# DECIDABLE LANGUAGES.

- Languages represent computational problems

eg: acceptance of DFA expressible as a language, $A_{DFA}$
(ie $w$ accepted by DFA?)

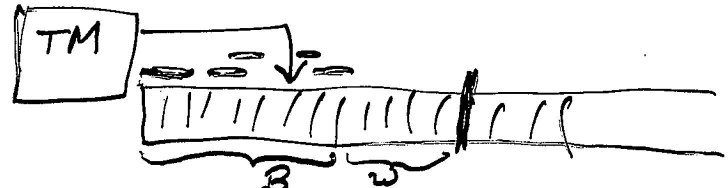
$$A_{DFA} = \{ \langle D, w \rangle \mid D \text{ is a DFA that accepts } w \}$$

Showing that a language (eg $A_{DFA}$) is decidable
$\longleftrightarrow$ showing corresponding computational problem is decidable

Th 1: $A_{DFA}$ is decidable. (ie, given any string $x = \langle D', w' \rangle$ encoding, decide ie Answer Yes/No using a TM, whether $D'$ (a DFA's encoding) accepts $w'$,)

// All problems requiring non-bool answers rephrased as (combination of) problems requiring bool. answers

Proof: (construction)




M : $\langle B, w \rangle$
1. Simulate $B$ on input $w$
2. If simulation ends in accept, then accept else reject

$A_{NFA} = \{\langle B, w\rangle \mid B$ is a NFA that accepts $w\}$

**Th 2:** $A_{NFA}$ is decidable

Proof : On input $\langle B, w\rangle$ :

1) Convert 'B' into 'C' (a DFA)
2) Run M on input w // procedure call
   if M accepts w then accept
   else reject

$A_{REX} = \{\langle R, w\rangle \mid R$ is a r.e. that generates $w\}$

**Th 3:** $A_{REX}$ is decidable

Proof : P : On input $\langle R, w\rangle$ :

1) Convert R into DFA A
2) Run M on w // procedure call
   if M accepts w, then accept
   else reject

$E_{DFA} = \{ \langle A \rangle \mid A \text{ is a DFA and } L(A) = \phi \}$

→ i.e., set of encodings of DFAs whose language is $\phi$

<u>Th 4</u>: $E_{DFA}$ is decidable

(i.e., given an encoding of a DFA, answer Y/N whether L(DFA given) is $\phi$ using a TM)

<u>Proof</u>:

[A] generate <u>all</u> strings over $\Sigma$ of the DFA
Run DFA on each string s
If any 's' is accepted by DFA, then reject DFA (from membership in $E_{DFA}$)
else keep looping

→ This TM is <u>not</u> a decider

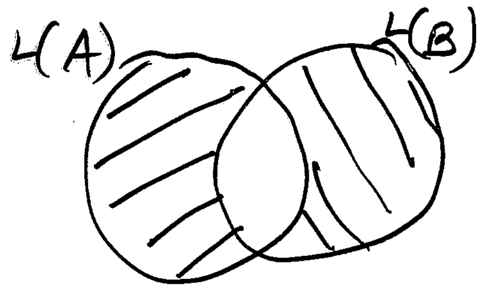
[B] observe: if path from $q_0$ to $q_{accept}$, then $L(DFA) \neq \phi$ $\therefore$ reject the DFA from membership in $E_{DFA}$

T: On input $\langle A \rangle$
1) Mark $q_0$
2) Repeat until no new state is marked
   for each unmarked $q' \in Q$
      if incoming edge from marked $q$, then mark $q'$
3) If no $q \in F$ is marked, then <u>accept</u> else <u>reject</u>

$$EQ_{DFA} = \{\langle A,B\rangle \mid L(A) = L(B) \text{ and } A, B \text{ are DFAs}\}$$

<u>Th 5</u> : $EQ_{DFA}$ is decidable

Proof : observe $L(A) = L(B)$ iff $L(A) - L(B) = \phi$
and $L(B) - L(A) = \phi$




<u>ie</u> $\left[L(A) - L(B)\right] \cup \left[L(B) - L(A)\right]$
$= L(C) = \phi$ ?

<u>ie</u> $\left[L(A) \cap \overline{L(B)}\right] \cup \left[L(B) \cap \overline{L(A)}\right]$
$= L(C) = \phi$ ?

Using closure of R.L under $\cup, \cap$, comp, conclude that $L(C)$ is a R.L.

(Construction:) On input $\langle A, B\rangle$,

1) Construct DFA $C \mid L(C) =$ // defined above.
2) invoke $E_{DFA}$ proof // procedure
3) Decide accordingly.

$A_{CFG} = \{ \langle G, w \rangle \mid G$ is a CFG that generates $w \}$

Th 6: $A_{CFG}$ is decidable. // Caution: $\infty$ly many derivations

Proof (construction): 1) Convert $G$ in $G'$ (in Chomsky N.F.)

// any derivation of $w$ occurs in $2|w|-1$ steps, guaranteed.

2) List all derivations using $G'$, having $2|w|-1$ steps.

3) If any of these derivations yield $w$, then accept
else reject.

$A_{PDA} = \{ \langle B, w \rangle \mid B$ is a PDA that accepts string $w \}$

Th 7: $A_{PDA}$ is decidable

$E_{CFG} = \{ \langle G \rangle \mid L(G) = \phi$ and $G$ is CFG $\}$

Th 8: $E_{CFG}$ is decidable. // Caution: $\infty$ly many $w^s$

TM R: On input $\langle G \rangle$:

1) Mark all terminals

2) Repeat until no new variables get marked.

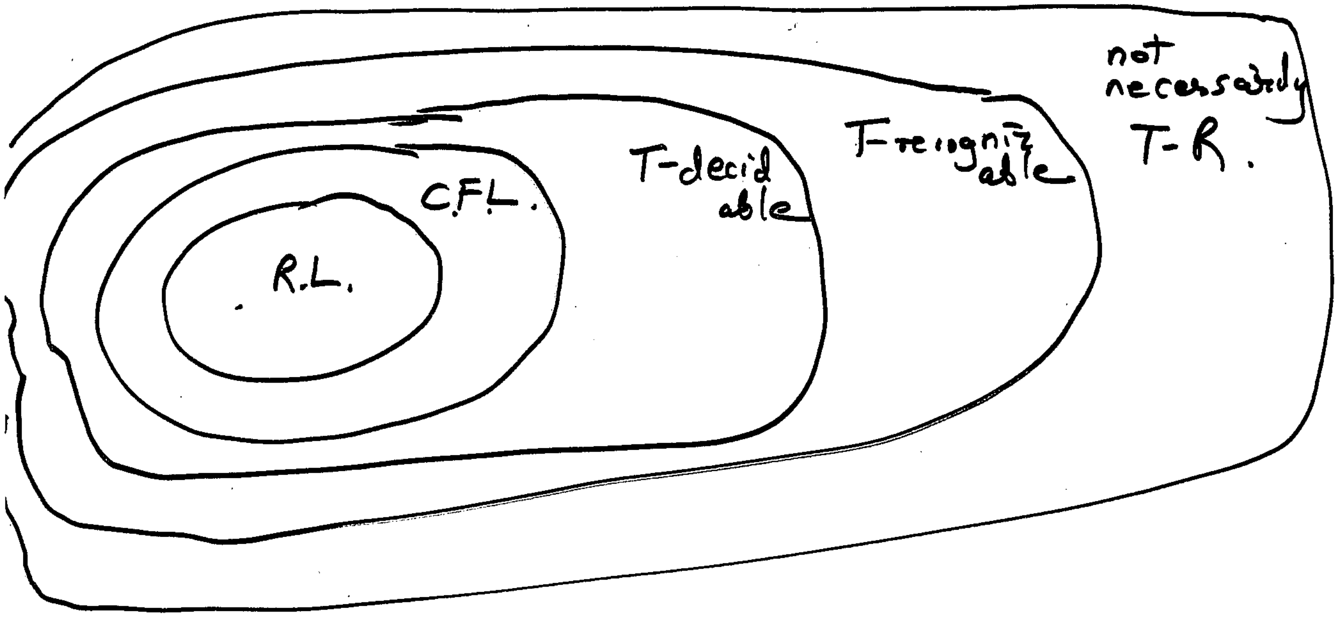
Mark any var $A$, where $A \rightarrow U_1 U_2 \cdots U_k$
and each symbol $u_1 \cdots u_k$ is marked

3) If $S$ is not marked then accept else reject

$EQ_{CFG} = \{ \langle G, H \rangle \mid L(G) = L(H) \text{ and } G, H \text{ are NPDAs} \}$

Th 9 : $EQ_{CFG}$ is NOT decidable

Th 10 (Corollary of Th. 6):
Every CFL is decidable




NB: R.L. $\equiv$ R.Exp. $\equiv$ R. Grammar $\equiv$ ~~PDA~~ DFA

CFL    CFG $\equiv$ PDA

CSL $\equiv$ Context-Sensitive G. $\equiv$ L.B.A. (NEW!) (Linear Bounded Automaton)

Restricted G. $\equiv$ TM

CHOMSKY HIERARCHY [read link on web page / GOOGLE]

# HALTING ~~ACCEPTANCE~~ PROBLEM

$$A_{TM} = \left\{ \langle M, w \rangle \;\middle|\; TM \; M \text{ accepts } w \right\}$$

<u>Th</u>: $A_{TM}$ is recognizable.

<u>Proof</u>: by construction.

$U$: On input "$\langle M, w \rangle$"

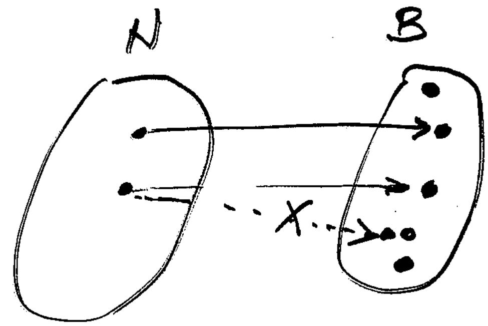
1) Simulate operation $M$ on $w$
2) If $M$ accepts $w$, then $U$ accepts $\langle M, w \rangle$
   else if $M$ rejects $w$, then $U$ rejects $\langle M, w \rangle$
   (else if) .....

// Digression: MATH!

which set is larger?

1) $N$ $(1, 2, 3, \cdots)$ <u>or</u> $\Sigma^*$ where $\Sigma = \{0, 1\}$
2) $N$ <u>or</u> $E = \{2, 4, 6, 8, \cdots\}$
3) $N$ <u>or</u> $Q$ (rational #s)
   $= \left\{ \frac{a}{b} \;\middle|\; a, b \in N \right\}$
4) $N$ or $R$ (real #s)
   $= \{\text{numbers expressible in decimals}\}$

<u>Defn</u>: Function : $f : N \longrightarrow B$




'into' fn : $\exists\, b \in B \mid$

$\nexists\, n \in N$ where

$f(n) = b$.

'onto' fn : $\forall\, b \in b,\ \exists\, n \in N \mid f(n) = b$

<u>Correspondence</u> : is a 1-1 onto function.

<u>Defn</u>: Set B is countable if B is finite <u>OR</u>

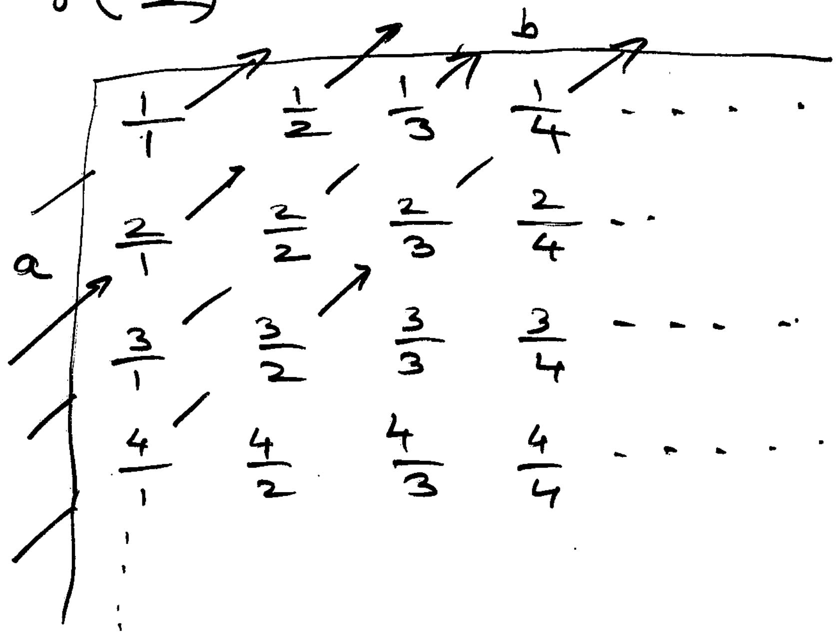
B has same size as N

(<u>ie</u>, $f : N \to B$ is a correspondence)

$\exists$ N vs E : $\forall\, b \in E,\ f\left(\frac{b}{2}\right) = b$

$\exists$ N vs $Q$ :

$\left(\frac{a}{b} \mid a, b \in N\right)$




- preimage of $\frac{a}{b}$ ?

Hint : $\sum_{i=1}^{?} i + j$

Th: $\mathbb{R}$ is uncountable.

<u>Proof</u>: by contradiction. (nested, use construction).

Assume $\mathbb{R}$ is countable.

| $n$ | $f(n)$ |
|---|---|
| 1 | $.\underline{7}31445$ ---- |
| 2 | $.8\underline{1}1226$ .... |
| 3 | $.34\underline{5}123$ .... |
| 4 | $.$ _ |
| 5 | $.$ _ |
| $\vdots$ | |

Now, we generate a real $x \mid \exists\, n$ for which $f(n) = x$,
(ie $x \neq f(n)$ for any $n$).

Let $j^{th}$ ~~fractional~~ decimal position value be $\neq$
$j^{th}$ ~~fractional~~ decimal position value of $f(j)$

"DIAGONALIZATION" technique

$x = .354 \ldots\ldots$

Claim: $x$ not in $f(n)$ for any $n$
(because) $x$ differs from $f(n)$ in its $n^{th}$ decimal position

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END DIGRESSION

<u>OBSERVE</u>: 1) Countably many TMs
2) Uncountably many languages.

$\Rightarrow$ more languages than TMs $\Rightarrow$ some languages will not be recognizable.

Th: There are languages that are not Turing-recognizable.

Proof:

① {TMs} is countable.

$\{TMs\} \subseteq \Sigma^*$ // $TM \in \Sigma^*$

$\Sigma^* = \{\epsilon, \underbrace{0, 1}, \underbrace{00, 01, 10, 11}, \underbrace{000, \cdots 111}, \underbrace{\cdots\cdots} \cdots\}$

show $f : N \to \Sigma^*$ is a corresspondence.

Hint: $s \in \Sigma^* \mid |s| = k$ has a pre-image $n$ in $N$, where ............

$$\sum_{i=0}^{k-1} 2^i + j$$

② $\mathcal{L} = \{$all languages$\}$ // show $\mathcal{L}$ is uncountable

Observe: {all infinite binary sequences} is uncountable.

(Proof technique:) like "R is uncountable"

Let $B = \{$infinite binary sequences$\}$

$f : \mathcal{L} \to B$ is a correspondence.

$\Sigma^* = \{\in, 0, 1, 01, 10, 11, 00, 000, \cdots\}$

example lang. $A = \{$ 0, 01, 00, 000 ... $\}$

string of $x^*$ $\chi_A$ = 0 1 0 1 0 0 1 1 ......

$f(A) \Rightarrow \chi_A$ is a 1-1 onto, ie, correspondence.
$\chi_A$ is an $\infty$ seq. called the "characteristic seq"
ie, $f: \mathcal{L} \longrightarrow \mathcal{B}$, is a correspondence

Here $f(A) = \chi_A$, where $A \in \mathcal{L}$ and $\chi_A \in \mathcal{B}$
(characteristic seq).

As $\mathcal{B}$ is uncountable, $\mathcal{L}$ is also uncountable.

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$\Sigma^*$: set of finite strings, // but $\infty$ number of members in it.
can be listed
$\therefore$ 1-1 corresp. with $N$ $\therefore$ countable

$\mathcal{B}$: cannot be listed.
$\therefore$ uncountable
// set of $\underline{\underline{\infty}}$ binary strings

$\mathcal{B} \not\subseteq \Sigma^*$

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$A_{TM} = \{\langle M, w\rangle \mid M \text{ accepts } w\}$

__Th__: $A_{TM}$ is undecidable.

Proof: (by contradiction):
Assume $H(\langle M, w\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } [M \text{ does not accept}]\ w \end{cases}$

$$D(\langle M\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ does not accept } \langle M\rangle \\ \text{rejects} & \text{if } M \text{ accepts } \langle M\rangle \end{cases}$$

$D$: On input $\langle M \rangle$

1] Runs $H(\langle M, \langle M \rangle \rangle)$

ie, Run $H$ on input $\langle M, \langle M \rangle \rangle$

// $\langle M \rangle$ is representation/encoding of $M$ using $\Sigma$

2] $H$ is a decider

$\therefore$ If $H$ accepts, then $D$ rejects $\langle M \rangle$

else $H$ rejects, then $D$ accepts $\langle M \rangle$

$$L(D) = \{ \langle M \rangle \mid M \text{ on input } \langle M \rangle \text{ is not accepted} \}$$

// decidable using call to $H$

Question: does $\langle D \rangle \in L(D)$ ?

$$\underbrace{D(\langle D \rangle) = \begin{cases} \text{accept} & \text{if } D \text{ does not accept } \langle D \rangle \\ \text{reject} & \text{if } D \text{ accepts } \langle D \rangle \end{cases}}$$

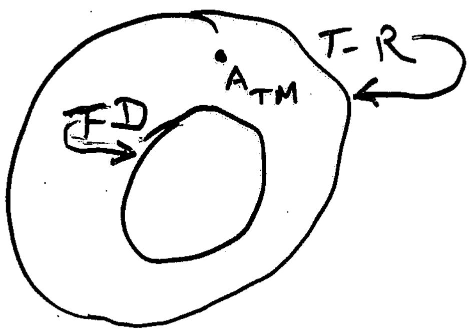
$\rightarrow$ absurdity ! $\therefore$ $H$ cannot exist.

| | $\langle M_1 \rangle$ | $\langle M_2 \rangle$ | $\langle M_3 \rangle$ | $\cdots$ | $\langle D \rangle$ | $\cdots$ |
|---|---|---|---|---|---|---|
| $M_1$ | A | rej | A | $\cdots$ | | |
| $M_2$ | A | A | A | $\cdots$ | | |
| $M_3$ | rej | A | rej | $\cdots$ | | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | | |
| $D$ | | | | | ? | |
| $\vdots$ | | | | | | |

Red color: behavior of $H$

$A_{TM}$ not decidable. $\underbrace{\mathcal{L}}_{\text{uncountable}} - \underbrace{\{\langle M \rangle\}}_{\text{countable}}$ is $\infty$

Use REDUCIBILITY to prove languages are not decidable.




What about not Turing-recognizable languages?

## NOT TURING-RECOGNIZABLE LANGUAGES

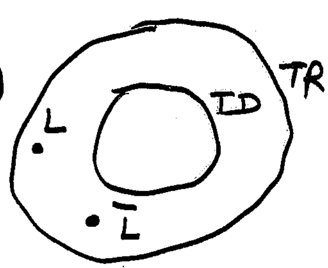
Defn: $L$ is co-Turing recognizable if $L$ is the complement of a Turing-recognizable language.

Th: $L$ is decidable $\Rightarrow \Leftarrow$ $L$ is T-recognizable and $L$ is co-T-recognizable.

($\Rightarrow$) $\bar{L}$ is decidable. $\therefore$ $\bar{L}$ is T-recognizable

$\therefore$ $L$ is co-T-recognizable

($\Leftarrow$)




Let $M_1$ recognize $L$

" $M_2$ " $\bar{L}$

M: On input $w$:

1) Run $M_1$ on $w$ and $M_2$ on $w$ alternating steps

2) If $M_1$ accepts $w$ then accept $w$;
If $M_2$ accepts $w$ then reject $w$.

Any $w$ : $w \in L$ or $w \in \overline{L}$

$\therefore$ $M_1$ accepts $w$ or $M_2$ accepts $w$

$\therefore$ $M$ is a decider

- accepts $w \in L$
- rejects $w \notin L$

$\therefore$ $L$ and $\overline{L}$ must reside in the set T-decidable (inner circle). $\square$.

<u>Cor</u>: $\overline{A_{TM}}$ is not T-recognizable.

<u>Proof</u>: (use contradiction). $A_{TM}$ is not decidable.