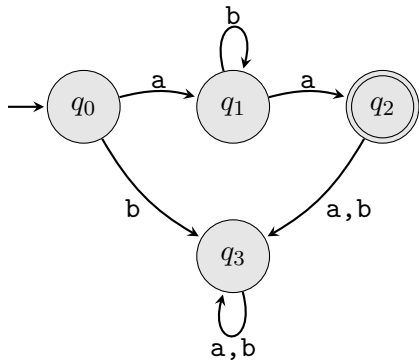


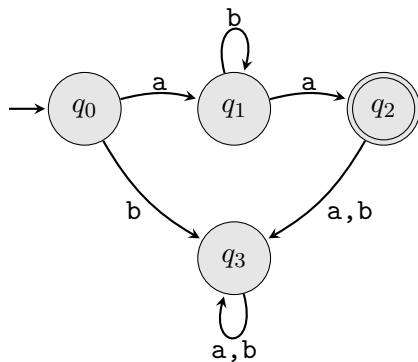
CS 301

Lecture 06 – Nonregular languages and the pumping lemma

DFA M_1



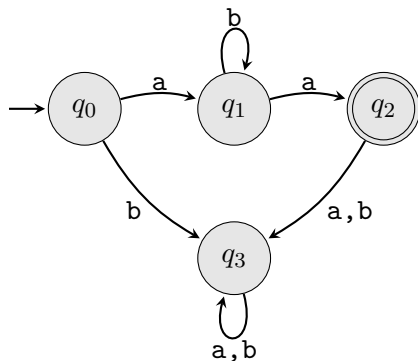
DFA M_1



Strings in the language

- aa
- aba
- abba
- abbba
- $ab^k a$ for all $k \geq 0$

DFA M_1



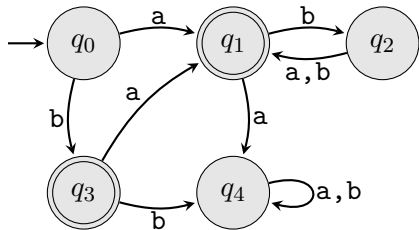
Strings in the language

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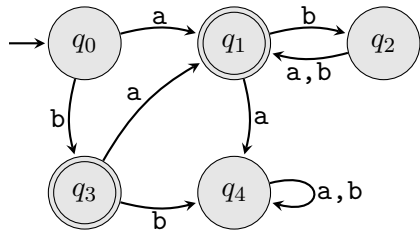
All of the strings $w \in L(M_1)$ s.t. $|w| \geq 3$ have a curious property: w can be written as $w = xyz$ where

- 1 $|y| > 0$ and
- 2 $xy^i z \in L(M_1)$ for all $i \geq 0$

DFA M_2



DFA M_2



Strings in the language include

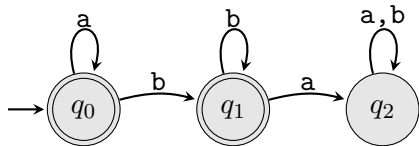
- a
- b
- ba
- aba
- abb
- baba
- abbba
- bababb

Again, strings $w \in L(M_2)$ s.t. $|w| \geq 3$ can be written as $w = xyz$ with $|y| > 0$ and $xy^iz \in L(M_2)$.

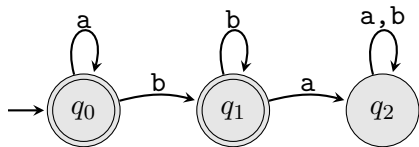
E.g., $x = ba$, $y = ba$, $z = \varepsilon$

- $xy^0z = ba$
- $xy^1z = baba$
- $xy^2z = bababa$
- ...

DFA M_3



DFA M_3



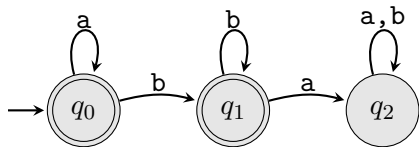
$$L(M_3) = \{a^m b^n \mid m, n \geq 0\}$$

Strings $w \in L(M_3)$ s.t. $|w| \geq 1$ have the same property.

E.g., $x = \varepsilon$, $y = a$, $z = abb$

- $xy^0 z = abb$
- $xy^1 z = aabb$
- $xy^2 z = aaabb$
- $xy^i z = a^{i+1}bb$

DFA M_3



$$L(M_3) = \{a^m b^n \mid m, n \geq 0\}$$

Strings $w \in L(M_3)$ s.t. $|w| \geq 1$ have the same property.

E.g., $x = \varepsilon$, $y = a$, $z = abb$

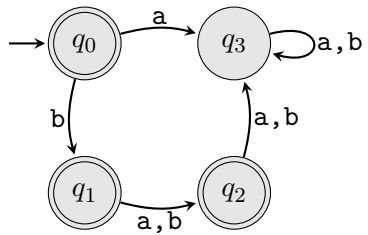
- $xy^0z = abb$
- $xy^1z = aabb$
- $xy^2z = aaabb$
- $xy^iz = a^{i+1}bb$

Not every way we split the strings works

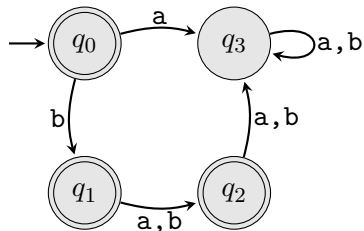
$x = a$, $y = ab$, $z = b$

- $xy^0z = ab \in L(M_3)$
- $xy^1z = aabb \in L(M_3)$
- $xy^2z = aababb \notin L(M_3)$

DFA M_4



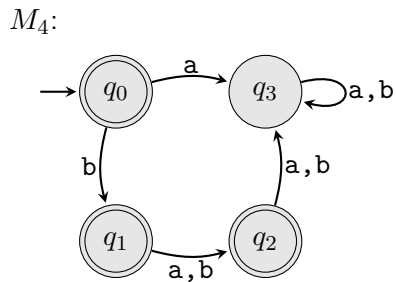
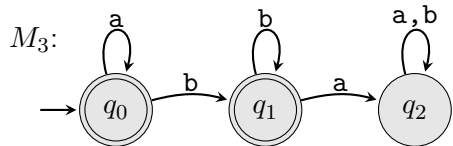
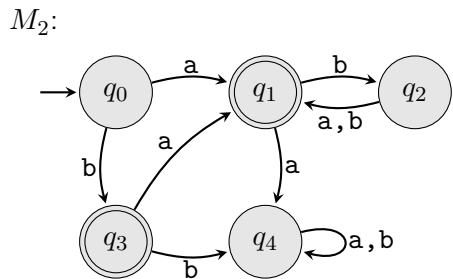
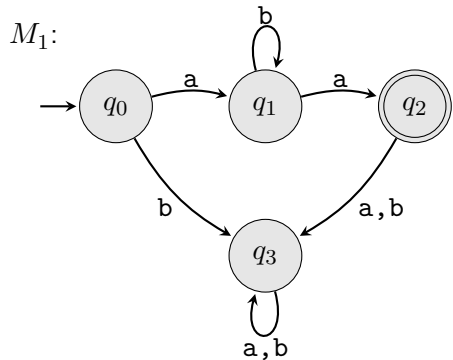
DFA M_4



$$L(M_4) = \{\varepsilon, b, ba, bb\}$$

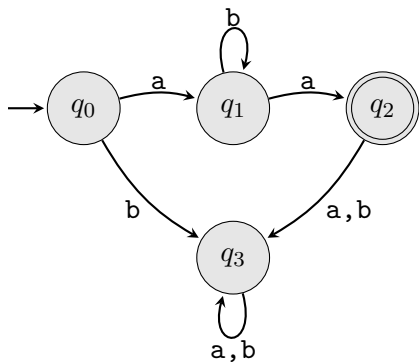
$L(M_4)$ doesn't appear to have this property (unless we say it holds for all strings in $L(M_4)$ with length at least 3 because there are no such strings)

What do M_1 , M_2 , and M_3 have that M_4 lacks?



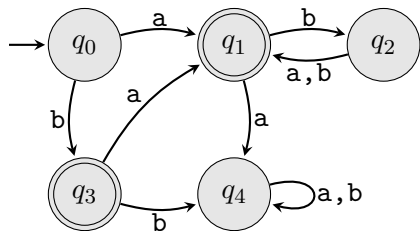
Repeated state for some string in the language

M_1 , M_2 , and M_3 all have a repeated state in some accepting computation



On input aba , M_1 goes through states q_0 , q_1 , q_1 , q_2

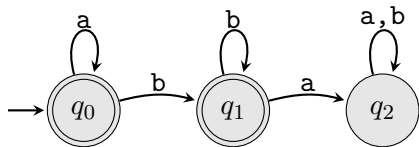
State q_1 is repeated so we can repeat it 0 or more times by following the loop on b

M_2 

On input baba, M_2 goes through states q_0, q_3, q_1, q_2, q_1

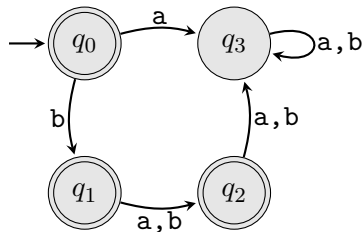
State q_1 is repeated so we can perform the $q_1 \rightarrow q_2 \rightarrow q_1$ sequence corresponding to input ba 0 or more times

M_3



On input aabb, M_2 goes through states q_0 , q_0 , q_0 , q_1 , q_1

State q_0 is repeated so we can perform the $q_0 \rightarrow q_0$ sequence corresponding to input a 0 or more times

M_4 

None of the strings in $L(M_4)$ lead to a repeated state

As mentioned, we can “cheat” and say that the property holds for strings of length at least 3 since $L(M_4)$ has no strings of length at least 3

Pumpable languages

A language A is said to be **pumpable** if there exists an integer $p > 0$ s.t. for all strings $w \in A$ with $|w| \geq p$, there exist strings $x, y, z \in \Sigma^*$ with $w = xyz$ s.t.

- ① $xy^iz \in A$ for all $i \geq 0$
- ② $|y| > 0$
- ③ $|xy| \leq p$

The integer p is called the **pumping length**

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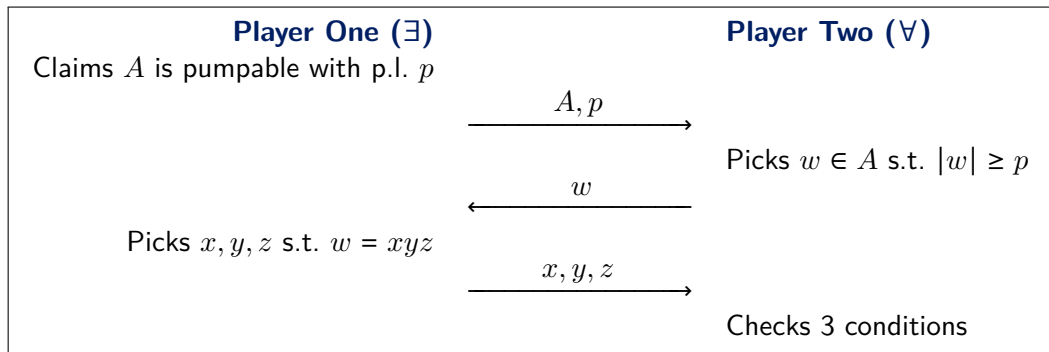
Almost certainly the most complicated mathematical definition you've seen:

$$\exists p > 0. \forall w \in A. \exists x, y, z \in \Sigma^*. \forall i \geq 0. [\dots]$$

Contrast with the definition of a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ from calculus

$$\forall \varepsilon > 0. \exists \delta > 0. [\dots]$$

A two-player game



Player One “wins” the game if

- ① $xy^i z \in A$ for all $i \geq 0$
- ② $|y| > 0$
- ③ $|xy| \leq p$

Player One can win if and only if A is pumpable

Pumping lemma for regular languages

Theorem (Pumping lemma)

Regular languages are pumpable.

Note: The converse is *not* true! There are pumpable languages that are not regular

Proof

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA with $L(M) = A$ and set $p = |Q|$.

If A contains no strings of length at least p , then we're finished since A is pumpable with pumping length p .

Proof

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA with $L(M) = A$ and set $p = |Q|$.

If A contains no strings of length at least p , then we're finished since A is pumpable with pumping length p .

Otherwise, let w be a string in A of length $n \geq p$.

Write $w = w_1 w_2 \cdots w_n$ where each $w_i \in \Sigma$.

Let r_0, r_1, \dots, r_n be the accepting computation of M on w .

By the pigeonhole principle, in the first $p + 1$ states (r_0, r_1, \dots, r_p) , there are states $r_j = r_k$ s.t. $0 \leq j < k \leq p$.

Proof

Set

$$x = w_1 w_2 \cdots w_j$$

$$y = w_{j+1} w_{j+2} \cdots w_k$$

$$z = w_{k+1} w_{k+2} \cdots w_n.$$

$$\begin{array}{l} \text{input:} \quad \overbrace{w_1 \ w_2 \ \cdots \ w_j}^x \ \overbrace{w_{j+1} \ w_{j+2} \ \cdots \ w_k}^y \ \overbrace{w_{k+1} \ w_{k+2} \ \cdots \ w_n}^z \\ \text{states:} \quad r_0 \ r_1 \ r_2 \ \cdots \ r_j \ r_{j+1} \ r_{j+2} \ \cdots \ r_k \ r_{k+1} \ r_{k+2} \ \cdots \ r_n \end{array}$$

Remember $\delta(r_{m-1}, w_m) = r_m$ for all $1 \leq m \leq n$

- ② $|y| = k - j > 0$
- ③ $|xy| \leq p$ because $k \leq p$

Proof

① $xy^iz \in A$

$$\delta(r_{m-1}, w_m) = r_m \quad \forall m$$

Proof

$$\textcircled{1} \quad xy^i z \stackrel{?}{\in} A$$

$i = 0$

$$\begin{array}{cccccccc} & \underbrace{}_x & & \underbrace{\phantom{w_{k+1} w_{k+2} \cdots w_n}}_z & & & & \\ & w_1 & w_2 & \cdots & w_j & w_{k+1} & w_{k+2} & \cdots & w_n \\ r_0 & r_1 & r_2 & \cdots & r_j & & & & \end{array}$$

$$\delta(r_{m-1}, w_m) = r_m \quad \forall m$$

Proof

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$i = 0$

$$\begin{array}{cccccccc} & \underbrace{\hspace{10em}}_x & & \underbrace{\hspace{10em}}_z & & & & \\ & w_1 & w_2 & \cdots & w_j & w_{k+1} & w_{k+2} & \cdots & w_n \\ r_0 & r_1 & r_2 & \cdots & r_j & & & & \end{array}$$

$$\begin{aligned} \delta(r_{m-1}, w_m) &= r_m \quad \forall m \\ \delta(r_j, w_{k+1}) &= \delta(r_k, w_{k+1}) = r_{k+1} \end{aligned}$$

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$$\begin{array}{cccccccc} & \underbrace{}_x & & \underbrace{\phantom{w_{k+1} \ w_{k+2} \ \cdots \ w_n}}_z & & & & \\ & w_1 & w_2 & \cdots & w_j & w_{k+1} & w_{k+2} & \cdots & w_n \\ r_0 & r_1 & r_2 & \cdots & r_j & r_{k+1} & r_{k+2} & \cdots & r_n \end{array}$$

$$i = 1$$

$$\begin{array}{cccccccccccc} & \underbrace{}_x & & \underbrace{\phantom{w_{j+1} \ w_{j+2} \ \cdots \ w_k}}_y & & \underbrace{\phantom{w_{k+1} \ w_{k+2} \ \cdots \ w_n}}_z & & & & & & & \\ & w_1 & w_2 & \cdots & w_j & w_{j+1} & w_{j+2} & \cdots & w_k & w_{k+1} & w_{k+2} & \cdots & w_n \\ r_0 & r_1 & r_2 & \cdots & r_j & r_{j+1} & r_{j+2} & \cdots & r_k & r_{k+1} & r_{k+2} & \cdots & r_n \end{array}$$

Proof

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$$\delta(r_j, w_{k+1}) = \delta(r_k, w_{k+1}) = r_{k+1}$$

$i = 0$

$$\begin{array}{cccccccc} & \underbrace{x} & & \underbrace{z} & & & & \\ & w_1 & w_2 & \cdots & w_j & w_{k+1} & w_{k+2} & \cdots & w_n \\ r_0 & r_1 & r_2 & \cdots & r_j & r_{k+1} & r_{k+2} & \cdots & r_n \end{array}$$

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$$\begin{array}{cccccccccccc} & \underbrace{x} & & \underbrace{y} & & \underbrace{z} & & & & & & & \\ & w_1 & w_2 & \cdots & w_j & w_{j+1} & w_{j+2} & \cdots & w_k & w_{k+1} & w_{k+2} & \cdots & w_n \\ r_0 & r_1 & r_2 & \cdots & r_j & r_{j+1} & r_{j+2} & \cdots & r_k & r_{k+1} & r_{k+2} & \cdots & r_n \end{array}$$

$i = 2$

$$\begin{array}{cccccccccccc} & \underbrace{x} & & \underbrace{y} & & \underbrace{y} & & \underbrace{z} & & & & & \\ & w_1 & w_2 & \cdots & w_j & w_{j+1} & w_{j+2} & \cdots & w_k & w_{j+1} & w_{j+2} & \cdots & w_k & w_{k+1} & w_{k+2} & \cdots & w_n \\ r_0 & r_1 & r_2 & \cdots & r_j & r_{j+1} & r_{j+2} & \cdots & r_k & & & & & & & & \end{array}$$

Proof

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$$\begin{array}{cccccccc} & \underbrace{x} & & \underbrace{z} & & & & \\ & w_1 & w_2 & \cdots & w_j & w_{k+1} & w_{k+2} & \cdots & w_n \\ r_0 & r_1 & r_2 & \cdots & r_j & r_{k+1} & r_{k+2} & \cdots & r_n \end{array}$$

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Proof

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$$\delta(r_{m-1}, w_m) = r_m \quad \forall m$$

$$\delta(r_j, w_{k+1}) = \delta(r_k, w_{k+1}) = r_{k+1}$$

$$\delta(r_k, w_{j+1}) = \delta(r_j, w_{j+1}) = r_{j+1}$$

$i = 0$

$$\begin{array}{cccccccc}
 & \underbrace{}_x & & \underbrace{\phantom{w_{k+1} \dots w_n}}_z & & & & \\
 w_1 & w_2 & \cdots & w_j & w_{k+1} & w_{k+2} & \cdots & w_n \\
 r_0 & r_1 & r_2 & \cdots & r_j & r_{k+1} & r_{k+2} & \cdots & r_n
 \end{array}$$

$i = 1$

$$\begin{array}{cccccccccccc}
 & \underbrace{}_x & & \underbrace{\phantom{w_{j+1} \dots w_k}}_y & & \underbrace{\phantom{w_{k+1} \dots w_n}}_z & & & & & & & \\
 w_1 & w_2 & \cdots & w_j & w_{j+1} & w_{j+2} & \cdots & w_k & w_{k+1} & w_{k+2} & \cdots & w_n \\
 r_0 & r_1 & r_2 & \cdots & r_j & r_{j+1} & r_{j+2} & \cdots & r_k & r_{k+1} & r_{k+2} & \cdots & r_n
 \end{array}$$

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$$\begin{array}{cccccccccccccc}
 & \underbrace{}_x & & \underbrace{\phantom{w_{j+1} \dots w_k}}_y & & \underbrace{\phantom{w_{j+1} \dots w_k}}_y & & \underbrace{\phantom{w_{k+1} \dots w_n}}_z & & & & & & \\
 w_1 & w_2 & \cdots & w_j & w_{j+1} & w_{j+2} & \cdots & w_k & w_{j+1} & w_{j+2} & \cdots & w_k & w_{k+1} & w_{k+2} & \cdots & w_n \\
 r_0 & r_1 & r_2 & \cdots & r_j & r_{j+1} & r_{j+2} & \cdots & r_k & r_{j+1} & r_{j+2} & \cdots & r_k & & & &
 \end{array}$$



Proof

① $xy^iz \in A$

$$\delta(r_{m-1}, w_m) = r_m \quad \forall m$$

$$\delta(r_j, w_{k+1}) = \delta(r_k, w_{k+1}) = r_{k+1}$$

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$$\begin{array}{cccccccc} & \underbrace{\hspace{2cm}}_x & & \underbrace{\hspace{2cm}}_z & & & & \\ & w_1 & w_2 & \cdots & w_j & w_{k+1} & w_{k+2} & \cdots & w_n \\ r_0 & r_1 & r_2 & \cdots & r_j & r_{k+1} & r_{k+2} & \cdots & r_n \end{array}$$

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$$\begin{array}{cccccccccccc} & \underbrace{\hspace{2cm}}_x & & \underbrace{\hspace{2cm}}_y & & \underbrace{\hspace{2cm}}_z & & & & & & & \\ & w_1 & w_2 & \cdots & w_j & w_{j+1} & w_{j+2} & \cdots & w_k & w_{k+1} & w_{k+2} & \cdots & w_n \\ r_0 & r_1 & r_2 & \cdots & r_j & r_{j+1} & r_{j+2} & \cdots & r_k & r_{k+1} & r_{k+2} & \cdots & r_n \end{array}$$

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$$\begin{array}{cccccccccccccccc} & \underbrace{\hspace{2cm}}_x & & \underbrace{\hspace{2cm}}_y & & \underbrace{\hspace{2cm}}_y & & \underbrace{\hspace{2cm}}_z & & & & & & & & & \\ & w_1 & w_2 & \cdots & w_j & w_{j+1} & w_{j+2} & \cdots & w_k & w_{j+1} & w_{j+2} & \cdots & w_k & w_{k+1} & w_{k+2} & \cdots & w_n \\ r_0 & r_1 & r_2 & \cdots & r_j & r_{j+1} & r_{j+2} & \cdots & r_k & r_{j+1} & r_{j+2} & \cdots & r_k & r_{k+1} & r_{k+2} & \cdots & r_n \end{array}$$



Proof

Starting in state r_j , when M reads y , it ends in state $r_k = r_j$.

Therefore, when M runs on xy^iz , it

- ① starts in state $r_0 = q_0$ and after reading x is in state r_j ;
- ② for each of the i copies of y , it is in state r_j , reads y , and moves to state $r_k = r_j$;
and
- ③ from state r_k , it reads z and ends in state $r_n \in F$

Therefore M accepts xy^iz so

- ① $xy^iz \in A$
- ② $|y| > 0$
- ③ $|xy| \leq p$.

Therefore, A is pumpable.



Proving languages are not regular

If we want to prove language A is not regular,

- 1 Assume A is regular

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- 1 Assume A is regular
- 2 Since A is regular, it is pumpable with pumping length p
- 3 Construct a string $w \in A$ of length at least p
- 4 Show that every partition of w into xyz such that $|xy| \leq p$ and $|y| > 0$ yields some i such that $xy^iz \notin A$
- 5 This contradicts the pumping lemma so our assumption must be false, namely A is not regular

Example

Let's prove $A = \{0^n 1^n \mid n \geq 0\}$ is not regular

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Now we need to pick a string $w \in A$ with length $|w| \geq p$

Let $w = 0^p 1^p$ which has length $2p \geq p$

Consider $xyz = w$ such that $|xy| \leq p$ and $|y| > 0$

We got to choose w , but we don't get to choose x , y , and z

We have to consider all possible choices!

What are the possible values of x , y , and z ?

Example

Let's prove $A = \{0^n 1^n \mid n \geq 0\}$ is not regular

Assume A is regular with pumping length p

Now we need to pick a string $w \in A$ with length $|w| \geq p$

Let $w = 0^p 1^p$ which has length $2p \geq p$

Consider $xyz = w$ such that $|xy| \leq p$ and $|y| > 0$

We got to choose w , but we don't get to choose x , y , and z

We have to consider all possible choices!

What are the possible values of x , y , and z ?

x and y consist solely of 0s and z has the rest of the p 0s followed by p 1s:

$x = 0^m$, $y = 0^n$, $z = 0^{p-m-n} 1^p$ where $n > 0$

Example

$$x = 0^m, y = 0^n, z = 0^{p-m-n}1^p \text{ where } n > 0$$

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Example

$x = 0^m$, $y = 0^n$, $z = 0^{p-m-n}1^p$ where $n > 0$

Now we need to find an $i \geq 0$ such that $xy^iz \notin A$

What value of i should we choose?

In this case any $i \neq 1$ works, so let's go with $i = 0$ ("pumping down")

$$xy^0z = xz = 0^{p-n}1^p$$

Since $n > 0$, $p - n \neq p$ so $xy^0z \notin A$ and thus A is not regular □

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Prove $B = \{w \mid w \in \{0, 1\}^* \text{ and } w = w^{\mathcal{R}}\}$ is not regular

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Let's "pump up" this time and try $i = 2$

$$xy^2z = 0^{p+n} 10^p \notin B$$

Therefore, B is not regular



Subsets of regular languages

True or false. If A is a regular language and $B \subseteq A$, then B is regular.

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Since A is regular, there is a DFA M that recognizes A

Since $B \subseteq A$, M accepts every string in B so B is regular

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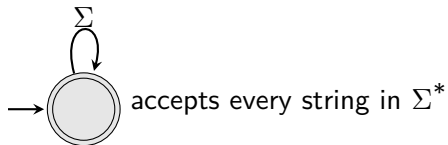
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Since $B \subseteq A$, M accepts every string in B so B is regular

It's missing the fact that for B to be regular there needs to be a DFA M' that accepts every string in B and rejects every string not in B



More nonregular languages

- $C = \{0^m 1^n 0^m \mid m, n \geq 0\}$
- $D = \{0^m 1^n \mid m \leq n\}$
- $E = \{w \mid w \in \{0, 1\}^* \text{ and } w \text{ has the same number of 0s and 1s}\}$