

# CS 301

## Lecture 01 – Introduction



# What is CS 301 all about?

This is a very mathematical course with a lot of practical applications

The overarching theme is **computation**

Three parts to the course

- ① Automata (singular is automaton) (8 weeks)
- ② Computability (4 weeks)
- ③ Complexity (3 weeks)

# Computation I

One main theme of this course is what can be computed and what can't

Which problems can be solved by computers and which can't

Here are some problems we can solve with computers

- Sort a finite list
- Check if an integer is prime
- Draw some triangles on a screen
- Determine the shortest path between two vertices (nodes) in a graph
- Factor a polynomial
- Never lose at tic-tac-toe
- Never lose at checkers (solved in 2007 but took 18 years!)
- ...

# Computation II

Everything you've learned in CS so far has been about solving problems

When you see a new problem, you might think,

“I know how to solve this problem”; or

“I don't know how to solve it right now, but I'm sure I can figure it out”; or maybe

“Eventually, someone will figure out how to solve it”

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Here are some problems we know we **can't** solve with computers

- Given a computer program, will the program crash when run on some input?
- Given two computer programs, do they compute the same answer when given the same input for all inputs?
- Given a multivariable, polynomial equation, determine if it has a solution in integers
- Find the cheapest airfare between two airports (this is surprisingly complicated)
- Lots of problems in mathematics
- ...

# Decision problems

In this course, we're going to focus on problems whose answers are Yes/No (or True/False)

These are **decision problems**

## Examples

- Is an integer  $n$  even?
- Does a directed graph  $G$  have a path of length  $n$  between vertices  $u$  and  $v$ ?
- Does a program  $P$  crash when run on input  $x$ ?
- Is  $x$  an element of a set  $S$ ?
- Does a string  $s$  end with a repeated letter?

# Models of computation

Real computers are frighteningly complex

They are much too complicated to reason about

Instead, we're going to focus on simpler models of computation

Finite automaton used in text-processing and compilers

Context-free grammar used in programming languages and compilers

Turing machines equivalent in power to general purpose computers

The models are progressively more powerful; they let us solve more problems

# Sets I

A **set** is a collection of objects (numbers, symbols, other sets, anything)

The objects in the set are the **elements** or **members**

E.g.,  $S = \{2, 3, 5, 7, 11\}$  is a 4-element set

We use  $\in$  and  $\notin$  to denote set membership and nonmembership

$5 \in S$  and  $9 \notin S$

Common sets:

The **empty set** is written  $\emptyset$

The set of **natural numbers** is written  $\mathbb{N} = \{1, 2, \dots\}$

The set of **integers** is written  $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

The set of **rational numbers** is written  $\mathbb{Q}$

The set of **real numbers** is written  $\mathbb{R}$



## Sets II

We can define sets by giving rules for sets

$$\text{PRIMES} = \{x \mid x \text{ is a prime number}\}$$

$$\text{ODDS} = \{2x + 1 \mid x \in \mathbb{Z}\}$$

$$T = \{n \mid n = m^2 \text{ for some } m \in \mathbb{N}\}$$

Set  $A$  is a **subset** of  $B$  (written  $S \subseteq B$ ) if every element of  $A$  is an element of  $B$

Set  $A$  is a **proper subset** of  $B$  (written  $S \subsetneq B$ ) if  $A \subseteq B$  and  $A \neq B$

## Set operations

**Union**  $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ ; elements of either  $A$  or  $B$

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**Power set**  $P(A) = \{S \mid S \subseteq A\}$ ; set of subsets of  $A$

## Set examples

$$A = \{0, 1, 2, 3, 4\}$$

$$B = \{2, 4, 5\}$$

$$C = \{3x \mid x \in \mathbb{Z}\}$$

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# Tuples

**Tuples** are finite sequences of objects in some order

$(2, 7, 8)$

$(a, a, b, a, b)$

$(\emptyset, \{0, 1\}, \{0\})$

Order of elements in a tuple matters, unlike in a set

Repeated elements in a tuple matter, unlike in a set

A tuple with  $k$  elements is called a  **$k$ -tuple**

A **pair** is a 2-tuple



# Cartesian product

A **Cartesian product** of two sets is defined by

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

A Cartesian product of  $k$  sets,  $A_1$  through  $A_k$ , is defined by

$$A_1 \times A_2 \times \cdots \times A_k = \{(x_1, x_2, \dots, x_k) \mid x_i \in A_i\}$$

If  $A = \{a, b\}$  and  $B = \{1, 2, 3\}$ , then

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

We can take a repeated Cartesian product of a set with itself  $k$  times

$$A^k = \underbrace{A \times A \times \cdots \times A}_k$$

# Functions

Functions are mappings of elements from one set, the **domain**, to another set, the **range**

The range is also called the codomain

Some texts use range for a related, but distinct concept

We write  $f : X \rightarrow Y$  where

$f$  – name of the function

$X$  – domain

$Y$  – range

To each element  $x \in X$ ,  $f$  assigns exactly one element  $y \in Y$ , written  $y = f(x)$

When the function  $f$  is clear (or unnamed), we can express that mapping as  $x \mapsto y$  (read “ $x$  maps to  $y$ ”)

## Function examples

- $inc : \mathbb{N} \rightarrow \mathbb{N}$  given by  $n \mapsto n + 1$  (equiv.  $inc(n) = n + 1$ )

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$$f(n, x) = \begin{cases} \text{true} & \text{if } |n - x| < 3 \\ \text{false} & \text{otherwise} \end{cases}$$

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- $g : \{q_0, q_1, q_2\} \times \{\mathbf{a}, \mathbf{b}\} \rightarrow \{q_0, q_1, q_2\}$  given by the table

$q$	$x$	$g(q, x)$	Equivalently,
$q_0$	$\mathbf{a}$	$q_1$	
$q_0$	$\mathbf{b}$	$q_0$	$g(q_i, \mathbf{a}) = q_{i \bmod 3}$
$q_1$	$\mathbf{a}$	$q_2$	$g(q_i, \mathbf{b}) = q_i$
$q_1$	$\mathbf{b}$	$q_1$	
$q_2$	$\mathbf{a}$	$q_0$	
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Equivalently,

$$g(q_i, \mathbf{a}) = q_{i \bmod 3}$$

$$g(q_i, \mathbf{b}) = q_i$$

Note that this completely specifies all 6 cases

## Function examples II

- Function of 3-tuples:  $h : \mathbb{N}^3 \rightarrow \mathbb{N}^2$  given by  $h(x, y, z) = (3x + z, yz)$



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- Bad example removed from slides, sorry! The key point was that a function  $f_1 : X \rightarrow Y$  is different from a function  $f_2 : P(X) \rightarrow Y$ . The first takes a single element of  $X$  as an argument whereas the second takes a *set* of elements from  $X$  (a subset of  $X$ ) as an argument.

# Functions in CS 301

We're going to see a lot of functions that look like

$$\delta_1 : Q \times \Sigma \rightarrow Q$$

$$\delta_2 : Q \times \Sigma \rightarrow P(Q)$$

where  $Q$  and  $\Sigma$  are (finite) sets

$\delta_1$  maps a pair  $(q, a)$  to an element of  $Q$

$\delta_2$  maps a pair  $(q, a)$  to a set of elements of  $Q$

## Functions later in CS 301

Later, we'll see a lot of functions that look like

$$\delta_3 : Q \times \Sigma \times \Gamma \rightarrow P(Q \times \Gamma)$$

$$\delta_4 : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

where  $\Gamma$  is some other set

$\delta_3$  maps a triple  $(q, a, b)$  to a set of pairs, e.g.,  $\delta_3(q, a, b) = \{(r, c), (s, d)\}$

$\delta_4$  maps a pair  $(q, a)$  to a triple, e.g.,  $\delta_4(q, a) = (r, b, L)$

# Alphabets, strings, and languages

Alphabets, strings, and languages are the key building blocks for this course

Almost everything in the course boils down to asking the question: Is the string  $s$  an element of the language  $L$ ?

# Alphabets and symbols

An **alphabet** is a nonempty, finite set

The members of an alphabet are the **symbols** of the alphabet

We (usually) denote alphabets with the capital Greek letters  $\Sigma$  and  $\Gamma$  (and various subscripts)

$$\Sigma_1 = \{0, 1\}$$

$$\Sigma_2 = \{a, b, c\}$$

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I will try my best to follow Sipser and write symbols in typewriter font

# Strings

A **string** (also called a **word**) is a finite, possibly empty sequence of symbols from a given alphabet

- 0110110 is a string over the alphabet  $\Sigma_1 = \{0, 1\}$
- aababacab is a string over the alphabet  $\Sigma_2 = \{a, b, c\}$
- 2100#xxy is a string over the alphabet  $\Gamma = \{\#, 0, 1, 2, x, y\}$

The **empty string** is a sequence of zero symbols and is denoted  $\varepsilon$

The **length** of a string  $w$ , written  $|w|$  is the number of symbols it contains

- $|0110110| = 7$
- $|aababacab| = 9$
- $|2100\#xxy| = 8$
- $|\varepsilon| = 0$

# String concatenation

We can **concatenate** two strings to produce a new string

- If  $x = \text{aab}$  and  $y = \text{ba}$ , then  $xy = \text{aabba}$
- Concatenating  $\varepsilon$  does not change the string:  $x\varepsilon = \varepsilon x = x$
- If  $x$  and  $y$  are strings, then  $|xy| = |x| + |y|$
- If  $x$  is a string and  $k$  is a nonnegative integer, then  $x^k = \underbrace{xx \cdots x}_k$  and  $|x^k| = k \cdot |x|$



# Substrings

A string  $s$  is a substring of  $w$  if all of the symbols in  $s$  appear consecutively in  $w$

- 000 is a substring of 001000
- 0100 is a substring of 001000
- 0000 is not a substring of 001000

# String reversal

If  $w = w_1w_2\cdots w_n$  is a string of length  $n$  where  $w_i \in \Sigma$ , then  $w^{\mathcal{R}} = w_nw_{n-1}\cdots w_1$  is the reversal of  $w$

- $abb^{\mathcal{R}} = bba$
- $a^{\mathcal{R}} = a$
- $\varepsilon^{\mathcal{R}} = \varepsilon$

# String prefix

String  $x$  is a **prefix** of string  $y$  if there exists string  $z$  such that  $xz = y$

A string of length  $n$  has  $n + 1$  prefixes

Prefixes of aaba

- ①  $\varepsilon$
- ② a
- ③ aa
- ④ aab
- ⑤ aaba

$\varepsilon$  has exactly one prefix:  $\varepsilon$  itself

String  $x$  is a **proper prefix** of string  $y$  if  $x$  is a prefix of  $y$  and  $x \neq y$

# Languages

A language is a (possibly infinite) set of strings over an alphabet  $\Sigma$

- $L_1 = \emptyset$ . The empty language
- $L_2 = \{\varepsilon\}$ . The language containing only the empty string
- $L_3 = \{a, aa, aba\}$
- $L_4 = \Sigma^*$ . The language of all strings (remember, strings have finite length)
- $L_5 = \Sigma^+$ . The language of all nonempty string ( $L_5 = L_4 \setminus L_2$ )
- $L_6 = \{a^n b^n \mid n \geq 0\} = \{\varepsilon, ab, aabb, aaabbb, \dots\}$

Languages  $L_1$ ,  $L_2$  and  $L_3$  are finite (meaning they have finitely many elements)

Languages  $L_4$ ,  $L_5$ , and  $L_6$  are infinite

# Operations on languages

Languages are sets, so the usual set operations like union and intersection have the normal meanings

The complement of a language  $L$  is the set of all strings over the alphabet that are not in  $L$ . In symbols  $\overline{L} = \Sigma^* \setminus L$

We'll see lots of operations defined on languages throughout the course

- Reversal.  $L^{\mathcal{R}} = \{w^{\mathcal{R}} \mid w \in L\}$
- Composition (or concatenation).  $L_1 \circ L_2 = \{xy \mid x \in L_1 \text{ and } y \in L_2\}$
- Kleene star.  $L^* = \{x_1x_2\cdots x_k \mid k \geq 0 \text{ and each } x_i \in L\}$
- ...

# Recap

Alphabets are finite, nonempty sets of symbols

E.g.,  $\Sigma = \{a, b\}$ ,  $\Gamma = \{0, 1, \dots, 9\}$

Strings are finite sequences of symbols from an alphabet

E.g.,  $\varepsilon$  (the empty string),  $aab$ ,  $b^5 aab = bbbbaab$

Languages are (possibly infinite) sets of strings

E.g.,  $\emptyset$ ,  $\{a\}$ ,  $\Sigma^*$ ,  $\{w \mid |w| \geq 3\}$

## Question 1

Are the sets  $\emptyset$  and  $\{\emptyset\}$  the same?

## Question 1

Are the sets  $\emptyset$  and  $\{\emptyset\}$  the same?

No.  $\emptyset$  is the set containing no elements.  $\{\emptyset\}$  is the set containing a single element, namely the empty set.



## Question 2

If  $S$  is a set, then  $\emptyset \subseteq S$ . True or false?

## Question 2

If  $S$  is a set, then  $\emptyset \subseteq S$ . True or false?

True.  $\emptyset$  is a subset of every set

### Question 3

Let  $S = \{1, 2, 3\}$ . Is  $1 \subseteq S$ ?

### Question 3

Let  $S = \{1, 2, 3\}$ . Is  $1 \subseteq S$ ?

No. 1 is not a set and so it certainly cannot be a subset of  $S$

## Question 4

Let  $S = \{1, 2, 3\}$ . Is  $\{2\} \in S$ ?

## Question 4

Let  $S = \{1, 2, 3\}$ . Is  $\{2\} \in S$ ?

No.  $\{2\}$  is a set but  $S$  doesn't contain any sets

However,  $\{2\} \subseteq S$

## Question 5

Let  $S = \{1, 2, 3\}$ . Is  $S \subseteq S$

## Question 5

Let  $S = \{1, 2, 3\}$ . Is  $S \subseteq S$

Yes. Every set is a subset of itself



## Question 6

Is  $\{0\}$  a valid alphabet?

## Question 6

Is  $\{0\}$  a valid alphabet?

Yes. It's called a unary alphabet because it has one symbol

## Question 7

Is  $\{0, 1\}$  a valid alphabet?

## Question 7

Is  $\{0, 1\}$  a valid alphabet?

Yes. It's called a binary alphabet because it has two symbols

## Question 8

Is  $\emptyset$  a valid alphabet?

## Question 8

Is  $\emptyset$  a valid alphabet?

No. Alphabets must be nonempty

## Question 9

Is  $\{0, 1, 2, \dots\}$  a valid alphabet?

## Question 9

Is  $\{0, 1, 2, \dots\}$  a valid alphabet?

No. Alphabets must have finitely many elements



## Question 10

Is the sequence of symbols  $a\#b$  a valid string over the alphabet  $\Sigma = \{a, b\}$ ?

## Question 10

Is the sequence of symbols  $a\#b$  a valid string over the alphabet  $\Sigma = \{a, b\}$ ?

No. All symbols in the string must come from the alphabet

It is a valid string over the alphabet  $\Sigma' = \{a, b, \#\}$ .

## Question 11

Is the empty-length sequence of symbols  $\varepsilon$  a valid string over the alphabet  $\Sigma = \{a\}$ ?

## Question 11

Is the empty-length sequence of symbols  $\varepsilon$  a valid string over the alphabet  $\Sigma = \{a\}$ ?

Yes. The empty string is a string over any alphabet

## Question 12

Is the infinite-length sequence of a symbols,  $aa \dots$ , a valid string over the alphabet  $\Sigma = \{a, b\}$ ?

## Question 12

Is the infinite-length sequence of a symbols,  $aa \dots$ , a valid string over the alphabet  $\Sigma = \{a, b\}$ ?

No. Strings must be finite length

## Question 13

Can a language have zero elements?

## Question 13

Can a language have zero elements?

Yes.  $\emptyset$  is a perfectly reasonable language



## Question 14

Can a language have infinitely many elements?

## Question 14

Can a language have infinitely many elements?

Yes.  $\Sigma^*$ , the set of all strings over  $\Sigma$  is finite for any alphabet  $\Sigma$

Consider  $\Sigma = \{a\}$ , then the language  $\Sigma^* = \{\varepsilon, a, aa, aaa, \dots\}$

Most of the languages in this course will be infinite

## Question 15

Every language  $L$  of strings over the alphabet  $\Sigma$  is a subset of  $\Sigma^*$ . True or false?

## Question 15

Every language  $L$  of strings over the alphabet  $\Sigma$  is a subset of  $\Sigma^*$ . True or false?

True.  $\Sigma^*$  is the set of all strings over  $\Sigma$  and  $L$  is some set of strings over  $\Sigma$ , so every element of  $L$  is an element of  $\Sigma^*$  and thus  $L \subseteq \Sigma^*$

## Next time

Read chapter 0 and section 1.1 of Sipser

We're going to talk about our first model of computation: deterministic finite automaton (DFA)

Components of a DFA:

- A finite set of **states**  $Q$
- An input **alphabet**  $\Sigma$
- A **transition function**  $\delta : Q \times \Sigma \rightarrow Q$  that controls how the DFA moves from state to state on a given input
- A **start state**  $q_0 \in Q$
- A **set of accept states**  $F \subseteq Q$  that control whether or not the DFA *accepts* an input string

