Variants of Quantified Linear Programming and Quantified Linear Implication

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Abstract

A Quantified Linear Program (QLP) consists of a set of linear inequalities and a corresponding quantifier string, in which each universally quantified variable is bounded. By extending the quantification of variables to implications of two linear systems, we explore Quantified Linear Implications (QLIs). QLPs and QLIs offer a rich language that is ideal for expressing specifications in real-time scheduling and for modelling reactive systems. In this paper, we show that the variants of QLP and QLI that arise when the universally quantified variables are partially bounded or unbounded can be decided in polynomial time. Moreover, we show that for each class of the polynomial hierarchy (**PH**), there exists a form of QLI that is complete for that class, thus providing a continuous analogue of the way Quantified Boolean Formulas cover the **PH**. We also prove that the generic QLI problem is **PSPACE-complete** and solve some open problems on QLIs with one quantifier alternation.

1 Introduction

Quantified linear programming is the problem of checking whether a linear system is satisfiable with respect to a given quantifier string that specifies which variables are existentially quantified and which are universally quantified. Hence, quantified linear programming is a (non-trivial) generalization of linear programming. Accordingly, a *Quantified Linear Program* (QLP) consists of a set of linear inequalities and a corresponding quantifier string, in which a bounded region of values is also specified for each universally quantified variable (Subramani, 2007). A study on QLPs with unbounded universally quantified variables appears in (Ruggieri et al., 2013). The problem that arises by extending the quantification of variables to implications of two linear systems is called *Quantified Linear Implication* (QLI) (Eirinakis et al., 2012a; Eirinakis et al., 2013).

QLPs represent a rich language that is ideal for expressing schedulability specifications in real-time scheduling (Gerber et al., 1995; Choi and Agrawala, 2000). In real-time scheduling, however, it may be the case that the dispatcher has already obtained a schedule (solution) but then some constraints are slightly altered. QLIs can be then utilized to decide whether the dispatcher needs to recompute a solution or can still use the current one. Moreover, QLPs and QLIs can be used to model reactive systems (Koo et al., 1999; Pfitzmann and Waidner, 2000; Kam et al., 2001; Hall, 2002), where the universally quantified variables represent the environmental input, while the existentially quantified variables represent the system's response.

In this paper, we examine some new variants of QLPs and QLIs, and establish their computational complexities. We also settle several open questions in the corresponding literature on QLIs. More specifically, we prove that the variants of QLP and QLI that arise when the universally quantified variables are partially bounded or unbounded can be decided in polynomial time. Moreover, with respect to each class of the polynomial hierarchy (**PH**), we show that there exists some instantiation to the QLI framework that is complete for this class. This is interesting due to the fact that QLIs cover the **PH** using only continuous (and not discrete) variables, thus providing a continuous analogue to the results of Stockmeyer (1977), where the **PH** is generated using Quantified Boolean Formulas (QBFs). We also show that the generic QLI problem is **PSPACE-complete**. Finally, we examine the computational complexities of some classes of QLIs with one quantifier alternation not identified by Eirinakis et al. (2013).

The rest of this paper is organized as follows: Section 2 formally describes the problems that we consider in this paper. Sections 3 and 4 examine polynomially-solvable variants of QLPs. Section 5 answers several open questions concerning QLIs, while Section 6 concludes the paper.

2 Statement of Problems

A QLP is a conjunctive system of linear constraints in which each variable is either existentially or universally quantified according to a given quantifier string:

$$\exists \mathbf{x}_1 \forall \mathbf{y}_1 \in [\mathbf{l}_1, \mathbf{u}_1] \ldots \exists \mathbf{x}_n \forall \mathbf{y}_n \in [\mathbf{l}_n, \mathbf{u}_n]\ \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$$

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where $\mathbf{x}_1 \ldots \mathbf{x}_n$ is a partition of $\mathbf{x}$ with, possibly, $\mathbf{x}_1$ empty; $\mathbf{y}_1 \ldots \mathbf{y}_n$ is a partition of $\mathbf{y}$ with, possibly, $\mathbf{y}_n$ empty; and $\mathbf{l}_i$, $\mathbf{u}_i$ are lower and upper bounds in $\Re$ for $\mathbf{y}_i,\ i = 1, \ldots, n$. Note that in a QLP each universally quantified variable is bounded from above and below.

Let us introduce two variants of QLP both of which change the nature of the bounds on the universal variables. A *Partially bounded Quantified Linear Program* (PQLP) is a QLP in which each universally quantified variable is only bounded on one side. Without loss of generality, we can assume this single bound forces each such variable to be nonnegative:

$$\exists \mathbf{x}_1 \forall \mathbf{y}_1 \in [\mathbf{0}, \infty) \ldots \exists \mathbf{x}_n \forall \mathbf{y}_n \in [\mathbf{0}, \infty)\ \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$$

An *Unbounded Quantified Linear Program* (UQLP), also studied by Ruggieri et al. (2013), is a QLP where there are no bounds on any universal variable:

$$\exists \mathbf{x}_1 \forall \mathbf{y}_1 \ldots \exists \mathbf{x}_n \forall \mathbf{y}_n\ \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$$

QLIs extend the notion of inclusion of two linear systems to arbitrary quantifiers:

$$\exists \mathbf{x}_1 \forall \mathbf{y}_1 \ldots \exists \mathbf{x}_n \forall \mathbf{y}_n \quad [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d}]$$

We say that a QLI *holds* if it is true as a first-order formula over the domain of the reals. The *decision problem for a QLI* consists of checking whether it holds or not.

A nomenclature is introduced in (Eirinakis et al., 2012a) to represent the classes of QLIs. Consider a triple $\langle A, Q, R \rangle$. Let A denote the number of quantifier alternations in the quantifier string and Q the first quantifier. Also, let R be an $(A+1)$-character string, which specifies whether each quantified set of variables in the quantifier string appears on the **L**eft, on the **R**ight, or on **B**oth sides of the implication. For instance, $\langle 1, \forall, \mathbf{BL} \rangle$ indicates a problem described by:

$$\forall \mathbf{y}\ \exists \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \rightarrow \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d}]$$

We extend the notion of partially bounded and unbounded universally quantified variables to QLIs. Note that due to the nature of implications, only the constraints in the Left Hand Side (LHS) restrict the values that universally quantified variables can take. Hence, a *Partially bounded Quantified Linear Implication* (PQLI) is a QLI in which each universally quantified variable is only bounded by a single absolute constraint. Similarly to PQLPs, we can assume (without loss of generality) that this single constraint is a nonnegativity constraint. Thus, a PQLI has the following form:

$$\exists \mathbf{x}_1 \forall \mathbf{y}_1 \ldots \exists \mathbf{x}_n \forall \mathbf{y}_n \quad [(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \wedge \mathbf{y} \geq \mathbf{0}) \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d}]$$

Accordingly, an *Unbounded Quantified Linear Implication* (UQLI) is a QLI in which each universally quantified variable does not appear in the LHS of the implication at all. Thus, a UQLI has the following form:

$$\exists \mathbf{x}_1 \forall \mathbf{y}_1 \ldots \exists \mathbf{x}_n \forall \mathbf{y}_n\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d}]$$

It is important to note that 2-person game semantics have been proposed both for QLPs (Subramani, 2007) and QLIs (Eirinakis et al., 2013). More specifically, for QLPs such a game includes an existential player $\mathbf{X}$, who chooses values for the existentially quantified variables, and a universal player $\mathbf{Y}$, who chooses values for the universally quantified variables. $\mathbf{X}$ and $\mathbf{Y}$ make their choices according to the order of the variables in the quantifier string. If, at the end, the instantiated linear system in the QLP is true, then $\mathbf{X}$ wins the game (and we say that $\mathbf{X}$ has a *winning strategy*). Otherwise, $\mathbf{Y}$ wins the game (and we say that $\mathbf{Y}$ has a winning strategy). In the case of QLIs, an existential player $\mathbf{X}$ and a universal player $\mathbf{Y}$ also choose their moves according to the order of the variables in the quantifier string. In any game of this form, the goals of the players are the following: $\mathbf{X}$ selects the values of the existentially quantified variables so as to violate the constraints in the LHS or to satisfy the constraints in the Right-Hand Side (RHS) of the implication. On the other hand, $\mathbf{Y}$ selects the values of the universally quantified variables so as to satisfy the constraints of the LHS and at the same time to violate the constraints of the RHS of the implication. $\mathbf{X}$ wins the game (and has a winning strategy) if at the end of the game the instantiated implication in the QLI is true. Otherwise, $\mathbf{Y}$ wins the game (and has a winning strategy). In both cases, the QLP or the QLI holds precisely when $\mathbf{X}$ has a winning strategy. It is apparent that these game semantics can be readily applied to the aforementioned variants.

3 Partially bounded variants

In this section, we utilize the notion of direction of a convex set (Ignizio and Cavalier, 1993) to show that both PQLP and PQLI are in $\mathbf{P}$. First, we obtain an intermediate result for PQLPs.

Theorem 3.1 *PQLP with a fixed number of quantifier alternations is in* $\mathbf{P}$.

Proof: Let $\mathbf{L}_n$ denote the following PQLP problem:

$$\exists \mathbf{x}_n\ \forall \mathbf{y}_n \in [\mathbf{0}, \infty)\ \exists \mathbf{x}_{n-1}\ \forall \mathbf{y}_{n-1} \in [\mathbf{0}, \infty)\ \ldots\ \exists \mathbf{x}_1\ \forall \mathbf{y}_1 \in [\mathbf{0}, \infty)\ \exists \mathbf{x}_0\ \ \sum_{i=0}^{n} (\mathbf{A}_i \cdot \mathbf{x}_i) + \sum_{i=1}^{n} (\mathbf{B}_i \cdot \mathbf{y}_i) \leq \mathbf{c}$$

For $n = 0$, $\mathbf{L}_0$ is a Linear Program (LP), hence in $\mathbf{P}$ (Khachiyan, 1979). Moreover, assume that $\mathbf{L}_k$ is in $\mathbf{P}$ and consider the problem $\mathbf{L}_{k+1}$:

$$\exists \mathbf{x}_{k+1}\ \forall \mathbf{y}_{k+1} \in [\mathbf{0}, \infty)\ \exists \mathbf{x}_k\ \forall \mathbf{y}_k \in [\mathbf{0}, \infty)\ \ldots\ \exists \mathbf{x}_1\ \forall \mathbf{y}_1 \in [\mathbf{0}, \infty)\ \exists \mathbf{x}_0\ \ \sum_{i=0}^{k+1} (\mathbf{A}_i \cdot \mathbf{x}_i) + \sum_{i=1}^{k+1} (\mathbf{B}_i \cdot \mathbf{y}_i) \leq \mathbf{c} \quad (1)$$

We examine whether it is possible to choose in $\mathbf{P}$ an $\mathbf{x}_{k+1}$ that makes (1) feasible. Such an $\mathbf{x}_{k+1}$ needs to handle the case where $\mathbf{y}_{k+1} = \mathbf{0}$. Thus, the following instance of $\mathbf{L}_k$ needs to be feasible:

$$\exists \mathbf{x}_{k+1}\ \exists \mathbf{y}_{k+1}\ \exists \mathbf{x}_k\ \forall \mathbf{y}_k \in [\mathbf{0}, \infty)\ \ldots\ \exists \mathbf{x}_1\ \forall \mathbf{y}_1 \in [\mathbf{0}, \infty)\ \exists \mathbf{x}_0 \sum_{i=0}^{k+1} (\mathbf{A}_i \cdot \mathbf{x}_i) + \sum_{i=1}^{k+1} (\mathbf{B}_i \cdot \mathbf{y}_i) \leq \mathbf{c},\ \mathbf{y}_{k+1} = \mathbf{0}$$

Moreover, $\mathbf{y}_{k+1}$ needs to be unbounded. Let m denote the dimension of $\mathbf{y}_{k+1}$ and each $\mathbf{e}_j, j = 1 \dots m$, denote a direction of the system after $\mathbf{x}_{k+1}$ is chosen. Note that the directions of that system do not depend on the value of $\mathbf{x}_{k+1}$. This is because the choice of $\mathbf{x}_{k+1}$ only changes the value of the constant term and hence does not affect whether $\mathbf{y}_{k+1} = \mathbf{e}_j$ is a direction or not. So we only need to check whether the following m instances of $\mathbf{L}_k$ are feasible:

$$\exists \mathbf{y}_{k+1}\ \exists \mathbf{x}_k\ \forall \mathbf{y}_k \in [\mathbf{0}, \infty)\ \dots\ \exists \mathbf{x}_1\ \forall \mathbf{y}_1 \in [\mathbf{0}, \infty)\ \exists \mathbf{x}_0$$
$$\sum_{i=0}^{k} (\mathbf{A}_i \cdot \mathbf{x}_i) + \sum_{i=1}^{k+1} (\mathbf{B}_i \cdot \mathbf{y}_i) \leq \mathbf{0},\ \mathbf{y}_{k+1} = \mathbf{e}_j : j = 1 \dots m$$

Thus, the feasibility of (1) can be decided based on the feasibility of $m+1$ instances of $\mathbf{L}_k$, which have been assumed to be in $\mathbf{P}$; so, $\mathbf{L}_{k+1}$ is also in $\mathbf{P}$. □

The inductive proof of Theorem 3.1 implies a recursive procedure for deciding any PQLP with a finite number of quantifier alternations in $\mathbf{P}$. However, if the number of alternations is unbounded, an exponential number of sub-problems is generated. However, many of these sub-problems are superfluous and hence need not be calculated. Next, we show that even if the number of alternations is unbounded, only polynomially-many LPs are generated.

Theorem 3.2 *PQLP is in* $\mathbf{P}$.

Proof: Let $\mathbf{L}$ be the following PQLP problem:

$$\exists x_n\ \forall y_n \in [0, \infty)\ \exists x_{n-1}\ \forall y_{n-1} \in [0, \infty)\ \dots\ \exists x_1$$
$$\forall y_1 \in [0, \infty)\ \exists x_0\ \ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c} \quad (2)$$

Using the recursive method implied by the proof of Theorem 3.1, $\mathbf{L}$ would take exponential time to solve since 2^n sub-problems are generated. We show that many of these sub-problems are the same and so they can be precomputed.

Each time we check whether $y_i = 1$ is a direction of the system, we always compute:

$$\exists y_i\ \exists x_{i-1}\ \forall y_{i-1} \in [0, \infty)\ \dots\ \exists x_1\ \forall y_1 \in [0, \infty)\ \exists x_0$$
$$\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{0} : y_i = 1 \quad (3)$$

This is because prior choices only change the value of the constant term and hence do not affect whether $y_i = 1$ is a direction or not. Checking whether $y_i = 1$ is a direction will involve subsequently checking whether $y_{i-1} = 1$ through $y_1 = 1$ are also directions; this can be handled by performing these computations first. Thus, we first check whether $y_1 = 1$ is a direction by solving:

$$\exists y_1\ \exists x_0\ \ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{0} : y_1 = 1$$

Then, when computing whether $y_2 = 1$ is a direction, we need to solve:

$$\exists y_2\ \exists x_1\ \forall y_1 \in [0, \infty)\ \exists x_0\ \ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{0} : y_2 = 1$$

Since we already know that $y_1 = 1$ is a direction, we only need to solve:

$$\exists y_2\ \exists x_1\ \exists y_1\ \exists x_0\ \ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{0} : y_2 = 1,\ y_1 = 0$$

Now assume that we know that $y_{i-1} = 1$ through $y_1 = 1$ are directions. Recall that in order to check whether $y_i = 1$ is a direction, we need to solve (3). Since $y_{i-1} = 1$ is a direction, we only need to solve:

$$\exists y_i\ \exists x_{i-1}\ \exists y_{i-1}\ \dots\ \exists x_1\ \forall y_1 \in [0, \infty)\ \exists x_0$$
$$\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{0} : y_i = 1,\ y_{i-1} = 0$$

Applying the same for $y_{i-2} = 1$ through $y_1 = 1$, we get:

$$\exists y_i\ \exists x_{i-1}\ \exists y_{i-1}\ \dots\ \exists x_1\ \exists y_1\ \exists x_0$$
$$\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{0} : y_i = 1,\ y_{i-1}, \dots, y_1 = 0$$

Thus, to show that each $y_i = 1$ is a direction, we only need to solve n LPs, each of the form above.

Let us now show how $\mathbf{L}$ can be solved in $\mathbf{P}$. Since $y_n = 1$ is a direction, we only need to check:

$$\exists x_n\ \exists y_n\ \exists x_{n-1}\ \forall y_{n-1} \in [0, \infty)\ \dots\ \exists x_1\ \forall y_1 \in [0, \infty)$$
$$\exists x_0\ \ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c} : y_n = 0$$

Since $y_{n-1} = 1$ through $y_1 = 1$ are also directions, the problem can be simplified down to:

$$\exists x_n\ \exists y_n\ \exists x_{n-1}\ \exists y_{n-1}\ \dots\ \exists x_1\ \exists y_1\ \exists x_0$$
$$\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c} : \mathbf{y} = \mathbf{0} \quad (4)$$

Thus, to solve $\mathbf{L}$, we only need to solve $n+1$ LPs. Furthermore, we can use these LPs to generate the function which determines the value of each x_i. Let $\hat{\mathbf{x}}^n$ contain the values of $\mathbf{x}$ that solve (4), which is equivalent to $\hat{\mathbf{x}}^n$ satisfying:

$$\exists x_n\ \exists x_{n-1}\ \dots\ \exists x_1\ \exists x_0\ \ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{c} \quad (5)$$

Moreover, for each $i = 1, \dots, n$, let $\hat{\mathbf{x}}^{i-1}$ contain the values of $\mathbf{x}$ that solve:

$$\exists y_i\ \exists x_{i-1}\ \exists y_{i-1}\ \dots\ \exists x_1\ \exists y_1\ \exists x_0$$
$$\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{0} : y_i = 1,\ y_{i-1} \dots y_1 = 0$$

which is equivalent to $\hat{\mathbf{x}}^{i-1}$ satisfying:

$$\exists x_{i-1}\ \dots\ \exists x_1\ \exists x_0\ \ \mathbf{A} \cdot \mathbf{x} + \mathbf{B}_i \leq \mathbf{0} \quad (6)$$

Now we expand the $\hat{\mathbf{x}}^{i-1}$s with zeroes so that for each $\hat{\mathbf{x}}^{i-1}, i = 1, \dots, n$, we have $|\hat{\mathbf{x}}^{i-1}| = |\mathbf{x}|$. Then, the following function can be used to determine the values of $\mathbf{x}$:

$$\mathbf{x} = \hat{\mathbf{x}}^n + \sum_{i=1}^{n} y_i \cdot \hat{\mathbf{x}}^{i-1} \quad (7)$$

Note that the value that each x_i takes depends only on the y_js that precede it in the quantifier string. But then:

$$\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} = \mathbf{A} \cdot \left(\hat{\mathbf{x}}^n + \sum_{i=1}^{n} y_i \cdot \hat{\mathbf{x}}^{i-1} \right) + \sum_{i=1}^{n} \mathbf{B}_i \cdot y_i$$
$$= \mathbf{A} \cdot \hat{\mathbf{x}}^n + \sum_{i=1}^{n} y_i \cdot (\mathbf{A} \cdot \hat{\mathbf{x}}^{i-1} + \mathbf{B}_i) \leq \mathbf{c} + \sum_{i=1}^{n} y_i \cdot \mathbf{0} = \mathbf{c}$$

since $\mathbf{A} \cdot \hat{\mathbf{x}}^n \leq \mathbf{c}$ by (5) and $\mathbf{A} \cdot \hat{\mathbf{x}}^{i-1} + \mathbf{B}_i \leq 0$ for any $i = 1, \dots, n$ by (6). But this means that the original PQLP (2) can also be satisfied by the values given to $\mathbf{x}$ by (7).

Note that if instead of variables y_i in (2), we have a vectors $\mathbf{y}_i$, then when checking for directions, instead of solving one LP, we need to solve $|\mathbf{y}_i|$ LPs. Thus, in that situation the total number of LPs needed to be solved is $|\mathbf{y}|+1$, which can also be done in $\mathbf{P}$. □

We will utilize this result to show that PQLI is also in $\mathbf{P}$.

Corollary 3.1 *PQLI is in* **P**.

Proof: Consider the following PQLI problem:

$$\exists \mathbf{x_1}\ \forall \mathbf{y_1}\ \ldots\ \exists \mathbf{x_n}\ \forall \mathbf{y_n}$$
$$[\mathbf{A}\cdot\mathbf{x}\leq\mathbf{b},\ \mathbf{y}\geq\mathbf{0}\rightarrow\mathbf{C}\cdot\mathbf{x}+\mathbf{D}\cdot\mathbf{y}\leq\mathbf{f}] \quad (8)$$

If $\forall\mathbf{x}\ \ \mathbf{A}\cdot\mathbf{x}\leq\mathbf{b}$ is infeasible (i.e., if $\mathbf{A}\cdot\mathbf{x}\leq\mathbf{b}$ contains even one constraint), then (8) is trivially satisfied. Otherwise, (8) is equivalent to:

$$\exists \mathbf{x_1}\ \forall \mathbf{y_1}\ \ldots\ \exists \mathbf{x_n}\ \forall \mathbf{y_n}\ \ [\mathbf{y}\geq\mathbf{0}\rightarrow\mathbf{C}\cdot\mathbf{x}+\mathbf{D}\cdot\mathbf{y}\leq\mathbf{f}]$$

which can be rewritten as

$$\exists \mathbf{x_1}\ \forall \mathbf{y_1}\in[\mathbf{0},\infty)\ldots\ \exists \mathbf{x_n}\ \forall \mathbf{y_n}\in[\mathbf{0},\infty)$$
$$\mathbf{C}\cdot\mathbf{x}+\mathbf{D}\cdot\mathbf{y}\leq\mathbf{f}$$

The latter is a PQLP and hence in **P** by Theorem 3.2. □

4 Unbounded variants

In this section, we show that UQLP and UQLI are also in **P**.

Theorem 4.1 *UQLP is in* **P**.

Proof: Consider the following UQLP problem:

$$\exists \mathbf{x_n}\ \forall \mathbf{y_n}\ \exists \mathbf{x_{n-1}}\ \forall \mathbf{y_{n-1}}\ \ldots\ \exists \mathbf{x_1}\ \forall \mathbf{y_1}\ \exists \mathbf{x_0}$$
$$\mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y}\leq\mathbf{c} \quad (9)$$

To show that this can be solved in polynomial time, we will reduce it to a PQLP. We start by creating additional vectors $\mathbf{y_i'}$ and $\mathbf{y_i''}$ for each $\mathbf{y_i}$. Then, for each $\mathbf{y_i}$, we add constraints $\mathbf{y_i}=\mathbf{y_i'}-\mathbf{y_i''}$. By applying universal quantification to $\mathbf{y_i'}$ and $\mathbf{y_i''}$ and existential quantification to $\mathbf{y_i}$, we construct the following PQLP:

$$\exists \mathbf{x_n}\ \forall \mathbf{y_n'}\in[\mathbf{0},\infty)\ \forall \mathbf{y_n''}\in[\mathbf{0},\infty)\ \exists \mathbf{y_n}\ \exists \mathbf{x_{n-1}}$$
$$\forall \mathbf{y_{n-1}'}\in[\mathbf{0},\infty)\ \forall \mathbf{y_{n-1}''}\in[\mathbf{0},\infty)\ \exists \mathbf{y_{n-1}}\ \ldots$$
$$\exists \mathbf{x_1}\ \forall \mathbf{y_1'}\in[\mathbf{0},\infty)\ \forall \mathbf{y_1''}\in[\mathbf{0},\infty)\ \exists \mathbf{y_1}\ \exists \mathbf{x_0}$$
$$\mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y}\leq\mathbf{c},\ \ \mathbf{y}=\mathbf{y}'-\mathbf{y}'' \quad (10)$$

where $\mathbf{y_1}',\ldots,\mathbf{y_n}'$ is a partition of $\mathbf{y}'$ and $\mathbf{y_1}'',\ldots,\mathbf{y_n}''$ is a partition of $\mathbf{y}''$. Now consider the corresponding 2-person game for (10). The universal player has full control over the values assumed by $\mathbf{y_i}$, (which can take any value in the range $(-\infty,\infty)$), since $\mathbf{y_i}=\mathbf{y_i'}-\mathbf{y_i''}$ and both $\mathbf{y_i'}$ and $\mathbf{y_i''}$ are in the range $[\mathbf{0},\infty)$. Thus, the quantifier sequence $\forall\mathbf{y_i'}\in[\mathbf{0},\infty)\ \forall\mathbf{y_i''}\in[\mathbf{0},\infty)\ \exists\mathbf{y_i}$ is equivalent to $\forall\mathbf{y_i}$. Therefore, (10) is equivalent to (9), which means that UQLP is in **P**. □

Observe that the above theorem has been proven independently and using different techniques in (Ruggieri et al., 2013). We will use this result to show that UQLI is also solvable in polynomial time.

Corollary 4.1 *UQLI is in* **P**.

Proof: Consider the following UQLI problem:

$$\exists \mathbf{x_1}\ \forall \mathbf{y_1}\ \ldots\ \exists \mathbf{x_n}\ \forall \mathbf{y_n}$$
$$[\mathbf{A}\cdot\mathbf{x}\leq\mathbf{b}\rightarrow\mathbf{C}\cdot\mathbf{x}+\mathbf{D}\cdot\mathbf{y}\leq\mathbf{f}] \quad (11)$$

Similarly to the proof of Corollary 3.1, if $\forall\mathbf{x}\ \ \mathbf{A}\cdot\mathbf{x}\leq\mathbf{b}$ is infeasible, then (11) is trivially satisfied. Otherwise, (11) is equivalent to:

$$\exists \mathbf{x_1}\ \forall \mathbf{y_1}\ \ldots\ \exists \mathbf{x_n}\ \forall \mathbf{y_n}\ \ \mathbf{C}\cdot\mathbf{x}+\mathbf{D}\cdot\mathbf{y}\leq\mathbf{f}$$

which is a UQLP and hence in **P** by Theorem 4.1. □

5 QLI and the polynomial hierarchy

In this section, we prove that for each class of the **PH**, there exists a form of QLI that is complete for that class. This is interesting, since QLIs use continuous variables and not discrete. Hence, we provide a continuous analogue to the results in Stockmeyer (1977), where the **PH** is generated using QBFs. Moreover, we show the generic QLI problem itself is **PSPACE-complete**.

Let $\mathbf{B}^{k+1}$ denote the string $\underbrace{\mathbf{B}\ldots\mathbf{B}}_{k+1}$. The following results were obtained in (Eirinakis et al., 2013, Theorems 14 and 15):

1. $\langle k,\exists,\mathbf{B}^{k+1}\rangle$ with k odd is $\mathbf{\Sigma_k^P}$**-hard**.
2. $\langle k,\forall,\mathbf{B}^{k+1}\rangle$ with k even is $\mathbf{\Pi_k^P}$**-hard**.

To establish the computational complexities of $\langle k,\exists,\mathbf{B}^{k+1}\rangle$ when k is even and $\langle k,\forall,\mathbf{B}^{k+1}\rangle$ when k is odd, we first provide a reduction from Q3DNF (see Appendix A) to QLI.

Theorem 5.1 *Q3DNF can be reduced to QLI.*

Proof: We will reduce the Q3DNF problem to a QLI. Consider a Q3DNF instance $Q(\mathbf{x},\mathbf{y})\ \ \phi(\mathbf{x},\mathbf{y})$, where $Q(\mathbf{x},\mathbf{y})$ represents the quantifier string, $\mathbf{x}$ is the set of existentially quantified variables, $\mathbf{y}$ is the set of universally quantified variables, and ϕ is a disjunction of 3-literal terms. We want to produce a corresponding QLI which will hold if and only if $Q(\mathbf{x},\mathbf{y})\ \ \phi(\mathbf{x},\mathbf{y})$ is satisfiable. Let E represent the set of constraints on the LHS of the implication and F the set of constraints on the RHS of the constructed implication.

For each existentially quantified variable x_i in the instance of Q3DNF, we add an existentially quantified variable x_i and a universally quantified variable r_i. We also add the constraints $r_i\leq x_i$ and $r_i\leq 1-x_i$ to E and the constraints $r_i\leq 0$ to F. Note that these constraints are equivalent to $r_i\leq\min(x_i,1-x_i)\rightarrow r_i\leq 0$. Moreover, we add $0\leq x_i\leq 1$ to F.

For each universally quantified variable y_i in the instance of Q3DNF, we add an existentially quantified variable s_i and a universally quantified variable y_i. We also add the constraints $0\leq y_i\leq 1$ to E and the constraints $0\leq s_i\leq 1$, $2y_i-1\leq s_i$, and $s_i\leq 2y_i$ to F. Note that these are the only constraints that use y_i variables, since the clause constraints will only use x_i and s_i variables.

For each clause ϕ_j in the instance of Q3DNF, we add the existentially quantified variable w_j, and 3 corresponding constraints to F. These constraints ask for w_j to be less than or equal to the existential variables corresponding to the literals of ϕ_j. Note that these constraints contain only existential variables. Even in the case of a universal variable y_i in ϕ_j, the constraint contains the existential variable s_i of the QLI. Negated literals are also treated appropriately. For example, if $\phi_j=(x_i,y_k,\bar{x_l})$, we add $w_j\leq x_i$, $w_j\leq s_k$, and $w_j\leq 1-x_l$ to F, while if $\phi_j=(x_i,\bar{y_k},\bar{x_l})$, we add $w_j\leq x_i$, $w_j\leq 1-s_k$, and $w_j\leq 1-x_l$ to F.

Furthermore, we add $w_1+w_2+\cdots+w_m\geq 1$ to F.

We create the quantifier string of the QLI according to $Q(\mathbf{x},\mathbf{y})$: For $\exists\mathbf{x}_i$ in $Q(\mathbf{x},\mathbf{y})$, we introduce $\exists\mathbf{x}_i\forall\mathbf{r}_i$; for $\forall\mathbf{y}_i$ in $Q(\mathbf{x},\mathbf{y})$, we introduce $\forall\mathbf{y}_i\exists\mathbf{s}_i$; finally, we introduce $\exists\mathbf{w}$.

To complete the proof, we will show that the constructed QLI holds if and only if $Q(\mathbf{x},\mathbf{y})\ \phi(\mathbf{x},\mathbf{y})$ is satisfiable. We start by establishing that all variables that participate in the constraints corresponding to the clauses of the Q3DNF instance are effectively restricted to values 0 or 1. To do so, we utilize the game semantics introduced in Section 2.

First note that the existential player $\mathbf{X}$ will only choose x_i from the set $\{0,1\}$. This is because if $x_i \notin [0,1]$, then at least one of the constraints in F would be violated. But then the implication would not hold (hence the universal player $\mathbf{Y}$ would win the game), since $\mathbf{X}$ cannot cause any constraint in E to be violated. On the other hand, if $x_i \in (0,1)$, then $\mathbf{Y}$ could choose to set $r_i = \min(x_i, 1-x_i) > 0$, which would cause the implication not to hold (causing the existential player to lose the game). To sum up, any choice of $x_i \notin \{0,1\}$ would cause $\mathbf{X}$ to lose the game.

The universal player $\mathbf{Y}$ will also choose y_i from the set $\{0,1\}$. We will show why this assumption is not restrictive. Suppose that $\mathbf{Y}$ can win by choosing $y_i \notin [0,1]$. Then at least one constraint of E is violated (i.e., $0 \leq y_i \leq 1$) and the implication holds (i.e., a contradiction). Suppose now that $\mathbf{Y}$ can win by choosing $y_i \in (0,1)$. If $y_i \in (0, \frac{1}{2}]$, then we have that $s_i \in [0, 2y_i]$. However, if instead $\mathbf{Y}$ had chosen $y_i = 0$, then $s_i \in \{0\} \subseteq [0, 2y_i]$, thus restricting the possible responses of $\mathbf{X}$. Since y_i only appears in the constraints described above, we have that this is a strictly better move for $\mathbf{Y}$. Similarly, choosing $y_i = 1$ is strictly better for $\mathbf{Y}$ than choosing $y_i \in [\frac{1}{2}, 1)$. To sum up, we can safely assume that the universal player would only choose $y_i \in \{0,1\}$. Since y_i is in the set $\{0,1\}$, we have that the existential player is forced to set $s_i = y_i$. Any other choice of s_i would violate at least one constraint of F, causing (again) the existential player to lose the game. Hence, s_i variables are also restricted in the set $\{0,1\}$.

Let us show that for the QLI obtained from a Q3DNF instance of the form $Q(\mathbf{x},\mathbf{y})\ \ \phi(\mathbf{x},\mathbf{y})$, the existential player has a winning strategy for the QLI if and only if $Q(\mathbf{x},\mathbf{y})\ \phi(\mathbf{x},\mathbf{y})$ is satisfiable.

Only-if part. Assume that the existential player $\mathbf{X}$ has a winning strategy U for the constructed QLI. This means that for every sequence of moves V made by the universal player $\mathbf{Y}$, the implication holds. From constraint $w_1 + w_2 + \cdots + w_m \geq 1$, we must have that at least one $w_j > 0$. This w_j corresponds to the clause ϕ_j of the original Q3DNF, and we can assume wlog that it has the form (x_i, y_k, x_l). Since the x_is and s_is are restricted to the set $\{0,1\}$, the constraints $w_j \leq x_i$, $w_j \leq s_k$, and $w_j \leq x_l$ force each variable to be 1. Thus at least one term of the original Q3DNF formula is satisfied, meaning that the entire formula is.

If part. Assume that $Q(\mathbf{x},\mathbf{y})\ \phi(\mathbf{x},\mathbf{y})$ is satisfiable. Then there exist values $\mathbf{x}'$ for $\mathbf{x}$ such that

$$\mathbf{x}' = [c_1, f_1(y_1), f_2(y_1,y_2), \ldots, f_{n-1}(y_1,y_2,\ldots,y_{n-1}]^T$$

and for any values $\mathbf{y}' = [y_1, y_2, \ldots, y_n]^T$ given to $\mathbf{y}$, the Q3SAT expression is satisfied. Note that $f_i()$ are Skolem functions and are used to represent that the values of the elements of $\mathbf{x}'$ depend on the values of the corresponding elements of $\mathbf{y}'$.

Since the Q3DNF expression is satisfied, at least one clause, say ϕ_j, must be satisfied. Consider the constraints constructed from ϕ_j, assuming wlog that it is of the form (x_i, y_k, x_l). Since ϕ_j is satisfied, we must have that x_i, y_k, x_l are all true. This means that $x_i = s_k = x_l = 1$. Thus, $\mathbf{X}$ can set $w_j = 1$ and the other clause variables to 0, satisfying the constraint $w_1 + w_2 + \cdots + w_m \geq 1$. Since the x_i and y_i variables can be restricted to the set $\{0,1\}$ and since $s_i = y_i$, we have that for each i, $r_i \leq x_i$ and $r_i \leq 1 - x_i$ imply that $r_i \leq 0$ (thus satisfying the corresponding constraint of F). For the same reason, for any i the constraints $0 \leq x_i \leq 1$, $0 \leq y_i \leq 1$, $0 \leq s_i \leq 1$, $2y_i - 1 \leq s_i$, and $s_i \leq 2y_i$ are all satisfied. Hence, $E \rightarrow F$ is satisfied and so $\mathbf{X}$ has a winning strategy for the corresponding QLI. □

Since the last quantifier of a Q3DNF formula is always $\forall$, this reduction increases the number of quantifier alternations by one and the produced QLI always has $\exists$ as the final quantifier. This allows us to obtain the following two results.

Corollary 5.1 *$\langle k, \exists, \mathbf{B}^{k+1} \rangle$ with k even is* $\mathbf{\Sigma_k^P}$**-hard**.

Proof: Consider the class of Q3DNF formulas with k quantifiers starting with an existential one, i.e., with a quantifier string of the form $\exists\forall\ldots\forall$. Such a class is $\mathbf{\Sigma_k P}$-complete (the assumption that k is even is essential). The previous proof reduces such a class to a QLI with a quantifier string obtained by adding an existential quantifier at the end, namely to a $\langle k, \forall, \mathbf{B}^{k+1} \rangle$ formula. Hence, the result. □

Corollary 5.2 *$\langle k, \forall, \mathbf{B}^{k+1} \rangle$ with k odd is* $\mathbf{\Pi_k^P}$**-hard**.

Proof: Consider the class of Q3DNF formulas with k quantifiers starting with a universal one, i.e., with the quantifier string of the form $\forall\exists\ldots\forall$. Such a class is $\mathbf{\Pi_k P}$-complete (the assumption that k is odd is essential). The previous proof reduces such a class to a QLI with a quantifier string obtained by adding an existential quantifier at the end, namely to a $\langle k, \forall, \mathbf{B}^{k+1} \rangle$ formula. Hence, the result. □

Corollaries 5.1 and 5.2 when paired with (Eirinakis et al., 2013, Theorems 14 and 15) show the importance of the final quantifier in a QLI. If the final quantifier is $\forall$, then the QLI corresponds to a Q3SAT instance with one fewer quantifier alternation and so covers the corresponding class in the polynomial hierarchy. Similarly, if the final quantifier is $\exists$, then the QLI corresponds to a Q3DNF instance with one fewer quantifier alternation and so covers the corresponding class in the polynomial hierarchy. However these results cover only the hardness of these forms of QLI; to show completeness, we need prove to that each of these problems is also contained within its level of the polynomial hierarchy.

Theorem 5.2 *$\langle k, \exists, \mathbf{B}^{k+1} \rangle$ is in $\mathbf{\Sigma_k^P}$, $\langle k, \forall, \mathbf{B}^{k+1} \rangle$ is in $\mathbf{\Pi_k^P}$.*

Proof: Sontag (1985) showed that deciding an arbitrary boolean combination of linear constraints under a given quantifier string is contained within **PSPACE**. QLIs are a sub-problem of this (since it can be rewritten as a quantified conjunction of the RHS constraints and the negation of the LHS constraints) and thus also in **PSPACE**. Hence, QLIs can be decided even if each variable is restricted to values that are polynomially-sized with respect to the input. Consider problem $\langle k, \exists, \mathbf{B}^{k+1} \rangle$. After k rounds of each player choosing polynomialy-sized values, the remaining

QLI, either $\langle 0, \exists, B\rangle$ or $\langle 0, \forall, B\rangle$, can be solved in **P** (Eirinakis et al., 2013, Theorems 1 and 6). Thus, $\langle k, \exists, B^{k+1}\rangle$ is in $\mathbf{\Sigma_k^P}$. Similarly, $\langle k, \forall, B^{k+1}\rangle$ is in $\mathbf{\Pi_k^P}$. □

The various forms of QLI cover the polynomial hierarchy as shown in Figure 1. Moreover, since QLIs are in general **PSPACE-hard** (Eirinakis et al., 2013, Theorem 5), the proof of Theorem 5.2 has the following implication.

Corollary 5.3 *QLI is* **PSPACE-complete**.

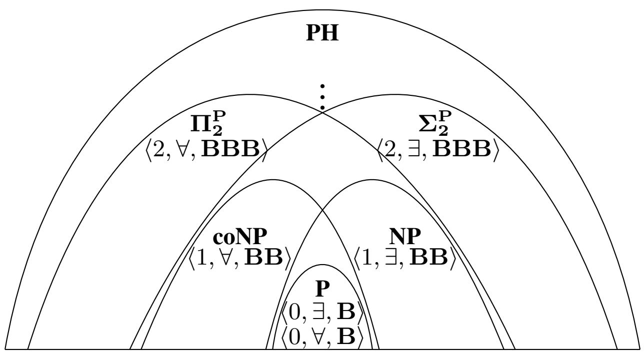

Figure 1: QLIs and the Polynomial Hierarchy

5.1 Related Problems

In this section, we examine the computational complexities of various classes of QLIs with one quantifier alternation that were left open in (Eirinakis et al., 2013).

Corollary 5.4 $\langle 1, \forall, RB\rangle$*, i.e., deciding whether* $\forall \mathbf{y}\ \exists \mathbf{x}\ [\mathbf{M} \cdot \mathbf{x} \le \mathbf{n} \rightarrow \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \le \mathbf{c}]$ *holds, is in* **P**.

Proof: First, we check whether $\forall \mathbf{x}\ \mathbf{M} \cdot \mathbf{x} \le \mathbf{n}$ is infeasible (i.e., whether $\mathbf{M} \cdot \mathbf{x} \le \mathbf{n}$ contains even one constraint). If it is, then the QLI trivially holds. If it is not, then $\forall \mathbf{y}\ \exists \mathbf{x}\ [\mathbf{M} \cdot \mathbf{x} \le \mathbf{n} \rightarrow \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \le \mathbf{c}]$ reduces to $\forall \mathbf{y}\ \exists \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \le \mathbf{c}$, which is a UQLP and hence in **P** by Theorem 4.1. □

The decision problem for formula $\exists \mathbf{x}\ \forall \mathbf{y}\ [\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \le \mathbf{c} \rightarrow \mathbf{M} \cdot \mathbf{y} \le \mathbf{n}]$ is **NP-complete** (Eirinakis et al., 2012a). However, by moving the existentially quantified variables $\mathbf{x}$ to the RHS, the problem becomes tractable.

Corollary 5.5 $\langle 1, \exists, BL\rangle$*, i.e., deciding whether* $\exists \mathbf{x}\ \forall \mathbf{y}\ [\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \le \mathbf{c} \rightarrow \mathbf{M} \cdot \mathbf{x} \le \mathbf{n}]$ *holds, is in* **P**.

Proof: First, we check whether $\exists \mathbf{x}\ \mathbf{M} \cdot \mathbf{x} \le \mathbf{n}$ holds, which can be done in **P** (Khachiyan, 1979). If $\exists \mathbf{x}\ \mathbf{M} \cdot \mathbf{x} \le \mathbf{n}$ holds, then $\exists \mathbf{x}\ \forall \mathbf{y}\ [\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \le \mathbf{c} \rightarrow \mathbf{M} \cdot \mathbf{x} \le \mathbf{n}]$ trivially holds. If $\exists \mathbf{x}\ \mathbf{M} \cdot \mathbf{x} \le \mathbf{n}$ does not hold, $\exists \mathbf{x}\ \forall \mathbf{y}\ [\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \le \mathbf{c} \rightarrow \mathbf{M} \cdot \mathbf{x} \le \mathbf{n}]$ reduces to $\neg \forall \mathbf{x}\ \exists \mathbf{y}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \le \mathbf{c}$, which can be checked to hold in **P** by Theorem 4.1. □

Let us turn our attention to the class $\langle 1, \forall, \mathbf{BR}\rangle$ and its super-class $\langle 1, \forall, \mathbf{BB}\rangle$, shown **coNP-hard** in (Eirinakis et al., 2013). Note that these classes are also in **coNP** by Theorem 5.2. We provide an alternative proof for the latter.

Lemma 5.1 $\langle 1, \forall, \mathbf{BR}\rangle$ *and* $\langle 1, \forall, \mathbf{BB}\rangle$ *are in* **coNP**.

Proof: Problems in $\langle 1, \forall, \mathbf{BR}\rangle$ are described by:

$$\forall \mathbf{y}\ \ \exists \mathbf{x}\ \ [\mathbf{N} \cdot \mathbf{y} \le \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \le \mathbf{d}] \tag{12}$$

We consider two cases: $\mathbf{N} \cdot \mathbf{y} \le \mathbf{b}$ being (a) bounded and (b) unbounded.

(a) Let $\mathbf{N} \cdot \mathbf{y} \le \mathbf{b}$ be bounded. Then, if (12) does not hold, there will exist some $\mathbf{y}'$ for which $\exists \mathbf{x}\ [\mathbf{N} \cdot \mathbf{y}' \le \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y}' \le \mathbf{d}]$ does not hold. Note that this means that $\mathbf{y}'$ satisfies $\mathbf{N} \cdot \mathbf{y} \le \mathbf{b}$ (otherwise the implication would be trivially true). But then, since $\mathbf{N} \cdot \mathbf{y} \le \mathbf{b}$ is bounded, there will also exist some extreme point $\mathbf{y}^*$ of $\mathbf{N} \cdot \mathbf{y} \le \mathbf{b}$, such that $\exists \mathbf{x}\ [\mathbf{N} \cdot \mathbf{y}^* \le \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y}^* \le \mathbf{d}]$ does not hold. Our claim follows from the fact that $\mathbf{y}^*$ is an extreme point of $\mathbf{N} \cdot \mathbf{y} \le \mathbf{b}$ and hence polynomially-sized.

(b) Now let $\mathbf{N} \cdot \mathbf{y} \le \mathbf{b}$ be unbounded. For each solution of $\mathbf{N} \cdot \mathbf{y} \le \mathbf{b}$ to satisfy $\mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \le \mathbf{d}$, we need to have that the extreme points of $\mathbf{N} \cdot \mathbf{y} \le \mathbf{b}$ satisfy $\mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \le \mathbf{d}$ and that the extreme directions of $\mathbf{N} \cdot \mathbf{y} \le \mathbf{b}$ are directions of $\mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \le \mathbf{d}$. For the latter, we need to check whether each solution of $\mathbf{N} \cdot \mathbf{y} \le \mathbf{0}$ also satisfies $\mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \le \mathbf{0}$, i.e., we need to decide $\forall \mathbf{y}\ \exists \mathbf{x}\ \ [\mathbf{N} \cdot \mathbf{y} \le \mathbf{0} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \le \mathbf{0}]$. But since we are searching for extreme directions, we can restrict the values of $\mathbf{y}$ to the interval $[-1, 1]$. Hence, consider a case where (12) does not hold. We would first decide $\forall \mathbf{y}\ \exists \mathbf{x}\ \ [\mathbf{N} \cdot \mathbf{y} \le \mathbf{0}, -\mathbf{1} \le \mathbf{y} \le \mathbf{1} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \le \mathbf{0}]$, which is in **coNP**, since the LHS is bounded (case (a)). If this QLI does not hold, we are done. Otherwise, there will exist a (polynomially-sized) extreme point $\mathbf{y}^*$ of $\mathbf{N} \cdot \mathbf{y} \le \mathbf{b}$ such that $\exists \mathbf{x}\ [\mathbf{N} \cdot \mathbf{y}^* \le \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y}^* \le \mathbf{d}]$ does not hold (case (a)).

Let us now consider problem $\langle 1, \forall, \mathbf{BB}\rangle$, i.e., $\forall \mathbf{y}\ \ \exists \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y}\ \le\ \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \le \mathbf{d}]$. If $\mathbf{A} \ne \mathbf{0}$, this QLI is trivially true; for any choice of values for the universally quantified variables, there will exist a choice of the existentially quantified variables such that the LHS of the QLI is false. Otherwise (if $\mathbf{A} = \mathbf{0}$), problem $\langle 1, \forall, \mathbf{BB}\rangle$ reduces to $\langle 1, \forall, \mathbf{BR}\rangle$, which was shown above to be in **coNP**. □

Corollaries 5.4 and 5.5, and Lemma 5.1 solve some open problems on the computational complexities of QLIs with one quantifier alternation (Eirinakis et al., 2013). A complete representation of all classes with one quantifier alternation QLIs starting with an existential or with a universal quantifier is given in Appendix B, in Figure 2 and Figure 3 respectively. The symmetry is apparent.

6 Conclusion

In this paper, we examined several variants of QLP and QLI that arise when the universally quantified variables are partially bounded or unbounded and showed that all these variants are in **P**. Furthermore, we proved that $\langle k, \exists, \mathbf{B}^{k+1}\rangle$ is $\mathbf{\Sigma_k^P}$**-complete** and $\langle k, \forall, \mathbf{B}^{k+1}\rangle$ is $\mathbf{\Pi_k^P}$**-complete**. Hence, we showed that any class of the **PH** can be represented by some instantiation to the QLI framework (using only continuous variables), thus providing a continuous analogue to the way QBFs cover the **PH**. Moreover, we proved that the generic QLI problem is **PSPACE-complete**. Finally, we answered several open questions on the computational complexities of classes of QLIs with one quantifier alternation, thus completing the map of the computational complexity of all such classes of QLIs.

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Appendix A

A quantified boolean expression is an instance of Q3DNF if it is a disjunction of terms each of which is a conjunction of three literals. Thus, the following problem is an instance of Q3DNF.

$$\exists x_1 \forall y_1 \exists x_2 \forall y_2 \exists x_3 \forall y_3 \quad (x_1 \wedge y_3 \wedge \neg x_2) \vee (x_3 \wedge y_2 \wedge \neg x_1) \vee (x_2 \wedge y_1 \wedge \neg x_3)$$

We can define the two player game semantics for this problem as follows. The existential and universal players make moves according to the quantifier string. The existential player wins a term if all the literals in the term are set to True. Conversely the universal player wins the term if at least on literal is set to False.

The existential player wins the game if he wins at least one term in the disjunction, thus causing the expression to evaluate to True. The universal player wins if he wins every term in the disjunction, thus causing the expression to evaluate to False.

This can be considered as a reversal of the players' objectives in an instance of Q3SAT, where the existential player needs to win *a*ll the clauses, and the universal player only needs to win a single clause. This occurs because the Q3DNF and Q3SAT problems are complements of each other.

Appendix B

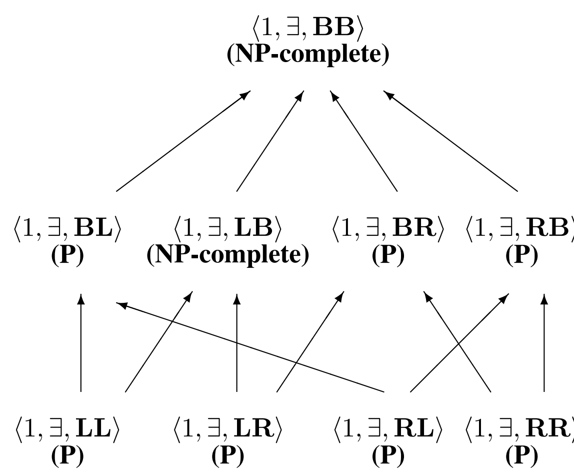

Figure 2: Complexity of ∃∀ classes of QLIs. Arrows denote inclusions.

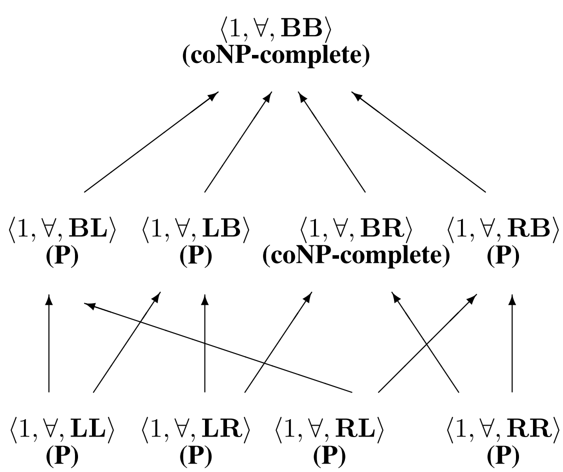

Figure 3: Complexity of $\forall\exists$ classes of QLIs. Arrows denote inclusions.