Turing Machine Runtime

- Let $f : \{0,1\}^* \to \{0,1\}^*$ be a function, and let M be a TM.

We say <u>M computes</u> f if $\forall x \in \{0,1\}^*$,

$$\underbrace{M(x) = f(x)}_{\text{M on input x halts with f(x) on the output tape}}.$$

Let $T : \mathbb{N} \to \mathbb{N}$. We say that M computes f in time T if $\forall x \in \{0,1\}^*$,

$M(x) = f(x)$ in at most $O(T(|x|))$ steps

- Time Constructible Function

$T : \mathbb{N} \to \mathbb{N}$ is time-constructible if $\exists$ TM $\hat{M}$ such that

$\hat{M}(n \in \mathbb{N}) = \langle T(n) \rangle_2$

in $O(T(n))$ time

Examples

$$T(n) = \begin{cases} n \\ 7 \\ n \log n \\ 2^n \end{cases} \qquad n^2$$

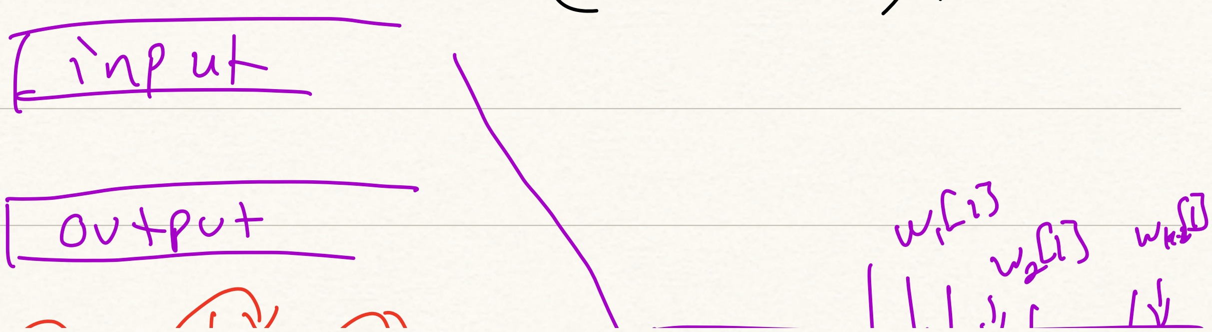
Equivalences

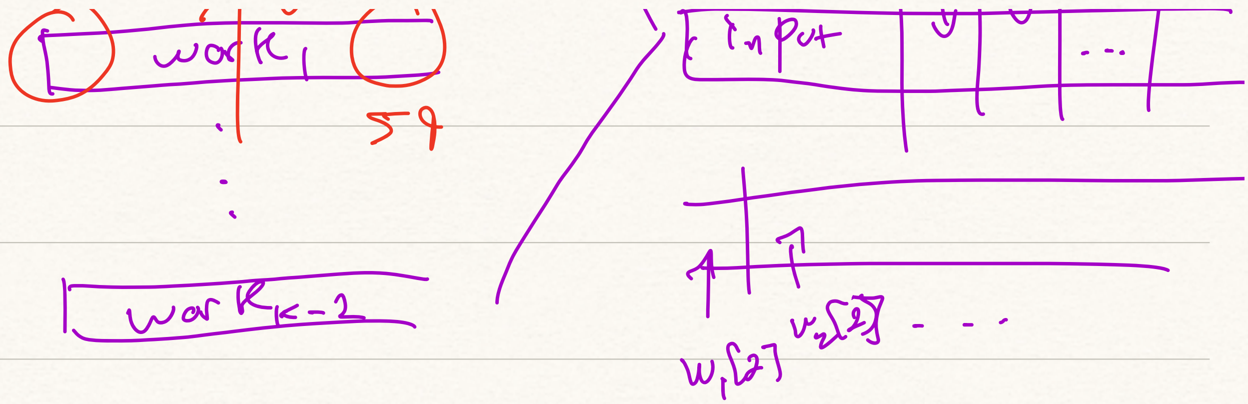
- Let $f : \{0,1\}^* \to \{0,1\}^*$ be a function and $T : \mathbb{N} \to \mathbb{N}$ be time-constructible
  - If f is computable by a time-T Turing Machine M_f with tape alphabet Γ
  - Then: f is computable

in time $O(\log_2(|\Gamma|) \cdot T)$ by a TM $\hat{M}_f$ with tape alphabet $\hat{\Gamma} = \{0, 1, \triangleright, \square\}$.

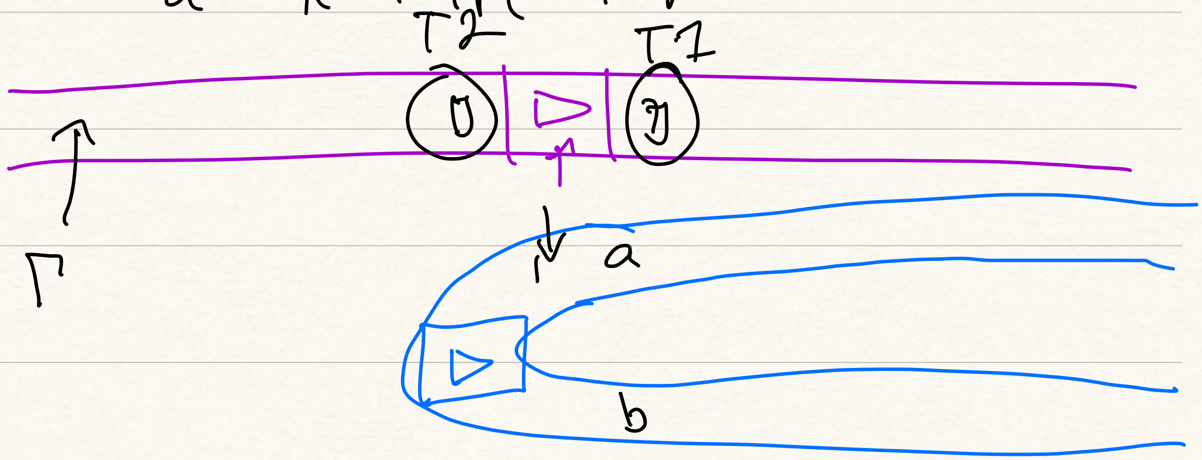
- Let f, T be as above with TM M_f, where M_f is a k-tape TM. Then, f is computable by a single-tape TM M^*_f in time

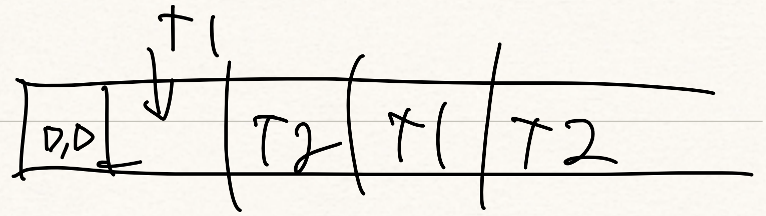
$$T^* = O(k \cdot T^2).$$





- Keep f, T above. M_f is computable by a k-bidirectional tape TM in time T
  $\Rightarrow$ $\hat{M}_f$ computes f in time $4T$, where $\hat{M}_f$ is a k-tape TM.



$(\Gamma \times \Gamma) \to$ 

Universal TMs

- All Turing Machines are represented by some string $\alpha \in \{0,1\}^*$

- we let $\langle M \rangle_2$ be the binary string rep of M.
- we also let M_α be the TM specified by $\alpha \in \{0,1\}^*$

- Want: give a program $\langle \alpha, x \rangle$ and output $M_\alpha(x)$

<u>Hennie & Stearns (1966)</u>

$\exists$ TM $\mathcal{U}$ s.t. $\forall \alpha, x \in \{0,1\}^*$,

$$\mathcal{U}(\alpha, x) = M_\alpha(x).$$

If $M_\alpha(x)$ halts in $T(|x|)$ steps, then

$\mathcal{U}(\alpha, x)$ halts in

$$O(T \log T) \text{ steps}$$

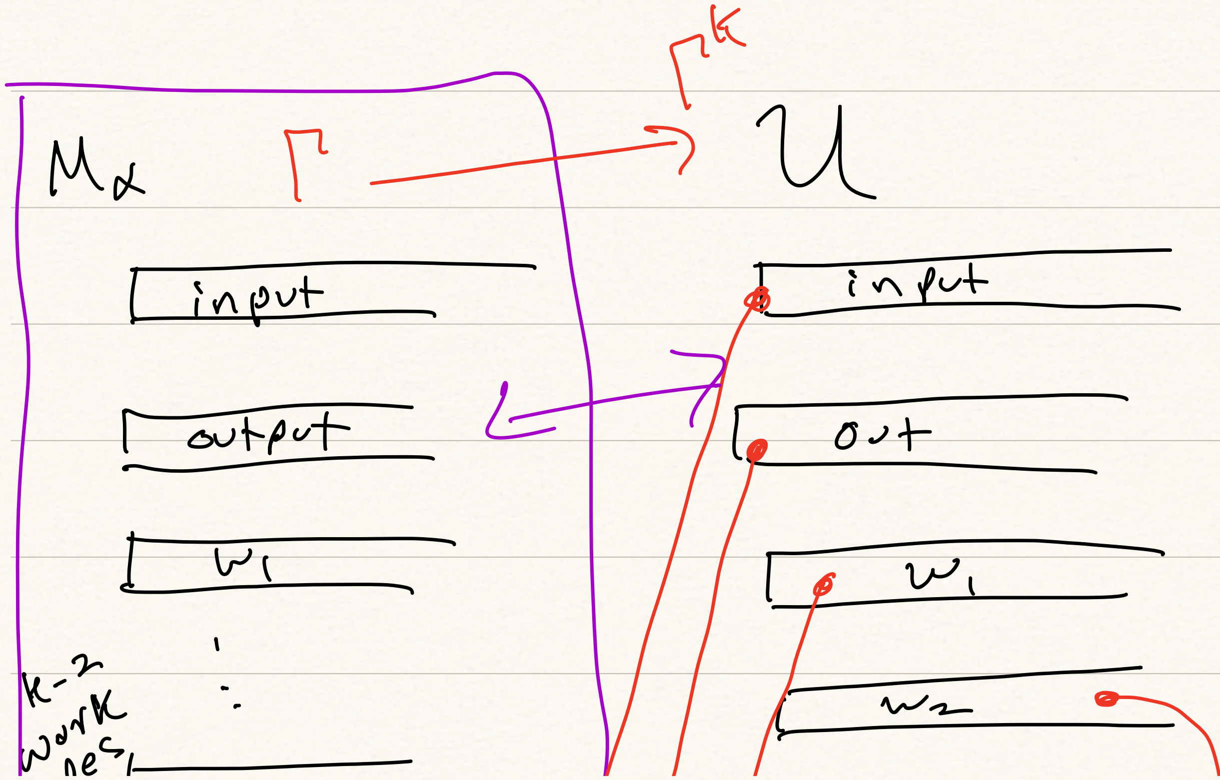
constants depend on M_α's $|\Gamma|$, # of tapes, # of states

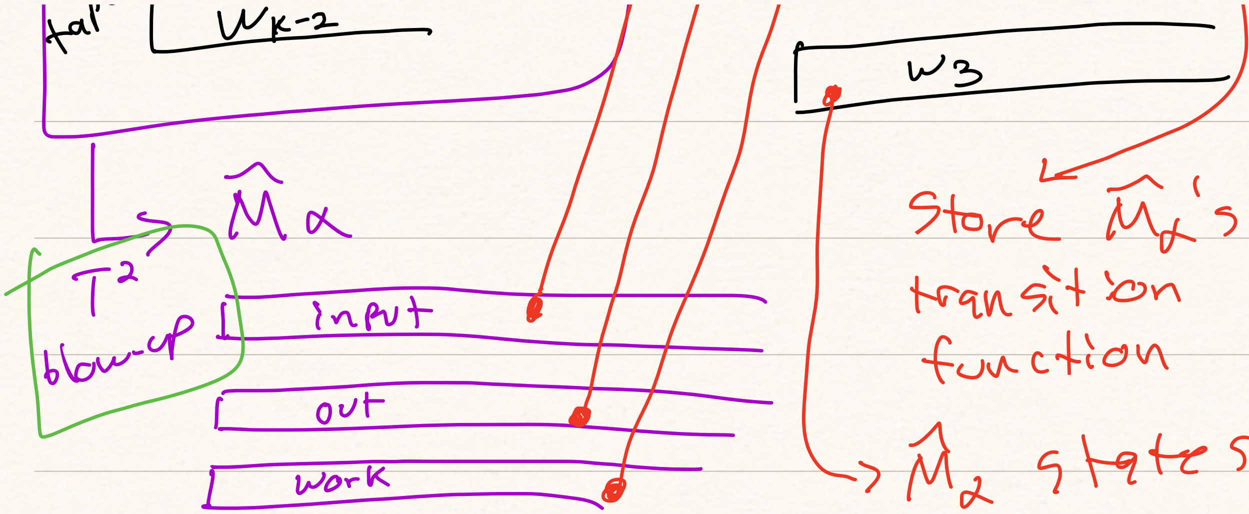
We'll show above for $O(T^2)$.

<u>Proof</u>: Let $\alpha, x \in \{0,1\}^*$. WLOG, assume M_α has tape alphabet $\Gamma = \{0, 1, \triangleright, \square\}$.

3 work tapes

• Let $\mathcal{U}$ be a 5-tape TM with $\Gamma_{\mathcal{U}} = \{0, 1, \triangleright, \square\}$





Simulate 1 step of $\hat{M}_\alpha$?

① Read current symbols under $\hat{M}_\alpha$'s work tape + input tape

constant time

② Read $\hat{M}_\alpha$ current state from WT_3.

$\hat{M}_\alpha$'s states

$O(\log(|Q|))$

③ Read WT_2 for the entry corresponding to $\hat{M}_\alpha$'s configuration.

$\hookrightarrow O(|Q| \log |Q|)$

(4) Execute transition function

- rewrite current state
- move WT1 & WT2 heads

$$\delta : \underbrace{Q \times \Gamma^3}_{|Q| \times |\Gamma|^3} \to \underbrace{Q \times \Gamma^2 \times \{L, R, S\}^3}_{(q,\ \gamma_1,\ \gamma_2,\ \omega)}$$

$|\Gamma|^3$: const

q: $\log |Q|$ — γ_1: const — γ_2: const

$O(|Q| \log |Q|)$
table size (in bits)

TM's and Languages

- A language $L \subset \{0,1\}^*$

$$f_L : \{0,1\}^* \to \{0,1\}$$

$$f(x) = 1 \iff x \in L$$
$$f(x) = 0 \iff x \notin L$$

Turing Recognizable:

A lang L is Turing Recognizable if $\exists$ TM M_L s.t.

$$\forall x \in L, \quad M_L(x) = 1$$

Turing Decidable:

A lang L is Turing Decidable if $\exists$ TM M_L $\forall x \in \{0,1\}^*$

1. $M_L(x)$ halts
2. $M_L(x)=1 \longleftrightarrow x \in L$
3. $M_L(x)=0 \longleftrightarrow x \notin L$

Thm: $\exists$ a language $L_{UC} \subset \{0,1\}^*$ that is undecidable.

Proof: Define language

$$\hat{L} = \{ \alpha \mid M_\alpha(\alpha) = 1 \}$$

Now define

$$L^* = \overline{\hat{L}}$$

$$= \{\alpha \mid \alpha \notin \hat{L}\}$$

Claim: L^* is undecidable.

Suppose not. So L^* is decidable. Thus $\exists$ TM M^* which decides L^*.

$\hookrightarrow$ $\forall \alpha \in L^*$: $M^*(\alpha) = 1$

$\forall \alpha \notin L^*$: $M^*(\alpha) = 0$

- Consider $M^*(\langle M^* \rangle)$

$$M^*(\langle M^* \rangle) = 1$$

$\Leftrightarrow \langle M^* \rangle \in L^*$

$\Leftrightarrow \langle M^* \rangle \notin L$

$\Leftrightarrow M^*(\langle M^* \rangle) = 0$

or never halts ↯

Impossible $\Rightarrow L^*$ is

undecidable. □

Halting Problem

$$L_H := \{ \langle \alpha, x \rangle : M_\alpha(x) \text{ halts in finite \# of steps} \}$$

L_H is undecidable

- Suppose L_H is <u>decidable</u>.
Then $\exists M_H$ which decides L_H.

$\forall x, \alpha$:

$M_H(x, \alpha) = 1$ if $M_\alpha(x)$ halts;

$M_H(x, \alpha) = 0$ if not.

We'll now build a decider for L_{UC}

- Define M_{UC} as

$M_{UC}(\alpha)$:

- Simulate

$M_H(\alpha, \alpha)$.

$$\text{pad}(x, k) = x\#^j$$

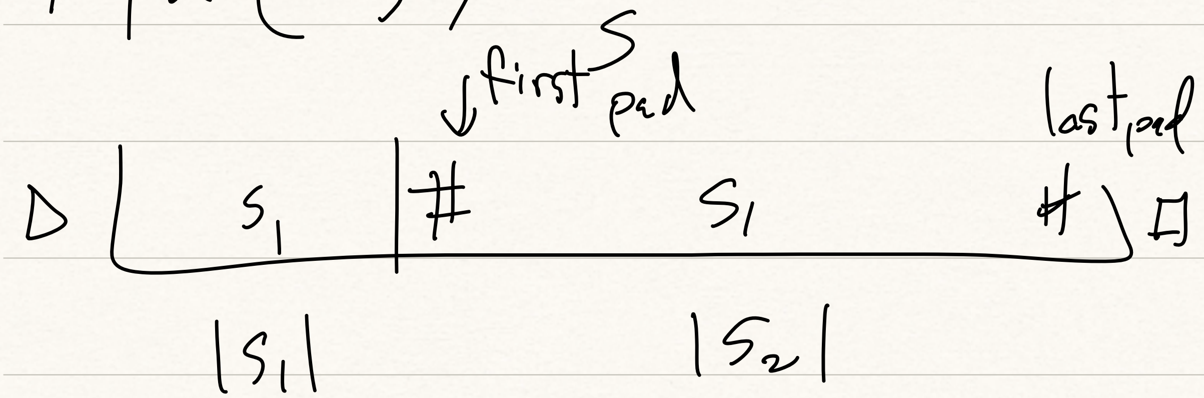
$$j = \max(0, k - |x|)$$

$$A \in \text{DTIME}(n^6)$$

$$\text{pad}(A, n^2) \in \text{DTIME}(n^3)$$

$$= \{ \text{pad}(x, |x|^2) : x \in A \}$$

$M_{pad}(s)$



1. Check all of s_2 is #
2. check that

$$|s_2| = |s_1|^2 - |s_1|$$

2. Check that

$$s_1 \in A.$$

Takes $O(|s_1|^6)$ time.

Let $|S| = n.$

$|S_1| + |S_2| = n$

$|S_1| + |S_1|^2 - |S_1| = n$

$\Rightarrow |S_1|^2 = n$

$\Rightarrow |S_1| = \sqrt{n}$

$\Rightarrow$ Decide $s_1 \in A$

in time $|S_1|^6$

$= |S_1|^{6/2} = |S_1|^3$

$= O(n^3)$ ✓

$HALT = \{ (\alpha, x) : \underset{\text{halts}}{M_\alpha(x)} \}$

Undecidable :

Suppose M_H decides $HALT$.
Define $\hat{M}$ s.t.

$$\hat{M}(\alpha) = \begin{cases} 1 \text{ if } M_H(\alpha,\alpha) = 0 \\ \text{loop forever if } M_H(\alpha,\alpha) = 1 \end{cases}$$

Run :

$\hat{M}(\langle \hat{M} \rangle)$

|

↳ Runs $M_H(\langle\hat{M}\rangle, \langle\hat{M}\rangle)$

If $M_H(\langle\hat{M}\rangle, \langle\hat{M}\rangle) = 1$
then $\hat{M}(\langle\hat{M}\rangle)$
loops forever
But $M_H(\langle\hat{M}\rangle, \langle\hat{M}\rangle) = 1$
means $\hat{M}(\langle\hat{M}\rangle)$ halts. ↯

If $M_H(\langle\hat{M}\rangle, \langle\hat{M}\rangle) = 0$
Then $\hat{M}(\langle\hat{M}\rangle)$ halts
w/ $\underline{1}$.
But $M_H = 0$ implies
it doesn't halt ↯