Cook-Levin

Lecture 5

Single-tape NTM

$\hookrightarrow \phi = \phi_{start} \wedge \phi_{accept} \wedge \phi_{cell} \wedge \phi_{move}$

Correctly captures correctness of the NTM

Claim: If the table has a correct starting config, and all 2×3 windows are legal, then

$row[i] \rightarrow row[i+1]$

is a valid transition

- Consider two rows,

upper config =

lower config q_2

c	b	c

center is the same

NO STATE

#		
#		

		#
		#

a	q_1	b
a	q_2	b

a	q_1	b
a	q_2	c

a	q_1	b
q_2	a	b

a	q_1	b
a	d	q_3

$$(n^{2k})$$

$$\Phi_{move} = \bigwedge_{\substack{1 \le i < n^k \\ 1 < j < n^k}} \left[(i,j)\text{-th window is legal} \right]$$

$$\begin{array}{c|ccc} & j-1 & j & j+1 \\ \hline i & a_1 & a_2 & a_3 \\ i+1 & a_4 & a_5 & a_6 \end{array}$$

const

$$\bigvee_{\binom{|C|}{6}} \Big(x_{i,j-1,a_1} \wedge x_{i,j,a_2} \wedge$$

$a_1, \dots, a_6 \in C$ s.t. the window w/ these is legal

$$\left(X_{i,j+1,a_3} \wedge X_{i+1,j-1,a_4} \wedge X_{i+1,j,a_5} \wedge X_{i+1,j+1,a_6} \right)$$

this is where you pick $a_1, \dots, a_6$ s.t.

$$\boxed{\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & a & a_6 \end{matrix}}$$

is a legal window

$$\phi = \phi_{start} \wedge \phi_{cell} \wedge \phi_{accept} \wedge \phi_{move}$$

is SAT iff. NTM accepts

Table: $n^k \times n^k$

ϕ is constructed in polynomial time

ϕ_{start} : n^k literals
$\to O(n^k)$

ϕ_{accept} : $O(n^{2k})$ size

ϕ_{cell} = $O(n^{2k})$
$O((n^k)^2)$

Φ_{move}: $O(n^{2k})$

$|\phi| = O(n^{2k})$ $\square$

NP-Complete Problems

Thm: If A is NP-comp. and $B \in NP$ satisfies $A \leq_p B$. Then B is NP-complete.

Proof:

Know: If

—

$X \le_p Y$, $Y \le_p Z$

then $X \le_p Z$.

- A is NP Complete
  - $A \in NP$
  - $\forall L \in NP$,

    $L \le_p A$
  - $A \le_p B$

$\Rightarrow L \le_p B$ $\square$

<u>3SAT</u>:

- CNF formula

  ↳ conj. normal form

$$\phi = \phi_1 \land \phi_2 \land \dots \land \phi_n$$

n clauses

$\forall \phi_i$

$$\phi_i = (x_i \vee \overline{x}_{i+1} \vee x_3)$$

$$\phi_2 = x_1$$

$$\phi_3 = (x_2 \vee \overline{x}_3)$$

<u>k-CNF</u>:

$$\phi = \phi_1 \wedge \dots \wedge \phi_n$$

s.t. ϕ_i contains exactly k-literals

& ϕ is CNF

<u>3SAT</u> $= \{ \langle \phi \rangle : \phi$ is 3CNF and satisfiable $\}$

<u>Ex</u>

$$\phi = (x_1 \vee x_3 \vee \overline{x_2}) \wedge (x_4 \vee x_5 \vee \overline{x_1})$$

<u>Thm</u>: 3SAT is NP-complete.

- 3SAT $\in$ NP ✓
- 3SAT is NP-hard

① SAT $\leq_p$ 3SAT

② Modify the proof for SAT (easier)

$$\phi_{start} = X_{1,1,\#} \wedge X_{1,2,q_0} \wedge X_{1,3,u_1} \wedge \cdots \wedge X_{1,n+2,u_n} \wedge X_{1,n+3,\sqcup} \wedge \cdots \wedge X_{1,n^k-1,\sqcup} \wedge X_{1,n^k,\#}$$

↳ in CNF form ✓

$$\phi_{accept} = \bigvee_{1 \leq i,j \leq n^k} X_{i,j,q_{accept}}$$

/

↳ Big OR

↳ one clause of a CNF

$$\phi_{cell} = \bigwedge_{1 \le i,j \le n^k} \Big[\underbrace{\Big(\bigvee_{s \in C} x_{i,j,s} \Big)}_{\text{ors}} \wedge \underbrace{\Big(\bigwedge_{\substack{s,t \in C \\ s \ne t}} \underbrace{(\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}})}_{\text{ors}} \Big)}_{\text{ands}} \Big]$$

↳ already in CNF

$$(a \vee b \vee c \vee d \vee e) \wedge (\bar{a} \vee \bar{b}) \wedge (\bar{c} \vee \bar{d})$$

$$\phi_{move} = \bigwedge_{i,j} \bigvee_{\substack{a_1,\dots,a_6 \\ \text{that are} \\ \text{legal}}} \left(a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5 \wedge a_6 \right)$$

$$\bigvee \left(\bigwedge \right) \longleftrightarrow \bigwedge \left(\bigvee \right)$$

Boolean equivalence

↳ convert ϕ_{move} to CNF

- ϕ is a CNF
  ↳ Turn into 3CNF

$\phi = \phi_1 \wedge \ldots \wedge \phi_m$

Look at ϕ_i $\forall i$:

- If ϕ_i has 3 literals, done ✓
- If ϕ_i has < 3 literals
  ↳ give it 3 literals

<u>Ex</u> $x_1 \mapsto (x_1 \vee x_1 \vee x_1)$

$x_1 \vee x_2 \longmapsto (x_1 \vee x_1 \vee x_2)$

$\mapsto (x_1 \vee x_2 \vee x_2)$

- If ϕ_i has > 3 literals
  - Split into 3CNFs using extra variable

Ex $\quad (\overset{1}{x_1} \vee \overset{0}{x_2} \vee \overset{0}{x_3} \vee \overset{0}{x_4})$

$$\hookrightarrow (\overset{1}{x_1} \vee \overset{0}{x_2} \vee \underset{0}{z}) \wedge$$

$$(\underset{1}{\bar{z}} \vee \underset{0}{x_3} \vee \underset{0}{x_4})$$

General statement $\ell \geq 4$

$$\phi_i = (a_1 \vee a_2 \vee \dots \vee a_\ell)$$

- Use $\ell - 3$ new variables $z_1, \dots, z_{\ell-3}$
- Transform into $\ell - 2$ 3CNF clauses

$$(a_1 \vee a_2 \vee z_1) \wedge (\bar{z}_1 \vee a_3 \vee z_2)$$
$$\wedge (\bar{z}_2 \vee a_4 \vee z_3) \wedge \dots$$
$$\wedge (\bar{z}_{\ell-3} \vee a_{\ell-1} \vee a_\ell) \qquad \square$$

INDSET

$\text{INDSET}_k =$

$\{ [G = (V, E), k] :$

$\exists S \subset V$ s.t. $|S| \geq k$

and $\forall u, v \in S$,

$(u,v) \notin E \}$

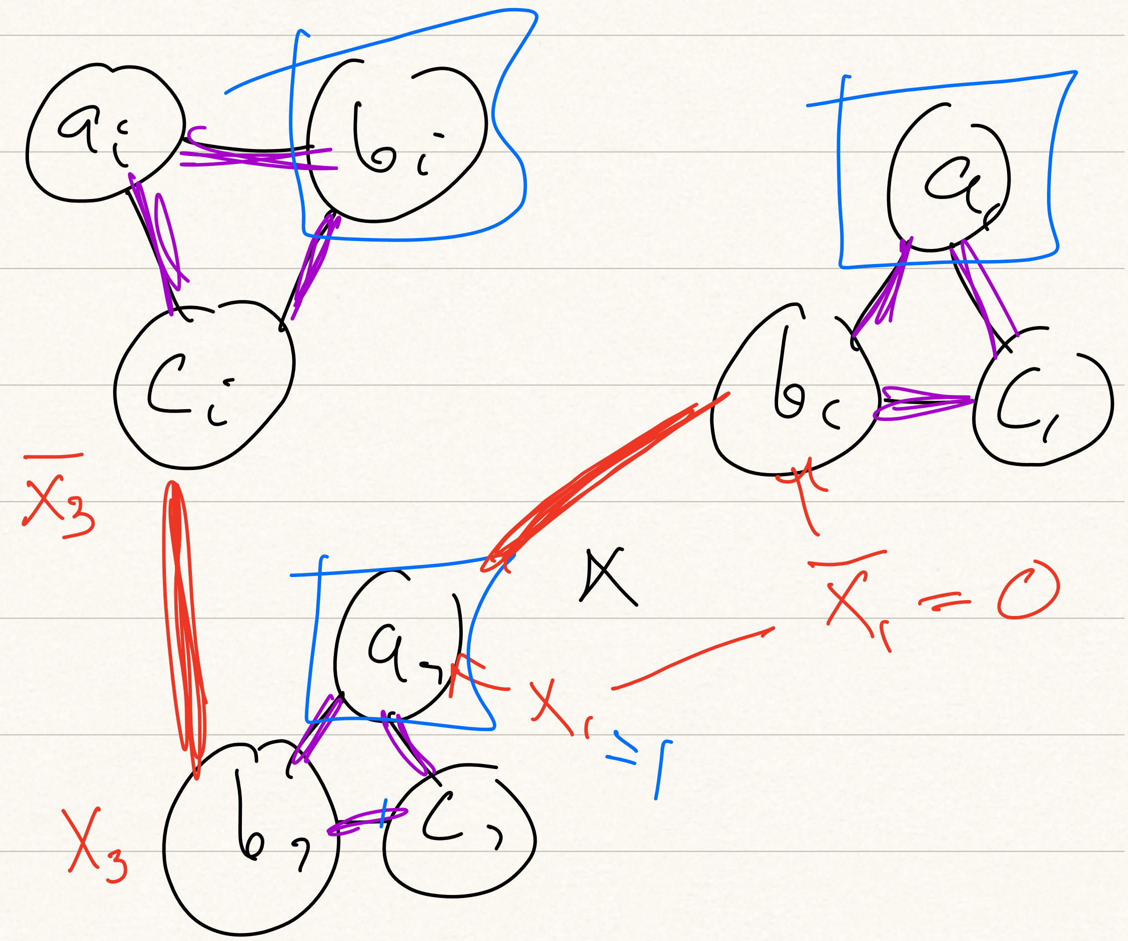
- $3SAT \leq_p \text{IND}_k$

$\phi = \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_k$

$(a_1 \vee b_1 \vee c_1) \wedge \cdots - - \wedge$

For each C_i:

- Create nodes a_i, b_i, c_i in G



- Connect every node in a clause

- Connect all nodes where it's a literal and its negation

$a_1 = x_1$

$b_1 = \bar{x}_1$

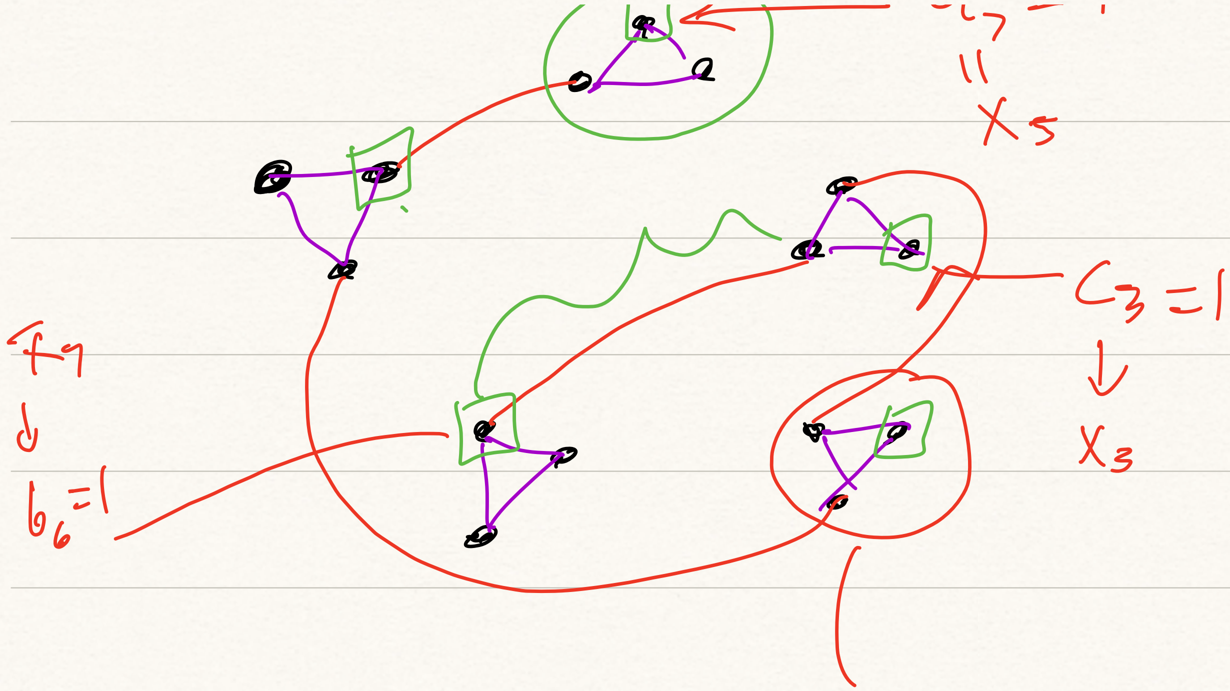
- ϕ is 3SAT iff G has k-ind set

- ϕ is 3SAT $\rightarrow$ find k-ind set

- ϕ being SAT
  $\Rightarrow$ at least 1 literal
  per clause $= 1$
- Choose one such literal
  from each of the
  k clauses
- Choose the corresponding
  node in the graph
- These nodes are of
  size k
- this is an ind. set.
  by picture above ✓

- k ind-set $\Rightarrow$ SAT assignment

$a = 1$



$$\phi_5 = (\underline{a_5} \vee b_5 \vee c_5)$$

labels

$$(x_4 \vee \bar{x}_3 \vee x_{15})$$

$$\phi(x_1, x_2, \ldots, x_n)$$

3SAT

↓

ind set

↓

$$\boxed{A \le_p B}$$

$$x \in A \iff f(x) \in B$$

reduction

$$L \le_p SAT$$

$$SAT \le_p L$$

$$IND \le_p 3SAT$$

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