PH (Polynomial Hierarchy)

Def. The class Σ_i^P $\forall i \in \mathbb{N}$ is the set of all lang. L s.t. $\exists$ a DTM M_L and polynomial q such that $\forall x$

$$
\begin{aligned}
x \in L \iff & \\
& \exists w_1 \in \{0,1\}^{q(|x|)} \\
& \quad \forall w_2 \in \{0,1\}^{q(|x|)} \\
& \qquad \ddots \\
& \qquad\quad Q_i w_i \in \{0,1\}^{q(|x|)} \\
& \qquad\quad M(x, w_1, w_2, \ldots, w_i) = 1
\end{aligned}
$$

$$Q_i = \begin{cases} \exists & \text{if } i \text{ is odd} \\ \forall & \text{if } i \text{ is even} \end{cases}$$

Def

$$PH = \bigcup_{i \geq 1} \Sigma_i^p$$

$$= \bigcup_{i \geq 1} \Pi_i^p$$

Def $\Pi_i^p = co\Sigma_i^p$

$L \in \Pi_i^p \Longleftrightarrow \exists$ DTM M and polynomial q s.t.

$\forall x$

$$x \in L \Longleftrightarrow \forall w_1 \exists w_2 \ldots Q_i \; M(x, w_1, \ldots, w_i) = 1$$

$$Q_i = \begin{cases} \exists & \text{if } i \text{ is even} \\ \forall & \text{if } i \text{ is odd} \end{cases}$$

* $NP = \Sigma_1^P$

$coNP = \Pi_1^P$

<u>Lemma</u>. $\forall i \geq 1 \quad \Sigma_i^P \subseteq \Pi_{i+1}^P \subseteq \Sigma_{i+2}^P$

$L \in \Sigma_i^P \iff \forall x$

$x \in L \iff \exists w_1 \forall w_2 \ldots Q_i w_i$

$M(x, w_1, \ldots, w_i) = 1$

$\forall w_0$

Facts about PH

- We believe

$$\Sigma_i^P \neq \Sigma_{i+1}^P, \quad \Pi_i^P \neq \Sigma_{i+1}^P$$

- We say PH collapses if $\exists\, i$ s.t.

$$\Sigma_i^P = \Sigma_{i+1}^P$$

or

$$\Pi_i^P = \Pi_{i+1}^P$$

- Theorem:

① If $\exists\, i$ s.t. $\Sigma_i^P = \Pi_i^P$ then

$$PH = \Sigma_i^P = \Pi_i^P$$

② If $P = NP$ then $PH = P$

Proof: of (2)

- If $P = NP$, then $NP = coNP$

$\Rightarrow \Sigma_1^P = \Pi_1^P$

(By (1), we can be done)

↳ Finish proof.

Proof by induction:

$\Sigma_i^P, \Pi_i^P \subseteq P \quad \forall i$

• Base Case: $\Sigma_1^P, \Pi_1^P \subseteq P$ ✓

since $P = NP = coNP$

- **Inductive Step:**

Assume

$$\Pi_i^P, \Sigma_i^P \subseteq P.$$

We show

$$\Sigma_{i+1}^P, \Pi_{i+1}^P \subseteq P$$

Let $L \in \Sigma_{i+1}^P$. // same thing for $L \in \Pi_{i+1}^P$

$x \in L \iff$

$$\exists w_1 \mid \forall w_2 \ldots Q_{i+1} w_{i+1}$$
$$M_L(x, w_1, \ldots, w_{i+1}) = 1$$

Let L' be the language

$(x, w_1) \in L' \iff$

$[\forall w_2 \exists w_3 \cdots Q_{i+1} w_{i+1}$

$M_L(x, w_1, \ldots, w_{i+1}) = 1$

$\Rightarrow L' \in \underbrace{\Pi_i^P = P}_{\text{inductive hypothesis}}$

$(x, w_1) \in L' \longleftrightarrow$ DTM

$M'(x, w_1) = 1$ in at most

poly-time

- So replace M_L in language L with M':

$x \in L \longleftrightarrow \exists w_1 \; \underline{M'(x, w_1) = 1}$

$\Rightarrow L \in NP = P$

$$L \in \underset{\substack{\text{\rotatebox{90}{$\subseteq$}} \\ P = NP}}{\Sigma_{i+1}^P} \longrightarrow L \in P, NP$$

$\square$

Complete Problems

- we define complete problems for Σ_i^P, Π_i^P, and PH
  $\rightarrow$ same as NP/CoNP
- $\forall i$: Σ_i^P and Π_i^P

have complete languages

* Do not believe $\exists L \in PH$ that is PH-complete

Theorem: If $L \in PH$ is PH-complete, then $\exists i$ s.t. $PH = \Sigma_i^P$.

Proof:

- $L \in PH \Rightarrow \exists i$ s.t. $L \in \Sigma_i^P$

$$PH = \bigcup_i \Sigma_i^P$$

- Assume L is PH complete

- Then $\forall L' \in PH$
  $L' \leq_p \underline{L}$.

- But $\underline{\underline{L}} \in \Sigma_i^P$

$\Rightarrow L' \in \Sigma_i^P$ $\square$

- Notice
  $PH \subseteq PSPACE$

<u>Corollary</u>: Unless PH collapses, $PH \neq PSPACE$

Complete problems for any Σ_i^p, Π_i^p

① for Σ_i^P

- Define $\Sigma_i SAT$

$$= \{ \phi : \exists u_1 \forall u_2 \ldots Q_i u_i \;\; \phi(u_1, \ldots, u_i) = 1 \}$$

$$Q_i = \begin{cases} \exists & i \text{ is odd} \\ \forall & i \text{ is even} \end{cases}$$

$\rightarrow \Sigma_i^P$ - complete

② For Π_i^P

$\Pi_i SAT = \{ \phi \mid$

$\forall u_1 \exists u_2 \ldots Q_i u_i$

$\phi(u_1, \ldots, u_i) = 1 \}$

$$Q_i = \begin{cases} \exists & i \text{ is even} \\ \forall & i \text{ is odd} \end{cases}$$

- Π_i^P - complete

(3) Σ_2^P-complete problem (Umans 1998)

- Succinct-Set-Cover

- Given $\{\phi_1, \ldots, \phi_m\}$ 3DNF on n variables, and integer k

- Does there exist $S \subset [m]$ s.t. $|S| \leq k$ and $\bigvee_{i \in S} \phi_i$ is a tautology

$$\exists S \subset [m] \; \forall x \in \{0,1\}^n \left(|S| \leq k \wedge \bigvee_{i \in S} \phi_i(x) = 1 \right)$$

$\Rightarrow \; \in \Sigma_2^P$

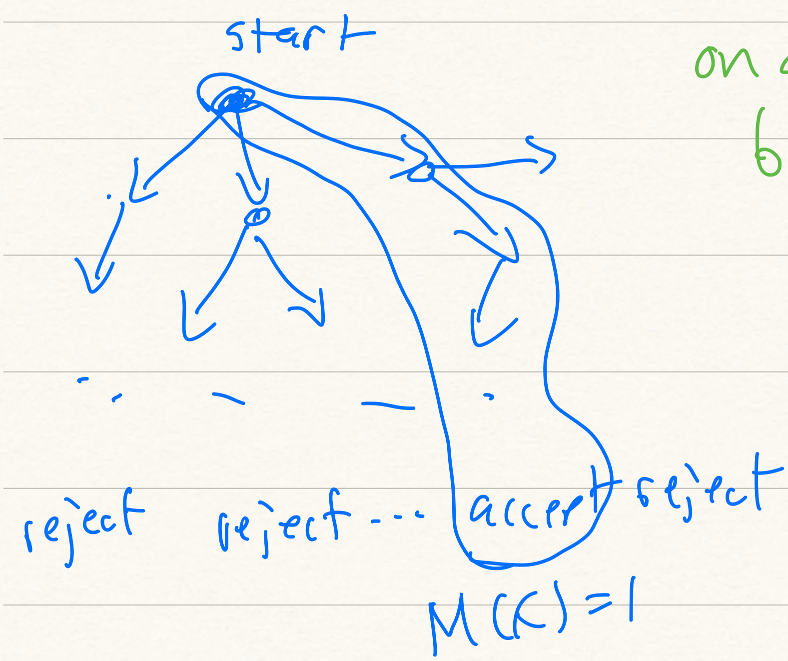
Alternating TMs: generalized non-determinism

- Think back to NTM for $L \in NP$

$$x \in L \longleftrightarrow \underbrace{M}_{\text{NTM}}(x) = 1$$

∃ at least one accepting path

NTM runs in at most poly-time on any computation branch

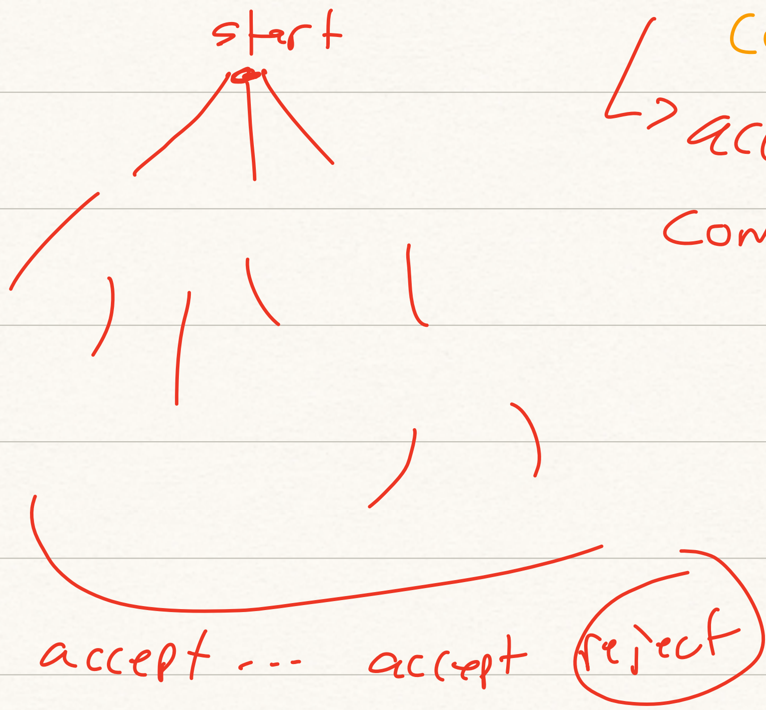


- NTM for coNP

$x \in L \longleftrightarrow M(x) = 1$

when does NTM for coNP output 1?

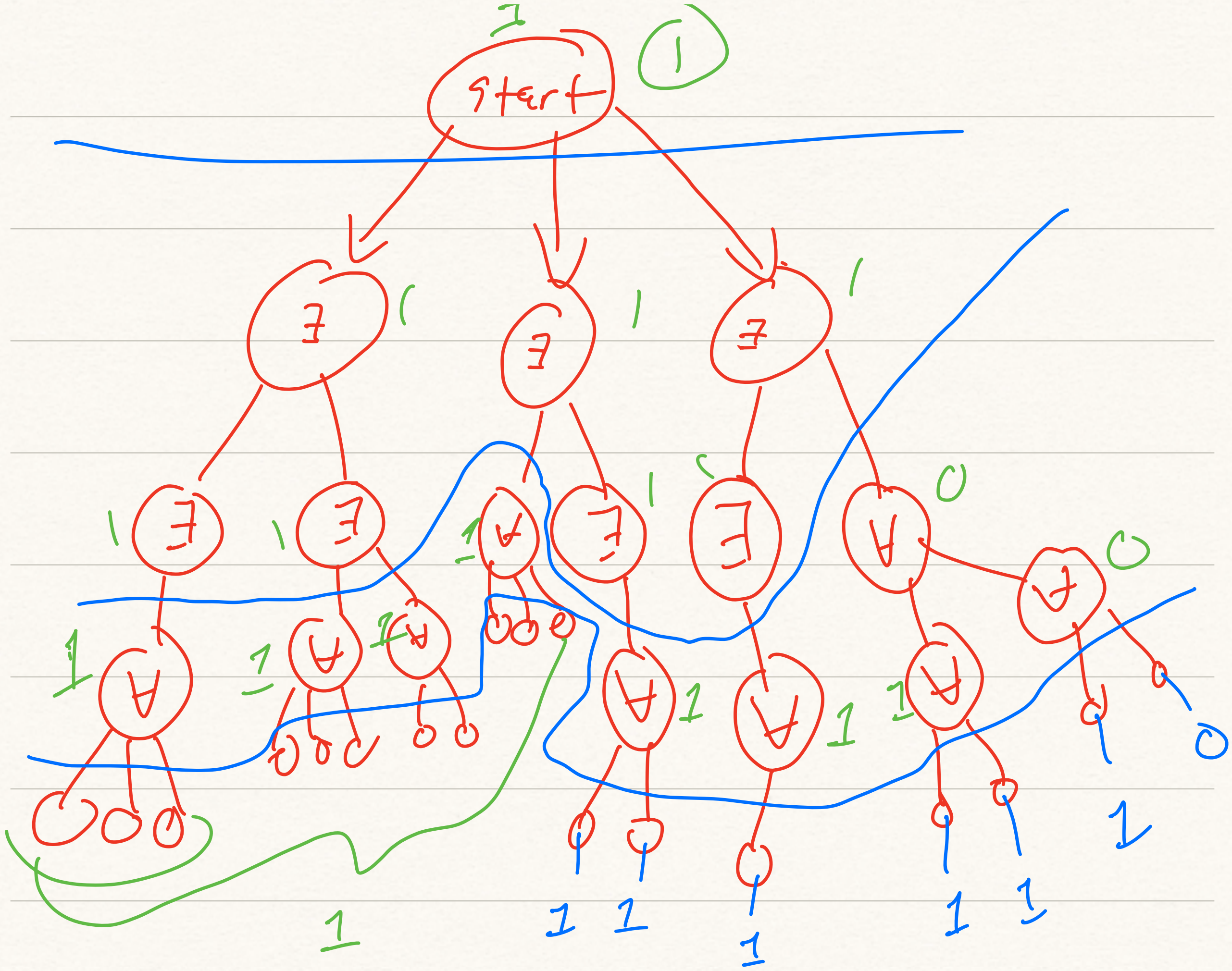
↳ accepts on all computation paths



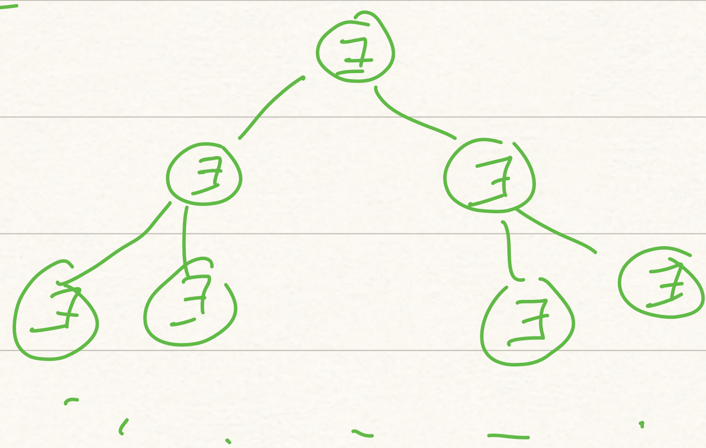
- Let's see NTM for Σ_2^P

take $L \in \Sigma_2^P$

$x \in L \longleftrightarrow \exists w_1 \forall w_2 \; M(x, w_1, w_2) = 1$



NP Tree



Alternating TMs

An NTM M is said to be an <u>alternating Turing Machine</u> if it additionally labels non-halting states with $\exists / \forall$ s.t.

$M(x) = 1$ if and only if the start configuration is labeled $\underline{1}$ after the following process

① Let $G_{M,x}$ be the config. graph for $M(x)$.

2. Label all accept nodes w/ 1 and all reject nodes w/ 0.
3. $\forall$ non-halt node v
   - If v is labeled $\exists$, label it 1 iff at least 1 child of v is labeled 1
   - If v is labeled $\forall$, label it 1 iff all children are labeled 1.

- ATM runs in time T if all paths are of length at most $O(T(|x|))$

$\forall x$

- ATM M is <u>i-alternating</u> if on any computational path, it alternates quantifiers at most $i-1$ times.

- $L \in ATIME(T)$ if $\exists$ ATM deciding L in at most $O(T)$ time

- $L \in ASPACE(S)$ if $\exists$ ATM deciding L in at most $O(S)$ space

lecture12