Lecture 7

Diagonalization

Say I have complexity classes C_1 and C_2.

- Want to show

$$C_1 \neq C_2$$

- Diagonalization
  - $L_1 \in C_1$ $\qquad$ $L_2 \in C_2$

    $\uparrow$ lang. $\qquad$ $\uparrow$ lang

→ If M_1 decides L_1, want to say M_1 is different from ANY Decider M_2 for L_2.

~ $\forall x$ if $M_2(x) = 1$ then $M_1(x) = 0$

$\rightarrow M_2(x) = 0$

then $M_1(x) = 1$

- Classic Diag. Showing that

$$|\underline{\mathbb{N}}| < |\mathbb{R}|$$

countable

$$|\mathbb{N}| = |\mathbb{Z}|$$

$0 \mapsto 0$

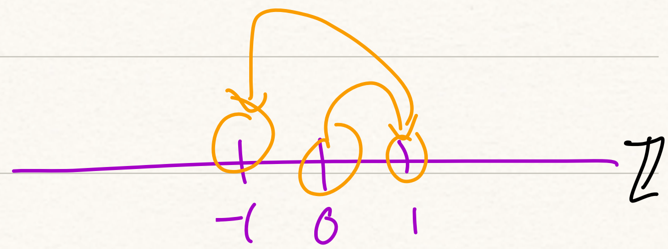
$1 \mapsto 1$

$2 \mapsto -1$

$3 \mapsto 2$

$4 \mapsto -2$

$\vdots$



- $|\mathbb{N}| < |\mathbb{R}|$ **theorem**
- $|\mathbb{N}| < |[0,1]|$

__Proof__: Suppose not.

$$|\mathbb{N}| = |[0,1]|$$

$\mathbb{N}$	$[0,1]$
0	$0.\boxed{1}2345$
1	$0.1\boxed{4}159$
2	$0.23\boxed{3}33$
3	$0.1\quad\boxed{0}$
4	$0.6732\boxed{1}$
	$\vdots$

$$r = 0.\boxed{2}76\boxed{1}4 \ldots$$

r is <u>not</u> in this table ✓

<u>Time Hierarchies</u>

<u>Thm</u>: Let f, g be time constructible such that

$$f = o\left(g / \log(g)\right).$$

Then

$$\mathrm{DTIME}(f) \subsetneq \mathrm{DTIME}(g)$$

- <u>Cor.</u> $\exists$ a language decidable in time f but not in time

$o(f/\log(f))$

<u>Proof</u>: Let D be a TM which does the following.

D (x) : $x \in \{0,1\}^n$

- Compute $\underbrace{k = \left\lceil \frac{g(n)}{\log(g(n))} \right\rceil}_{O(g(n)) \text{ time}}$
- Simulate $M_x(x)$
- * If $M_x(x)$ halts within k-steps,

  * output $1 - M_x(x)$
- Else output 0.

Let $A = L(D)$
$= \{x \mid D(x) = 1\}$

(1) D is a decider w/ lang A.

(2) $A \in \mathrm{DTIME}(g)$
$\to$ $D(x)$ decides in time $g(|x|)$

Claim: $A \notin \mathrm{DTIME}(f)$
for any $f = o\left(g/\log(g)\right)$

Suppose not. So A is $(A \in \mathrm{DTIME}(f))$
decidable in $O(f)$ time.

$\Rightarrow \exists M_A$ s.t. M_A decides A in time $O(f)$.

$x \in A \longleftrightarrow M_A(x) = 1$ in time $O(f(|x|))$

What happens with

$D(\langle M_A \rangle)$? $\rightarrow$ still runs in time $g(n)$

By construction,

$M_A(\langle M_A \rangle)$ halts in time

$$f(\underbrace{|\langle M_A \rangle|}_{\parallel \atop m})$$

$$f(m) = o\left(g(m) / \log(g(m))\right)$$

$$< \left\lceil \frac{g(m)}{\log(g(m))} \right\rceil$$

$\Rightarrow$ when D simulates
$\underline{M_A(\langle M_A \rangle)}$

↳ halts in $f(m)$

$< \lceil g(m) / \log(g(m)) \rceil$

steps

$$\Rightarrow D(\langle M_A \rangle) = 1 - M_A(\langle M_A \rangle)$$

$$\Rightarrow D(\langle M_A \rangle) \neq M_A(\langle M_A \rangle)$$

- But we assumed M_A decides A
- By construction, D decides A

$\Rightarrow$ M_A does not exist! $\square$

M_A runs in time $O(f)$

$\Rightarrow$ Simulate M_A in time $O(f\log(f))$

$$f = o\left(\frac{g}{\log(g)}\right)$$

$$\log(f) = o(\log(g) - \log\log(g))$$

$$= o(\log(g))$$

$f \log(f) =$

$$o\left(\frac{g}{\log(g)} \cdot \log(g)\right)$$

$$= o(g) \checkmark$$

$$< O(g)$$

Corollary: $\forall\, 1 \leq \varepsilon_1 < \varepsilon_2$

$$\mathrm{DTIME}(n^{\varepsilon_1}) \subsetneq \mathrm{DTIME}(n^{\varepsilon_2})$$

Example

$\varepsilon_1 = 1$

$\varepsilon_2 = 1.1$

$\varepsilon_2 = 1.0000 \cdots 01$

Corollary

$$P \subsetneq EXP$$

Thm. Let f, g be time-constructible such that

$$f(n+1) = o(g(n))$$

Then

$\mathrm{NTIME}(f) \subsetneq \mathrm{NTIME}(g)$.

<u>Proof</u>:

- For $i \in \mathbb{N}$, let M_i denote the machine given by $\langle i \rangle_2$.

- Let f be a function such that $f(1) = 2$, $f(i+1) = 2^{f(i)^2}$

- Let $g(n)$ be a function such that $f(n+1) = o(g(n))$

- Let D be an NTM which does the following

$$f(i+1) = 2^{f(i)^2}$$

$D(1^n)$:

- Compute i s.t. $\quad O(g(n))$

$$f(i) < n \le \underline{f(i+1)}$$

$\quad O(g(i+1))$

- If $\overset{f(i)}{\wedge}\, n < f(i+1)$
    - Simulate $M_i(1^{n+1})$ ← nondet sim.
    - run only for $g(n)$ steps.
    - If M_i halts within $g(n)$ steps, output $M_i(1^{n+1})$
    - Else output 1

$O(g(n))$ time

$D(1^n) = \hat{M}(1^{n+1})$

- If $n = f(i+1)$
  - Deterministically simulate $M_i(1^{f(i)+1})$ $f(i) < f(i+1) = 2^{f(i)^2}$
    $\to$ Deterministically try all computation paths $2^{f(i)}$ paths
  - Output $\underbrace{1 - M_i(1^{f(i)+1})}_{\log(n)+1}$. $= D(1^n)$

$|2^{f(i)^2} - f(i)|$

① D runs in time $O(g(n))$ $D($

Claim if $A = \mathcal{L}(D)$
then $A \notin \mathrm{NTIME}(f)$

Suppose not. Then $\exists$

NTM $\hat{M}$ s.t.

$$\mathcal{L}(\hat{M}) = \mathcal{L}(D)$$

and $\mathcal{L}(\hat{M}) = A \in NTIME(f)$

- Consider $|\langle \hat{M} \rangle_2| = n$

  s.t.

  $M_i = \hat{M}$ and

  $f(i) < n \leq f(i+1)$

Run $D(1^n)$:

- If $n < f(i+1)$

- Simulate $M_i(1^{n+1})$ only for $g(n)$ steps

$M_i(1^{n+1})$ runs in time $f(n+1) = o(g(n))$

$\Rightarrow$ Simulation halts before $g(n)$ steps

$\Rightarrow$ Output $M(1^{n+1})$

$$D(1^{\overset{n}{f(i)+1}}) = \hat{M}(1^{\overset{n+1}{f(i)+2}})$$

$$D(1^{\underset{n}{f(i)+2}}) = M_i(1^{f(i)+3})$$

$$\vdots \quad D(1^n) = \hat{M}(1^n)$$

$$D\left(1^{f(i+1)-2}\right) = \hat{M}_i\left(1^{f(i+1)-1}\right)$$

D		$\hat{M}$
$1^{f(i)+1}$	$=$	$\boxed{1^{f(i)+1}}$
	$=$	
$1^{f(i)+2}$	$=$	$1^{f(i)+2}$
	$=$	
$\neq$	$=$	$1^{f(i)+3}$
	$\vdots$	
$1^{f(i+1)-1}$	$=$	$1^{f(i+1)-1}$
	$=$	
$\boxed{1^{f(i+1)}}$	$=$	$1^{f(i+1)}$

By assumption,
D and $\hat{M}$ decide
the same language.

$\Rightarrow$

$$D(x) = \hat{M}(x) \ \forall x$$

$$\Rightarrow D(x) = D(x') \ \forall x$$
$$\hat{M}(x) = \hat{M}(x') \ \forall x$$

<u>BUT</u> If $n = \underline{f(i+1)}$

- $D(1^n)$ runs $D(1^{f(i+1)})$

$$\hat{M}(1^{f(i)+1}) \qquad ||$$

- outputs $\boxed{1 - \hat{M}(1^{f(i)+1})}$

$\Rightarrow \hat{M}$ does not exist!

$$D(1^{f(i+1)}) \neq \hat{M}(1^{f(i)+1})$$

$$D(1^n) \neq \hat{M}(1^{\log(n)+1})$$

$$\forall \; f(i) < n \leq f(i+1)$$

$$D(1^n) = 1 = \hat{M}(1^n)$$

If $D(1^n) = \hat{M}(1^n)$

and $D(1^n) = \hat{m}(1^{n+1})$

then

$D(1^n) = 1$ or zero

$f(x) = g(x) \ \forall x$

$f(x) = g(x+1) \ \forall x$

$\Rightarrow f(x) = c$ constant

$p(x) = \begin{cases} x & \text{and} \\ x+1 \end{cases}$

unless

$p(x) = c$

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