A graph $G = (V,E)$.
Question:	Does there exist a wonderfully stable partition of V for G?

WONDERFULLY STABLE PARTITION VERIFICATION (WSPV)	
Given:	A graph $G = (V,E)$ and a partition Π of V into cliques.
Question:	Does there exist a clique $P \subseteq V$ that blocks Π?

Just as the core stability problems considered by Woeginger (2013a), the latter two problems are related to each other, since there exists a wonderfully stable partition in a graph if and only if there exists a partition into cliques such that no clique blocks this partition. In other words, the verification problem can be characterized by an existential quantifier as follows: $(G,\Pi) \in \text{WSPV} \iff (\exists P)[P \text{ blocks } \Pi]$, and the existence problem can be characterized by an existential quantifier followed by a universal quantifier:

$$G \in \text{WSPE} \iff (\exists \Pi)(\forall P)[\neg(P \text{ blocks } \Pi)].$$

Complexity Theory

We assume the reader is familiar with the basic notions of complexity theory, such as the complexity classes P, NP, and coNP and the notions of hardness and completeness (based on the polynomial-time many-one reducibility, $\leq_m^p$). DP was introduced by Papadimitriou and Yannakakis (1984) as the class of differences of any two NP problems; DP is also known as the second level of the boolean hierarchy over NP (Cai et al. 1988; 1989). $P_{\parallel}^{NP}$ was introduced by Papadimitriou and Zachos (1983) as the class of problems that can be solved in polynomial time by asking $\mathcal{O}(\log n)$ sequential Turing queries to an NP oracle. This class is also known as capturing "parallel access to NP" and constitutes the Θ_2^p level of the polynomial hierarchy, which has been studied by Wagner (Wagner 1987; 1990) and others (see, e.g., (Köbler, Schöning, and Wagner 1987; Hemachandra 1989; Hemaspaandra, Hemaspaandra, and Rothe 1997; Rothe, Spakowski, and Vogel 2003; Hemaspaandra, Spakowski, and Vogel 2005; Baumeister et al. 2013)). $\Sigma_2^p = \text{NP}^{\text{NP}}$ is the second level of the polynomial hierarchy (Stockmeyer 1976). Recent Σ_2^p-completeness results on the complexity of core stability in hedonic games are due to Woeginger (2013b), see also (Woeginger 2013a). It holds that $\text{P} \subseteq \text{NP} \subseteq \text{DP} \subseteq \Theta_2^p \subseteq \Sigma_2^p$, and none of these inclusions is known to be strict.

The following two lemmas are due to Wagner (1987) and provide sufficient conditions for proving lower bounds for DP and Θ_2^p, respectively.

Lemma 2.6 (Wagner (1987)). *Let A be some* NP*-hard problem, and let B be any set. If there exists a polynomial-time computable function f such that, for any two instances x_1 and x_2 of A for which $x_2 \in A$ implies that $x_1 \in A$, we have*

$$|\{i \mid x_i \in A\}| \text{ is odd} \iff f(x_1,x_2) \in B, \tag{1}$$

then B is DP*-hard.*

Lemma 2.7 (Wagner (1987)). *Let A be some* NP*-hard problem, and let B be any set. If there exists a polynomial-time computable function f such that, for all $k \geq 1$ and any $2k$ instances $x_1,\dots,x_{2k}$ of A for which $x_j \in A$ implies that $x_i \in A$ for $i < j$, we have*

$$|\{i \mid x_i \in A\}| \text{ is odd} \iff f(x_1,x_2,\dots,x_{2k}) \in B, \tag{2}$$

then B is Θ_2^p-hard.

3 Hardness of WSPV, WSPE, and SCSCS

General Hardness Results

As it holds for the core stability problems, obviously, the verification problem for wonderfully stable partitions, WSPV, belongs to NP, since it can be tested in polynomial time whether a given subset of vertices is a clique and, if so, whether it blocks a given partition. Consequently, the existence problem, WSPE, belongs to Σ_2^p. As a (potentially) better upper bound, Woeginger (2013a) shows membership of WSPE in Θ_2^p and conjectures that WSPE is Θ_2^p-hard.

Let us first consider WSPV. To pinpoint its complexity, we make use of the same proof technique that Sung and Dimitrov (2006) used for the core stability problem in hedonic games with enemy-oriented preferences.

Theorem 3.1. WSPV *is* NP*-complete.*

Proof. NP membership is obvious, as stated above. NP-hardness is shown via a reduction from CLIQUE as in (Sung and Dimitrov 2006). Given an instance of CLIQUE (which, for an undirected graph $G = (V,E)$ and a positive integer k, asks whether G has a clique of size at least k), we construct the following graph $G' = (V',E')$. The vertex set V' is obtained from V by adding, for each $v \in V$, $k-2$ vertices. We connect each of the $k-2$ new vertices and v to form a clique of size $k-1$, for each $v \in V$. The edge set E' consists of these edges and all edges in E. Let Π be the partition into $|V|$ cliques such that each $(k-1)$-clique as constructed above forms one part. This can obviously be achieved in polynomial time. We claim that there is a clique of size k in G if and only if there exists a clique $P \subseteq V'$ that blocks Π in G'.

Only if: If there is a clique P of size k in G, the same clique can be found in G'. The vertices $v \in P$ thus have a clique number $\omega_{G'}(v)$ of at least k. Since the size of all cliques in Π is $k-1$, there exists a vertex v in the clique P with $\omega_{G'}(v) > |\Pi(v)|$; therefore, P blocks Π in G'.

If: If there is no clique of size k in G, there is no clique of size k in G' either and each vertex $v \in V'$ has a clique number of $k-1$. Furthermore, $|\Pi(v)| = k-1$ for each $v \in V'$. Thus, there is no blocking clique for Π in G'. ❑

We now turn to the problem WSPE, seeking to raise its lower bound step by step. We start by showing coNP-hardness.

Theorem 3.2. WSPE *is* coNP*-hard.*

Proof. Again, we reduce from CLIQUE, but this time to the complement of WSPE. Given an instance (G,k) of CLIQUE, we construct the same graph $G' = (V',E')$ as in the proof of Theorem 3.1 as an instance for the complement of WSPE. We may assume that $k \geq 3$; otherwise, we could test in polynomial time whether E is empty or not and reduce to an appropriate trivial instance. We now show that there is a clique of size k in G if and only if there is no wonderfully stable partition for G'.

Only if: If there is a clique P of size k in G, the same clique can be found in G'. As in the proof of Theorem 3.1, P blocks the partition that consists of the $|V|$ cliques of size $k-1$ that are constructed in the reduction. On the other hand, if a partition contains P, then each of the $(k-1)$-cliques mentioned above blocks this partition, since the new vertices are now in a clique of size at most $k-2$, but their clique number is $k-1$.

If: If there is no clique of size k in G, the partition as in the proof of Theorem 3.1 is wonderfully stable, since there is no blocking clique. ❑

Next, we show that WSPE is also NP-hard, which was mentioned without proof already by Woeginger (2013a). Thus, it is unlikely that the problem is in either NP or coNP (otherwise, the polynomial hierarchy would collapse).

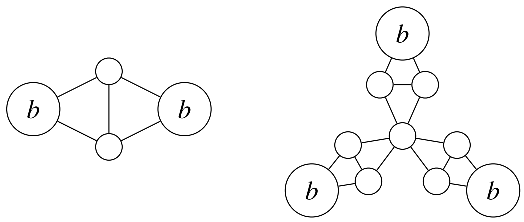

Figure 1: Construction between vertices labeled $b \in B$

Theorem 3.3 (Woeginger (2013a)). WSPE *is* NP*-hard.*

Proof. We show NP-hardness via a reduction from the well-known NP-hard problem EXACT COVER BY THREE SETS (see, e.g., (Garey and Johnson 1979)), which we abbreviate by X3C. The input of this problem is a base set $B = \{b_1, \ldots, b_{3k}\}$, $k > 0$, and a collection $\mathscr{S} = \{S_1, \ldots, S_m\}$ of 3-element subsets of B, and the question is whether B can be exactly covered by k sets from $\mathscr{S}$. Given an X3C instance $(B, \mathscr{S})$, we may assume that each element of B occurs at most three times in any of the sets in $\mathscr{S}$. Furthermore, we can assume that each element occurs at least once; otherwise, we could reduce to a trivial no-instance of WSPE.

We construct the following graph $G = (V, E)$ from $(B, \mathscr{S})$. For each $S_i \in \mathscr{S}$, add three vertices to V that are connected to each other as a 3-clique. Label the three vertices with the three elements of S_i. For each element $b \in B$, consider the following three cases. First, if b occurs only once in a set of $\mathscr{S}$, no changes are made. Second, if b occurs twice, the subgraph in Figure 1a is inserted between the two vertices labeled with b. Third, if b occurs three times, the subgraph in Figure 1b is inserted between the three vertices labeled with b. Since it is easy to determine how often an element of B occurs in a set of $\mathscr{S}$ and the number of new vertices is limited by $7|B|$, G can be constructed in polynomial time. We now show that there is an exact cover of B by sets in $\mathscr{S}$ if and only if there is a wonderfully stable partition for G.

Only if: If there exists an exact cover of B by $k = |B|/3$ sets in $\mathscr{S}$, include the 3-cliques corresponding to these sets into the partition Π that shall be wonderfully stable. The remaining vertices (those from the inserted connecting subgraphs, and those corresponding to the S_i that are not part of the exact cover) are distributed as follows. Again, consider the three cases of occurrence: If an element b occurs only once, the only vertex labeled with b is already in a clique in Π. If an element b occurs twice, one vertex labeled b remains. This vertex forms a 3-clique with the two connecting vertices as in Figure 1a. Put this 3-clique into Π. If an element b occurs three times, two vertices with the same label remain. From the structure of the connecting subgraph as in Figure 1b, the two vertices connected to the vertex that is already in a part of the partition, form a 3-clique with the vertex in the middle. The other two pairs of vertices again form 3-cliques with the remaining vertices labeled b. If these three cliques are added to Π, the partition is complete. It remains to show that Π is wonderfully stable. Since each part of Π is a clique of size 3 and each vertex in G has clique number 3, the conditions for a wonderfully stable partition are satisfied.

If: If there exists a wonderfully stable partition Π in G, all cliques in Π have size 3, since by construction each vertex $v \in V$ has a clique number $\omega_G(v) = 3$. Since the connecting subgraphs from Figures 1a and 1b are constructed such that exactly one labeled vertex is not part of a 3-clique, we have that, for each element $b \in B$, the one corresponding vertex has to be part of another 3-clique that does not contain an unlabeled vertex. Thus, there exist exactly $|B|/3$ cliques that consist of three labeled vertices, corresponding to sets in $\mathscr{S}$ in which each element of B occurs exactly once. That is, there exists an exact cover of B in $\mathscr{S}$. ❑

In order to prove DP-hardness of WSPE, we make use of Wagner's sufficient condition stated in Lemma 2.6 and Property 2.5.

Theorem 3.4. WSPE *is* DP*-hard.*

Proof. Again, consider the NP-hard problem X3C. Given two instances of X3C, $(B_1, \mathscr{S}_1)$ and $(B_2, \mathscr{S}_2)$, where $(B_2, \mathscr{S}_2) \in$ X3C implies $(B_1, \mathscr{S}_1) \in$ X3C, we construct the following graph $G = (V, E)$. G consists of two disconnected subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, that is, $G = (V_1 \cup V_2, E_1 \cup E_2)$. G_1 is obtained from $(B_1, \mathscr{S}_1)$ by the construction given in the proof of Theorem 3.3. G_2 is built in two steps. First, the X3C instance $(B_2, \mathscr{S}_2)$ is transformed into an instance of CLIQUE: For each set $S_i \in \mathscr{S}$, create a vertex v_i. If two sets S_i and S_j are disjoint, connect the corresponding vertices by an edge $\{v_i, v_j\}$. Let $k = |B|/3$. In the second step, add $k - 1$ vertices and edges as in the proof of Theorem 3.2. This construction can obviously be done in polynomial time. Note that, again, the proof only works for $k \geq 3$. If $k \leq 2$, reduce to an approriate trivial WSPE instance.

We claim that $(B_1, \mathscr{S}_1) \in$ X3C and $(B_2, \mathscr{S}_2) \notin$ X3C if and only if there exists a wonderfully stable partition for G. Note that, since $(B_2, \mathscr{S}_2) \in$ X3C implies $(B_1, \mathscr{S}_1) \in$ X3C, this is enough to establish equivalence (1) in Lemma 2.6.

Only if: Suppose $(B_1, \mathscr{S}_1) \in$ X3C and $(B_2, \mathscr{S}_2) \notin$ X3C. Since $(B_1, \mathscr{S}_1)$ is in X3C, G_1 has a wonderfully stable partition by the proof of Theorem 3.3. Since additionally $(B_2, \mathscr{S}_2) \notin$ X3C, there are no $k = |B|/3$ pairwise disjoint sets in $\mathscr{S}$, thus there is no clique of size k in G. By the proof of Theorem 3.2, G_2 then also has a wonderfully stable partition. Since G_1 and G_2 are not connected, that is, the clique number of each vertex remains unchanged ($\omega_G(v) = \omega_{G_1}(v)$ if $v \in V_1$, and $\omega_G(v) = \omega_{G_2}(v)$ if $v \in V_2$), and since there are no additional vertices in G, G has a wonderfully stable partition as well.

If: We prove the contrapositive, i.e., if $(B_1, \mathscr{S}_1) \notin$ X3C or $(B_2, \mathscr{S}_2) \in$ X3C, then there is no wonderfully stable partition for G. Indeed, if $(B_1, \mathscr{S}_1) \notin$ X3C, then by the proof of Theorem 3.3, there is no wonderfully stable partition for G_1.

On the other hand, if $(B_2, \mathscr{S}_2) \in$ X3C, there exists an exact cover of B in $\mathscr{S}$, that is, there are $k = |B|/3$ pairwise disjoint sets in $\mathscr{S}$. By construction, these sets are represented by k vertices in G_2, each connected to one another, thus forming a k-clique. By the proof of Theorem 3.2, it follows that there is no wonderfully stable partition for G_2. By construction, since there is there is no wonderfully stable partition for G_1 or G_2, there is no wonderfully stable partition for G either.

By Lemma 2.6, WSPE is DP-hard. ❑

We now turn to SCSCS, first showing its coNP-hardness by a reduction from CLIQUE to the complement of SCSCS.

Theorem 3.5. SCSCS *is* coNP-*hard.*

Proof. Let (G,k) be a CLIQUE instance with a graph $G = (V,E)$ and an integer $k \geq 4$. Construct an SCSCS instance represented by the graph $G' = (V',E')$. Let $V' = V \cup V_1 \cup V_2$, where V_1 contains $k-2$ new vertices for each of the vertices $v \in V$ and V_2 contains $k-3$ new vertices for each $v \in V$, so $|V'| = |V| + |V|(2k-5)$. Every vertex $v \in V$ is connected to its $k-2$ associated vertices from V_1, any two of which are also connected by an edge, thus forming a $(k-1)$-clique with "their" vertex v. Moreover, the $k-3$ vertices from V_2 associated with any $v \in V$ are connected to one of the vertices from V_1 in the $(k-1)$-clique containing v, and they are also connected among each other, thus forming a $(k-2)$-clique with the single vertex v' from V_1 they are connected to. E' contains all edges from E and the additional edges described above. See Figure 2 for an illustration.

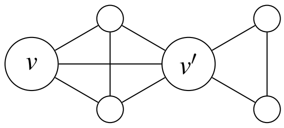

Figure 2: Construction of $G' = (V',E')$ from $G = (V,E)$: Connecting vertices from V_1 and V_2 to $v \in V$ for $k = 5$

We claim that G has a clique of size at least k if and only if there is no strictly core stable coalition structure in the game $\mathscr{G}'$ represented by G'.

Only if: Assuming that there is a clique P of size k in G, this clique also exists in G'. Every possible strictly core stable coalition structure Γ has to contain a coalition C corresponding to P, since otherwise C would block Γ. Consider an arbitrary vertex $v \in P$ and the vertices from $V_1 \cup V_2$ connected to v. The player corresponding to the single vertex v' from V_1 that is connected to v and vertices in V_2 can form a coalition of size $k-2$ with the players corresponding to v''s neighbors either in V_1 or in V_2. In both cases, the one coalition with the player corresponding to v' that is not contained in Γ weakly blocks Γ: While the player corresponding to v' is indifferent, the other players strictly prefer to be in a coalition with her. Thus, there can be no strictly core stable coalition structure for the game represented by G'.

If: Assuming that there is no clique of size k in G, there is no such clique in G' either. Construct a strictly core stable coalition structure Γ for $\mathscr{G}'$ by letting each player corresponding to $v \in V$ form a coalition with the players corresponding to v's neighbors in V_1, and letting the players corresponding to the vertices from V_2 form a coalition with the players corresponding to their $k-4$ neighbors from V_2. ❑

Recalling Lemma 2.4, we know that in graphs where all vertices have the same fixed clique number, every wonderfully stable partition Π of G corresponds also to a strictly core stable coalition structure in the game represented by G, and vice versa. Hence, NP-hardness for SCSCS follows straightforwardly from the NP-hardness proof for WSPE, see the proof of Theorem 3.3.

Theorem 3.6. SCSCS *is* NP-*hard.*

Proof. Use the reduction from the proof of Theorem 3.3 to construct a graph from a given X3C instance $(B, \mathscr{S})$. In this graph all vertices have the same clique number, so with Lemma 2.4 we have that $(B, \mathscr{S}) \in$ X3C if and only if the game represented by G has a strictly core stable coalition structure. ❑

By using Wagner's sufficient condition from Lemma 2.6, DP-hardness of SCSCS can be shown. We state this result without proof and refer to the proof of Theorem 3.4. The construction can be directly transferred to SCSCS by using the reduction showing CLIQUE $\leq_m^p \overline{\text{SCSCS}}$ (see the proof of Theorem 3.5) to construct G_2 from a given X3C instance.

Theorem 3.7. SCSCS *is* DP-*hard.*

A Result for a Special Graph Class

Consider the special class of graphs $G = (V,E)$ where all vertices have the same fixed clique number k (i.e., $\omega_G(v) = k$, for all $v \in V$). We can show NP membership of WSPE restricted to instances of this graph class (denoted by k-WSPE). Together with a lower bound that follows from the construction for proving Theorem 3.3, this gives NP-completeness.

Theorem 3.8. *For $k \geq 3$, k-*WSPE *is* NP-*complete.*

Proof. By assumption, all vertices in the given graph G have the same clique number k. The graph has to have $\ell \cdot k$ vertices for $\ell \in \mathbb{N}$; otherwise, a wonderfully stable partition could never be found. Thus, the problem of deciding whether G has a wonderfully stable partition is equivalent to the problem of deciding whether there is a clique cover of size ℓ for G, which is an NP-complete problem. Thus, NP membership of k-WSPE is shown by nondeterministically guessing a partition of the vertices into ℓ sets and, for each partition guessed, testing whether these sets are cliques.

For the lower bound, it follows from Theorem 3.3 that WSPE on graphs with a fixed clique number of $k = 3$ is NP-hard. We can extend this NP-hardness to any fixed clique number $k \geq 3$ by reducing k-WSPE to $(k+1)$-WSPE. We may assume that an instance for k-WSPE has $\ell \cdot k$ vertices (otherwise, we reduce to a trivial no-instance). Given such a graph, we construct an instance of $(k+1)$-WSPE by adding ℓ vertices to the original graph. We connect each new vertex to each original vertex and leave the new vertices unconnected among each other. It is easy to see that there is a wonderfully stable partition into ℓ k-cliques in the original graph if and only if there is a wonderfully stable partition into ℓ cliques of size $(k+1)$ each in the constructed graph. ❑

Since by Lemma 2.4 the problems WSPE and SCSCS are equivalent for graphs in this class, the stated result also holds for k-SCSCS.

4 Toward Θ_2^p-Hardness of WSPE and SCSCS

In this section, we discuss a way for how to approach the as yet open issues of showing that WSPE and SCSCS are Θ_2^p-hard. To apply Lemma 2.7, the idea would be to generalize the construction for showing DP-hardness of WSPE and SCSCS (see Theorems 3.4 and 3.7), which we will elaborate on exemplarily for WSPE.

From $2k$ given instances $x_1,\dots,x_{2k}$ of an NP-hard problem A such as X3C, we construct a WSPE instance as a graph G with $k+1$ independent components G_i, $1 \le i \le k+1$. Then again, we can use Property 2.5 to conclude that G has a wonderfully stable partition if and only if each G_i has one. The single components G_i are constructed in the following way: The first one, G_1, is constructed from the first A instance x_1, the last one, G_{k+1}, is constructed from the last A instance x_{2k}, and the remaining $k-1$ components $G_i, 2 \le i \le k$, are constructed from pairs (x_{2i-2}, x_{2i-1}) of A instances (see Figure 3 for an illustration). For the thus constructed subgraphs, we need the following properties to hold.

G_1	G_2	G_3	$\cdots$	G_{k-1}	G_k	G_{k+1}
$\uparrow$	$\uparrow$	$\uparrow$		$\uparrow$	$\uparrow$	$\uparrow$
x_1	(x_2,x_3)	(x_4,x_5)	$\cdots$	(x_{2k-4},x_{2k-3})	(x_{2k-2},x_{2k-1})	x_{2k}
$+$	$(+,+)$	$(+,+)$	$\cdots$	$(+,+)$	$(+,+)$	$+$
$+$	$(+,+)$	$(-,-)$	$\cdots$	$(-,-)$	$(-,-)$	$-$
$+$	$(+,+)$	$(+,+)$	$\cdots$	$(+,-)$	$(-,-)$	$-$
$-$	$(-,-)$	$(-,-)$	$\cdots$	$(-,-)$	$(-,-)$	$-$

Figure 3: Illustration of the reduction using Lemma 2.7. The last rows show possible cases of yes/no-instances due to the relation between the x_i, "+" denotes a yes-instance, and "−" denotes a no-instance.

Property 4.1. *Let $x_1,\dots,x_{2k}$ be given instances of an* NP-*hard problem A. Construct graphs $G_1,\dots,G_{k+1}$ as follows:*

1. *Construct G_1 from x_1 such that*
$$x_1 \in A \iff G_1 \in \text{WSPE}.$$
2. *Construct G_{k+1} from x_{2k} such that*
$$x_{2k} \in A \iff G_{k+1} \notin \text{WSPE}.$$
3. *Construct G_i, $2 \le i \le k$, from x_{2i-2} and x_{2i-1} such that*
$$(x_{2i-2}, x_{2i-1} \in A) \text{ or } (x_{2i-2}, x_{2i-1} \notin A) \iff G_i \in \text{WSPE}.$$

If we can find graphs $G_1,\dots,G_{k+1}$ satisfying Property 4.1 in polynomial time, we can prove Θ_2^p-hardness of WSPE using Lemma 2.7.

Theorem 4.2. *Let A be an* NP-*hard problem and let $x_1,\dots,x_{2k}$ be any $2k$ instances of A such that $x_j \in A$ implies $x_i \in A$ for $i < j$. If $G_1,\dots,G_{k+1}$ are graphs that can be constructed from $x_1,\dots,x_{2k}$ in polynomial time such that Property 4.1 is satisfied, then* WSPE *is Θ_2^p-hard.*

Proof. Let f be a polynomial-time computable function such that $f(x_1,\dots,x_{2k}) = G$, where G is the graph consisting of $k+1$ independent components $G_1,\dots,G_{k+1}$ that satisfy Property 4.1. To apply Lemma 2.7, we have to show (2):

$$|\{x_i \mid x_i \in A, 1 \le i \le 2k\}| \text{ is odd} \iff G \in \text{WSPE}.$$

Only if: Assume that $|\{x_i \mid x_i \in A, 1 \le i \le 2k\}|$ is odd. Since $x_j \in A$ implies that $x_i \in A$ for $i < j$, neither $x_1 \notin A$ nor $x_{2k} \in A$ can hold (see the top and the bottom row of Figure 3). By Property 4.1, we have that both G_1 and G_{k+1} have a wonderfully stable partition.

Since $x_1 \in A$ and $x_{2k} \notin A$, there exists an index s (which we call the *separation index*) such that $x_i \in A$ for $i \le s$, and $x_i \notin A$ for $i > s$. Again, since $x_j \in A$ implies that $x_i \in A$ for $i < j$, only the following three cases can occur for each pair (x_{2i-2}, x_{2i-1}) of the remaining instances:

Case 1: Both x_{2i-2} and x_{2i-1} are in A.

Case 2: Neither x_{2i-2} nor x_{2i-1} are in A.

Case 3: x_{2i-2} is in A, yet x_{2i-1} is not.

Case 3 implies that the separation index is of the form $s = 2i-2$ for some i (see the third row of Figure 3), which leads to a contradiction, since that would mean that there is an even number of yes-instances. So all pairs have to be of the form stated in Case 1 or Case 2 (see the second row of Figure 3). By Property 4.1, each component G_i, $2 \le i \le k$ has a wonderfully stable partition and so has G.

If: Assume that G has a wonderfully stable partition. This implies that every component G_i, $1 \le i \le k$, has a wonderfully stable partition. By Property 4.1, we have that x_1 is a yes-instance, x_{2k} is a no-instance, and that for all pairs (x_{2i-2}, x_{2i-1}), $2 \le i \le k-1$, either both x_{2i-2} and x_{2i-1} are in A, or neither x_{2i-2} nor x_{2i-1} are in A. In total, we have an odd number of yes-instances among $x_1,\dots,x_{2k}$.

By Lemma 2.7, WSPE is Θ_2^p-hard. ❑

With the reduction presented in the DP-hardness proof for WSPE (see Theorem 3.4), the subgraphs G_1 and G_{k+1} can be constructed from given X3C instances such that the desired first two properties of Property 4.1 hold. To complete the reduction, we have to construct the remaining subgraphs $G_2,\dots,G_k$ so as to satisfy the third property of Property 4.1.

Looking closely at this property and letting the NP-complete set A from Lemma 2.7 be 3-SAT, we are searching for a polynomial-time reduction f such that for two given 3-SAT instances, φ_1 and φ_2, we have:

$$\begin{aligned} & (\varphi_1, \varphi_2 \in \text{3-SAT}) \text{ or } (\varphi_1, \varphi_2 \notin \text{3-SAT}) \\ \iff & f(\varphi_1, \varphi_2) \in \text{WSPE}. \end{aligned} \tag{3}$$

Papadimitriou and Yannakakis (1984) introduced the well-known DP-complete problem SAT-UNSAT: Given two boolean formulas in 3-CNF, φ_1 and φ_2, is it true that φ_1 is satisfiable (i.e., $\varphi_1 \in$ 3-SAT) and φ_2 is not satisfiable (i.e., $\varphi_2 \notin$ 3-SAT)? We may assume that $\varphi_2 \in$ 3-SAT implies $\varphi_1 \in$ 3-SAT. By Lemma 2.6, this restriction of SAT-UNSAT is also DP-complete. Then (3) simplifies to:

$$(\varphi_1, \varphi_2) \notin \text{SAT-UNSAT} \iff f(\varphi_1, \varphi_2) \in \text{WSPE}$$

It follows that in order to prove Θ_2^p-hardness—and thus Θ_2^p-completeness—of WSPE, it suffices to show coDP-hardness of WSPE, and essentially the same argument works for SCSCS as well. We leave this as an open problem.

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