CS 594 – Advanced Crypto (Spring 2026)

Alex Block

Lecture 11

February 23, 2026

Zero-knowledge, Intuitively

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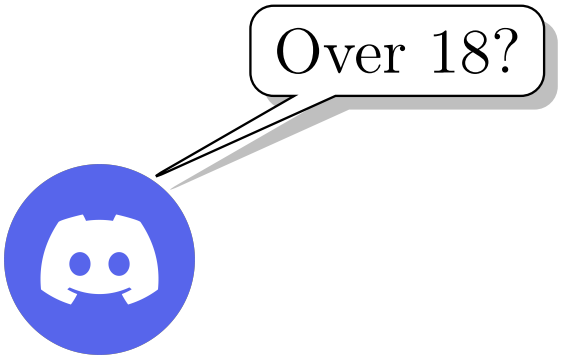

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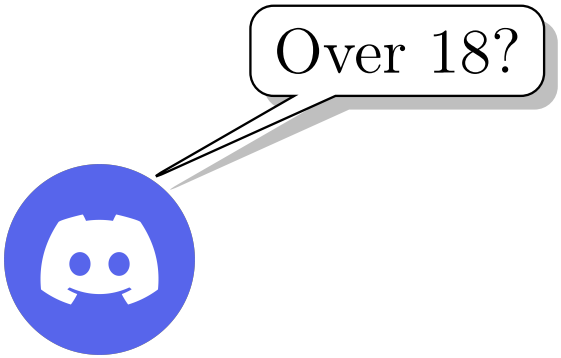

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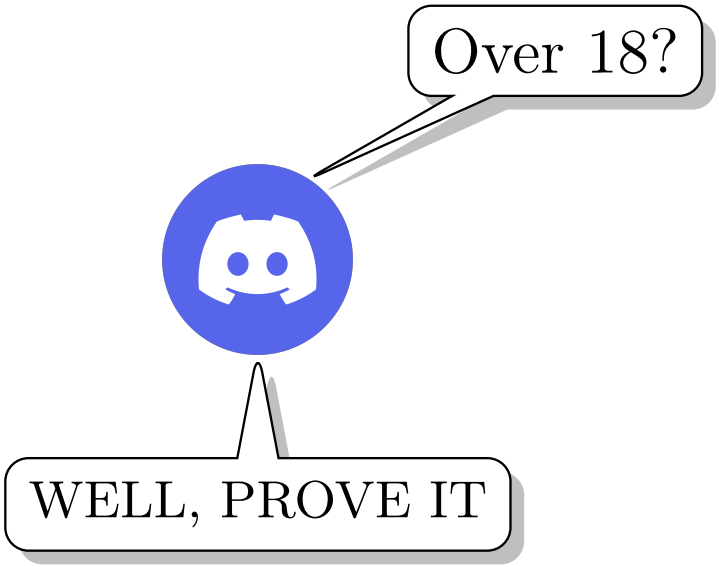

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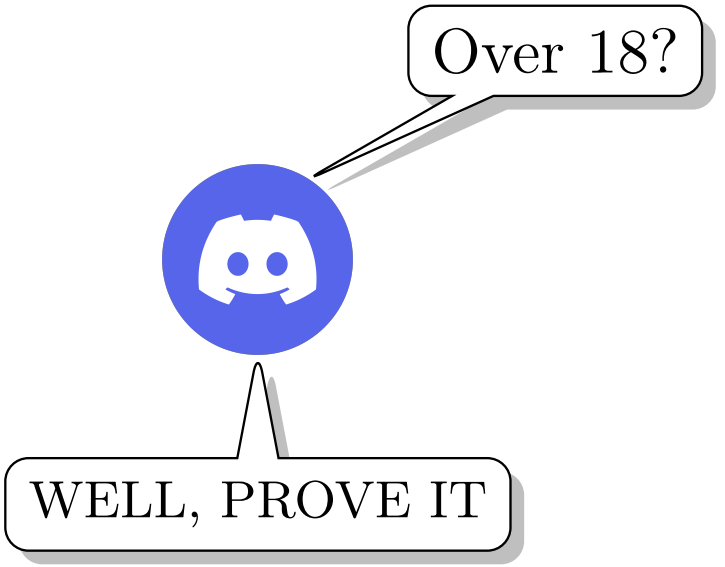

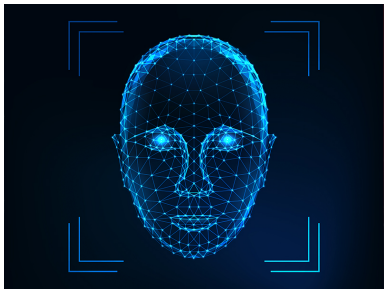

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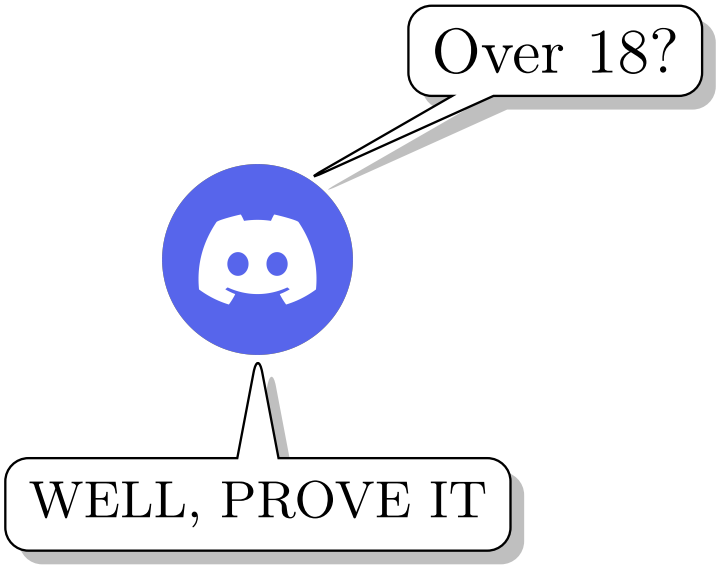

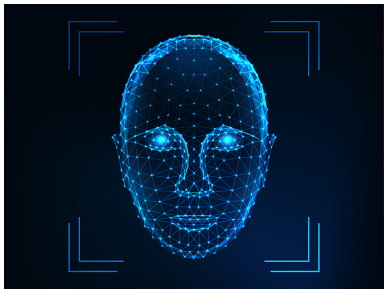

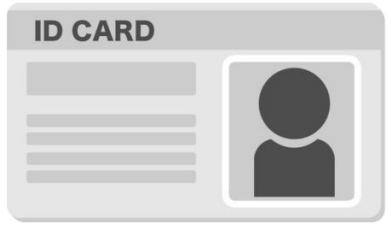

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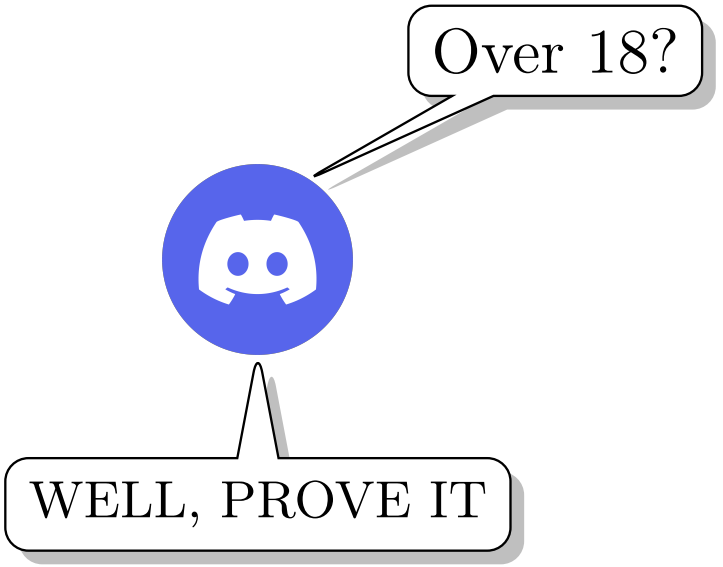

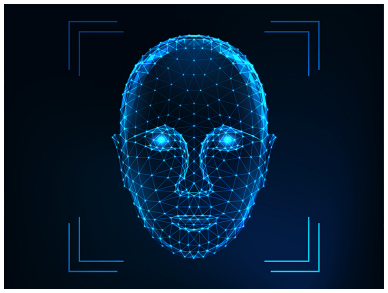

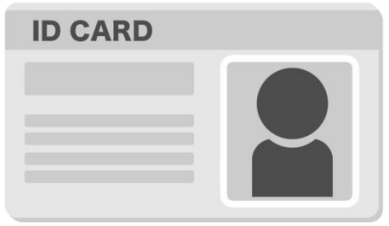

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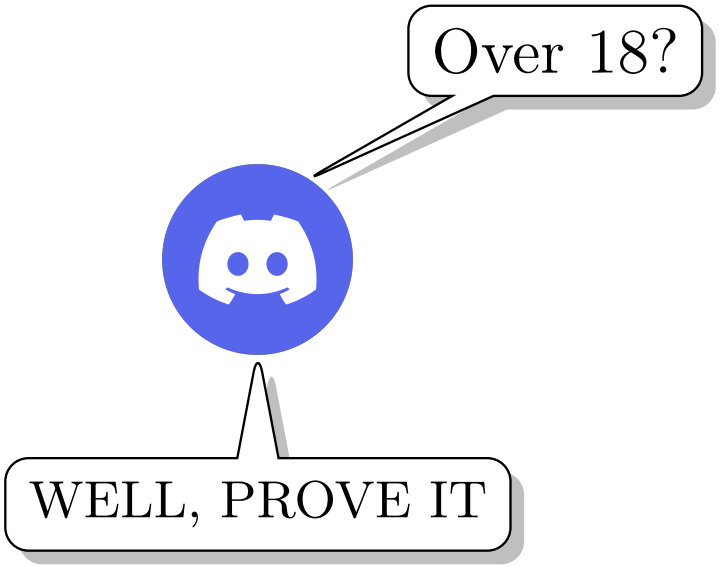

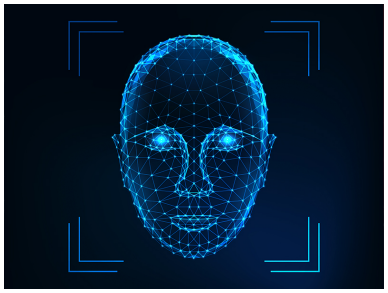

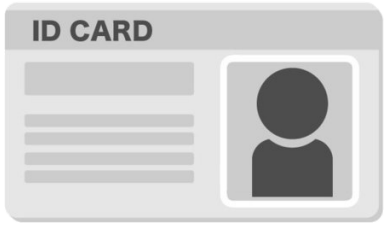

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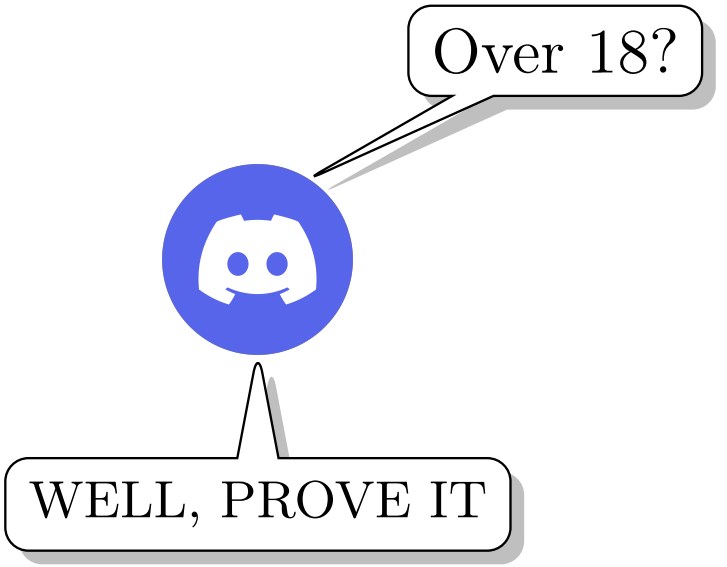

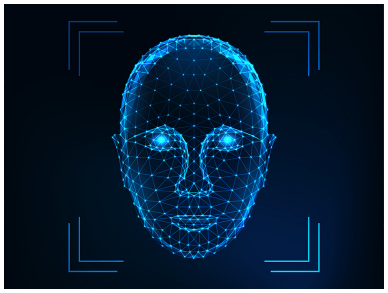

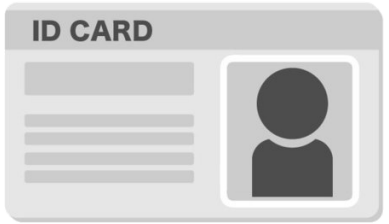

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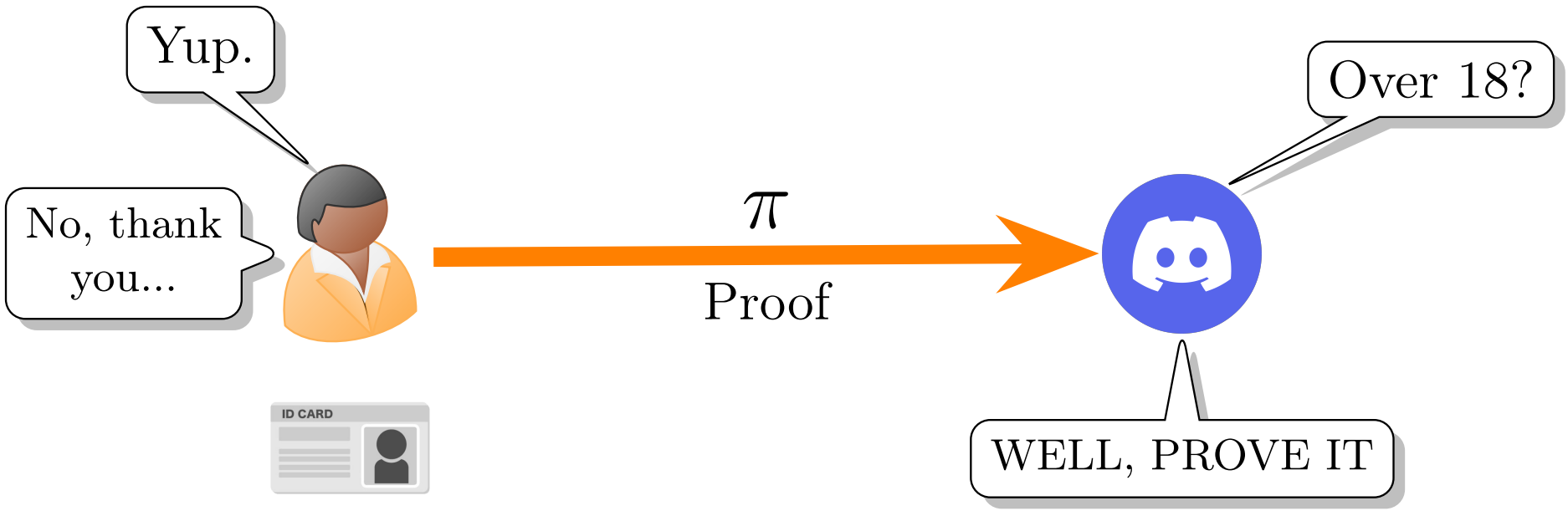

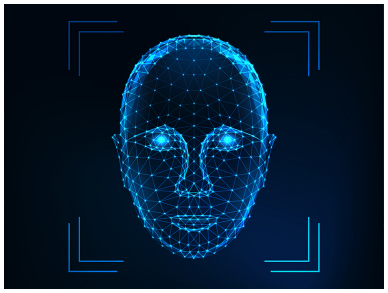

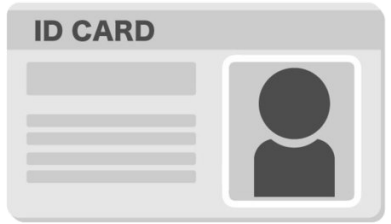

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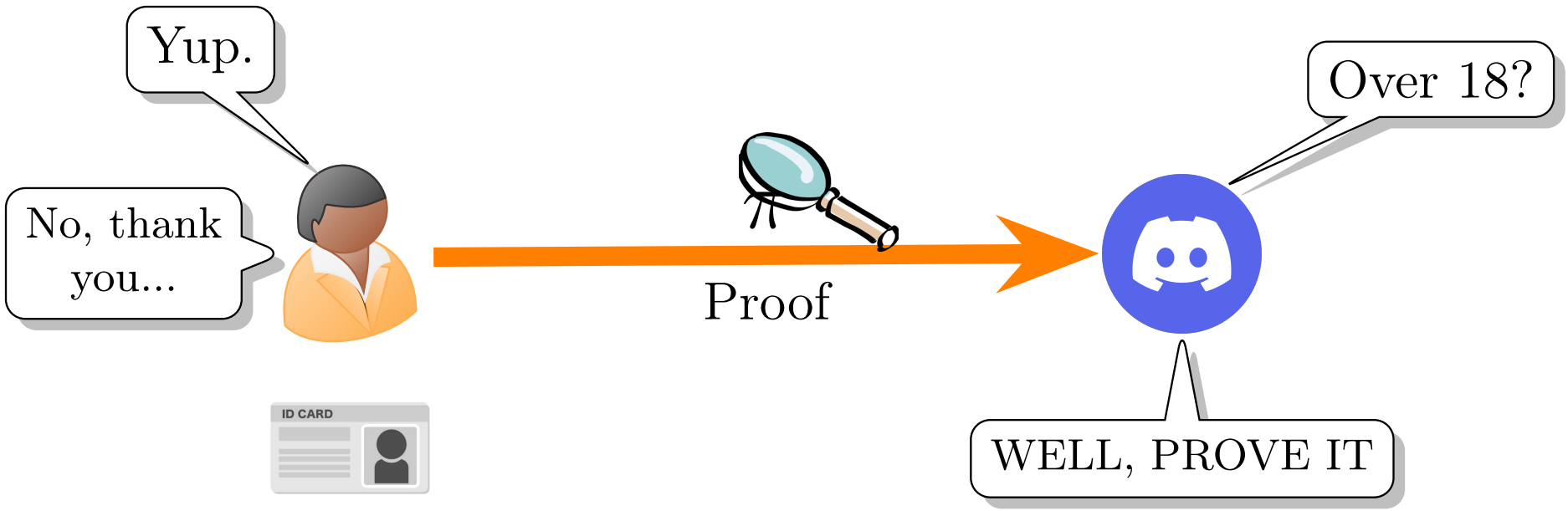

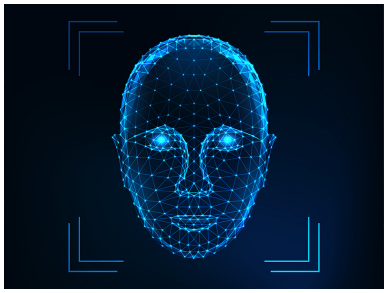

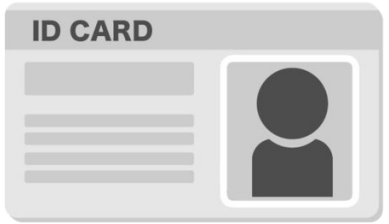

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1. Proves you are 18+
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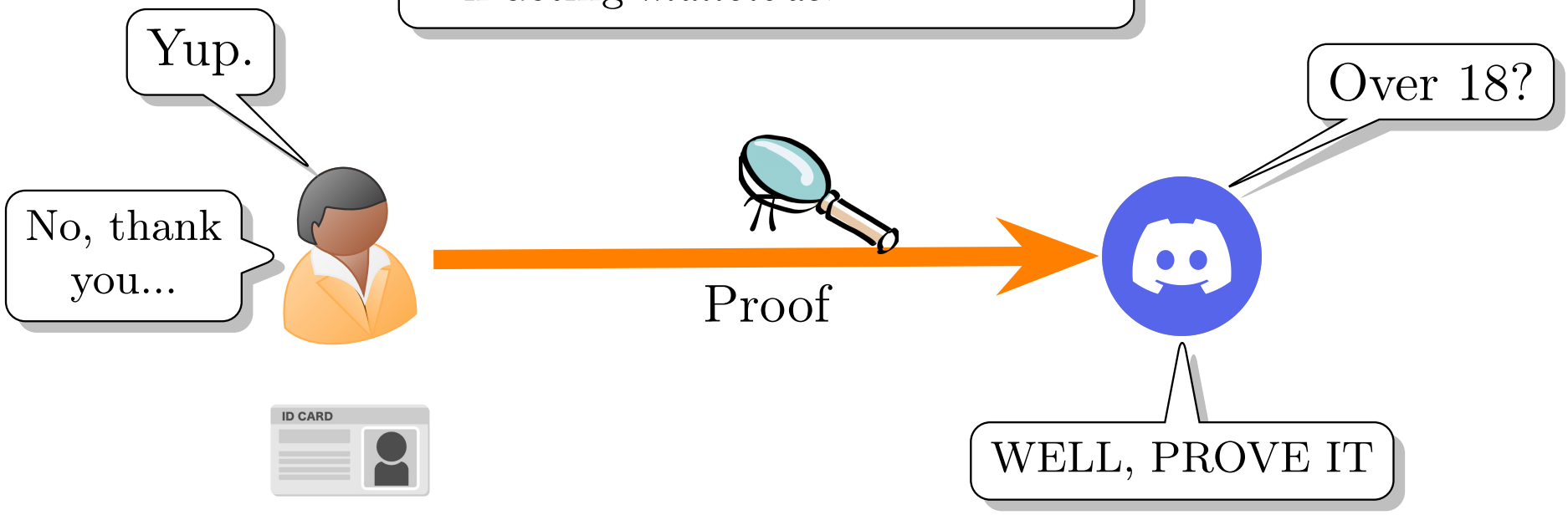

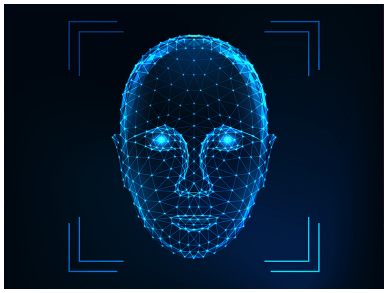

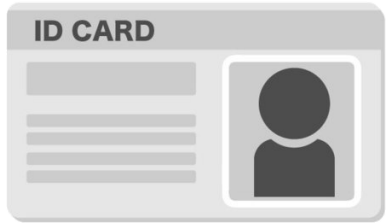

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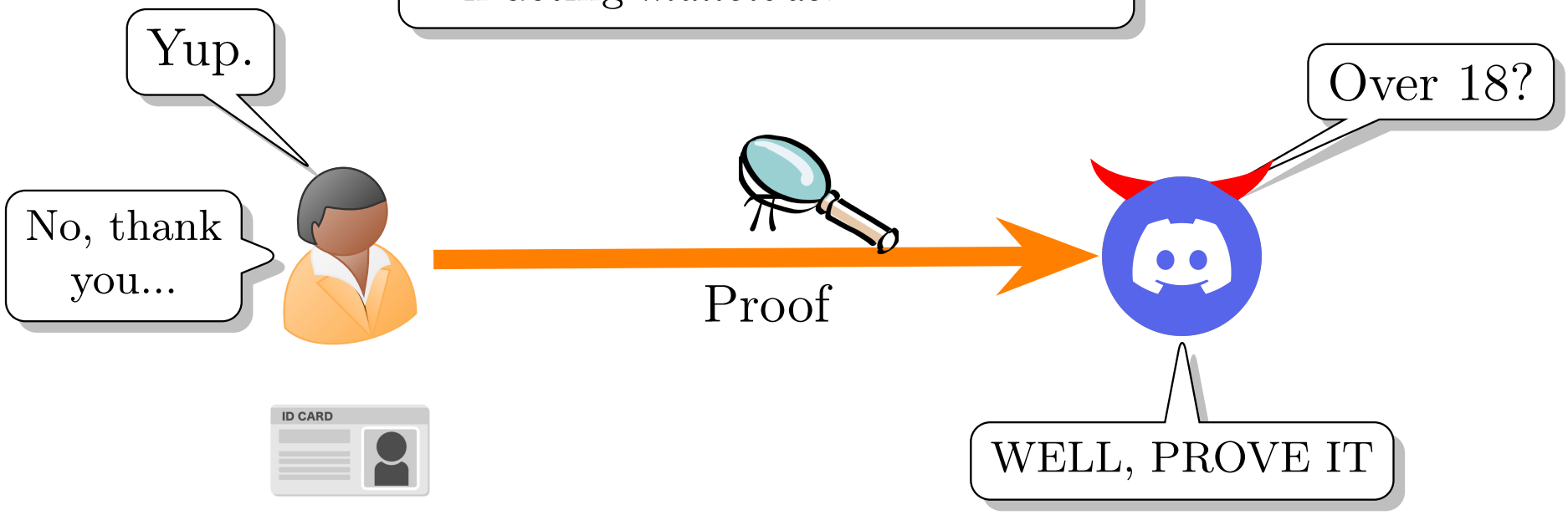

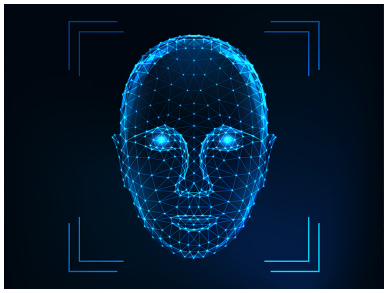

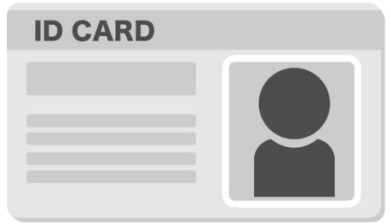

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- A *relation for language* L, denoted as R_L, is the set of all tuples (x,w) such that $M_L(x,w) = 1$.

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  - $\exists$ poly-time computable function f such that $x \in L' \iff f(x) \in L$.

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The "statements" we prove will be with respect to some fixed language.

Interactive Proofs and Arguments

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- *Interactive Proofs*: attempt to re-define **NP** using *interaction*.
- Goal: using interaction, for some fixed $x \in \{0,1\}^*$, can a (all-powerful) prover P with $w \in \{0,1\}^*$ such that $(x, w) \in R_L$ convince a (computationally-weak) verifier V that $x \in L$ while only exchanging $o(|w|)$ bits?

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$\chi = \phi(x_1, \ldots, x_n)$ $w = (a_1, \ldots, a_n)$ s.t. $\phi(a_1, \ldots, a_n) = 1$

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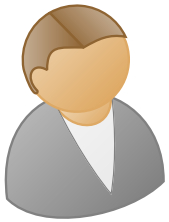

Prover P

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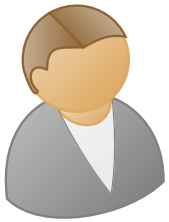

Prover P

Verifier V

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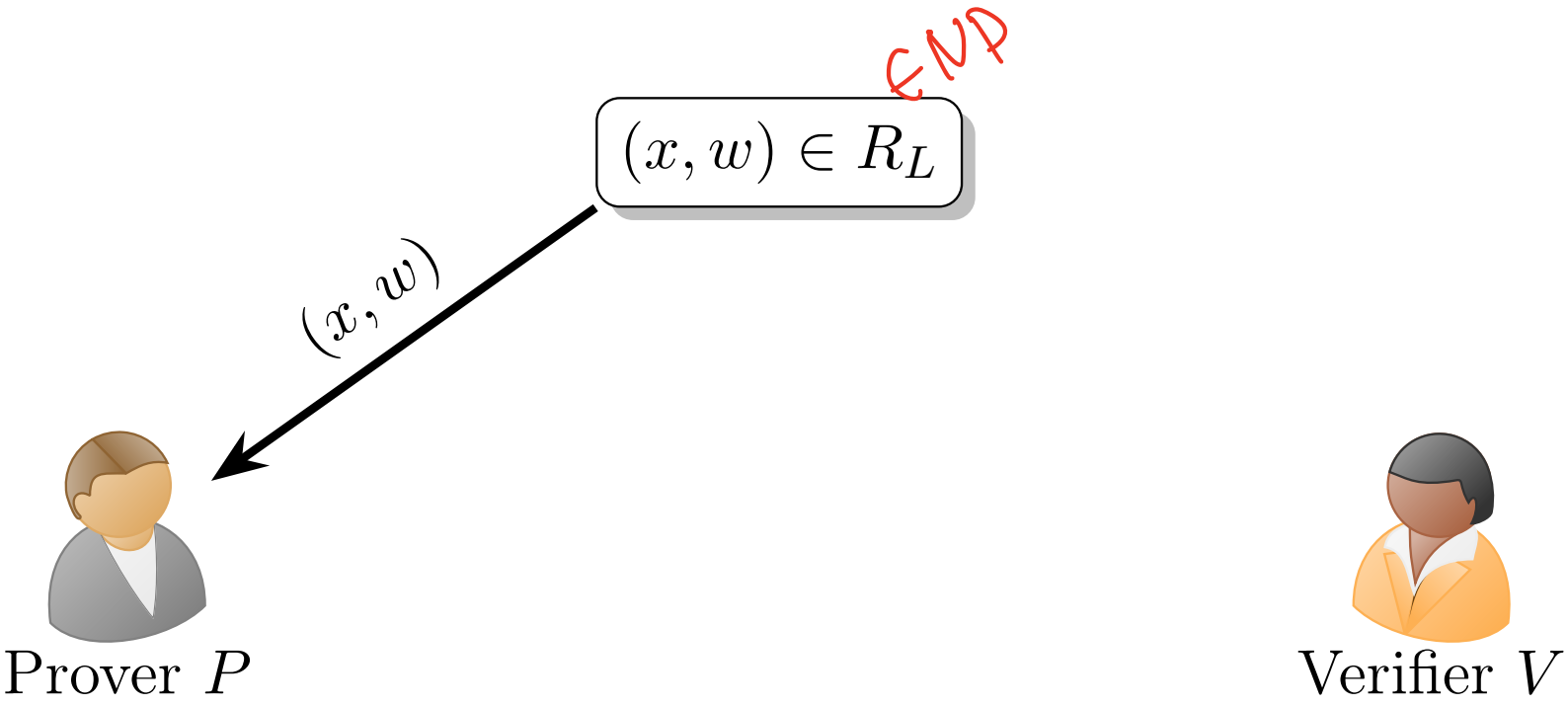

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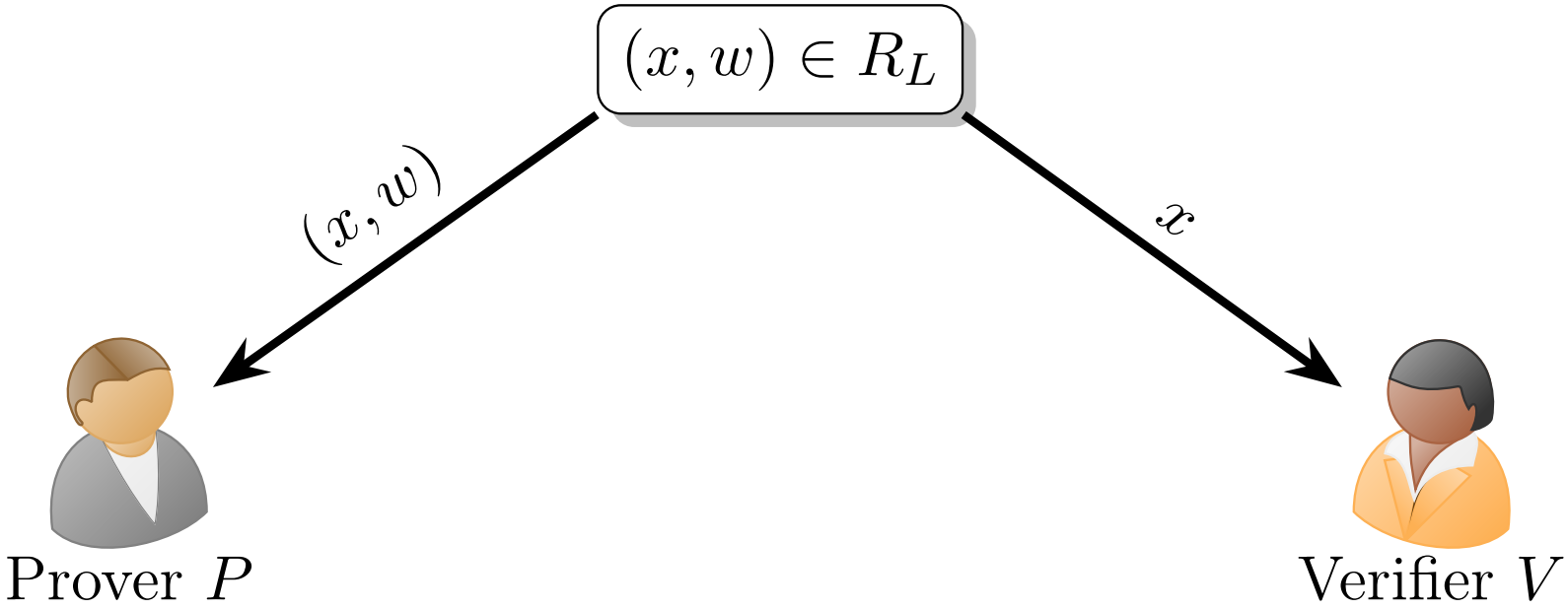

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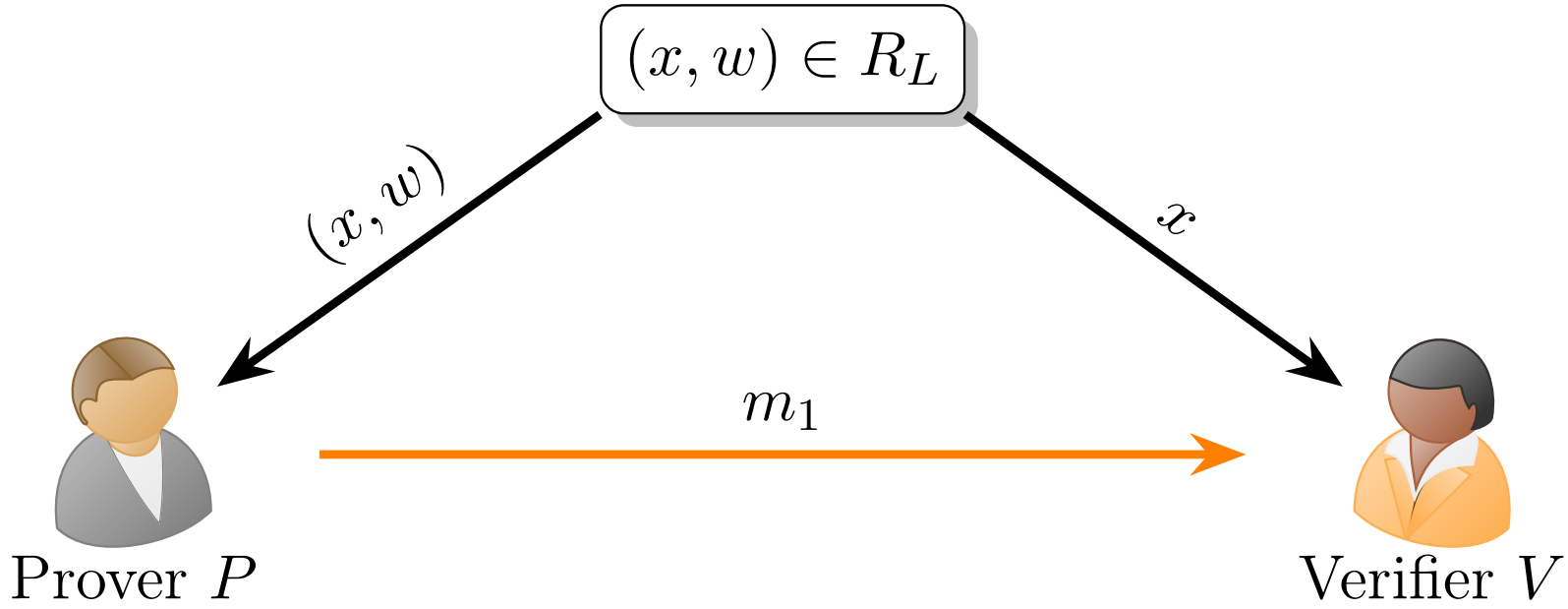

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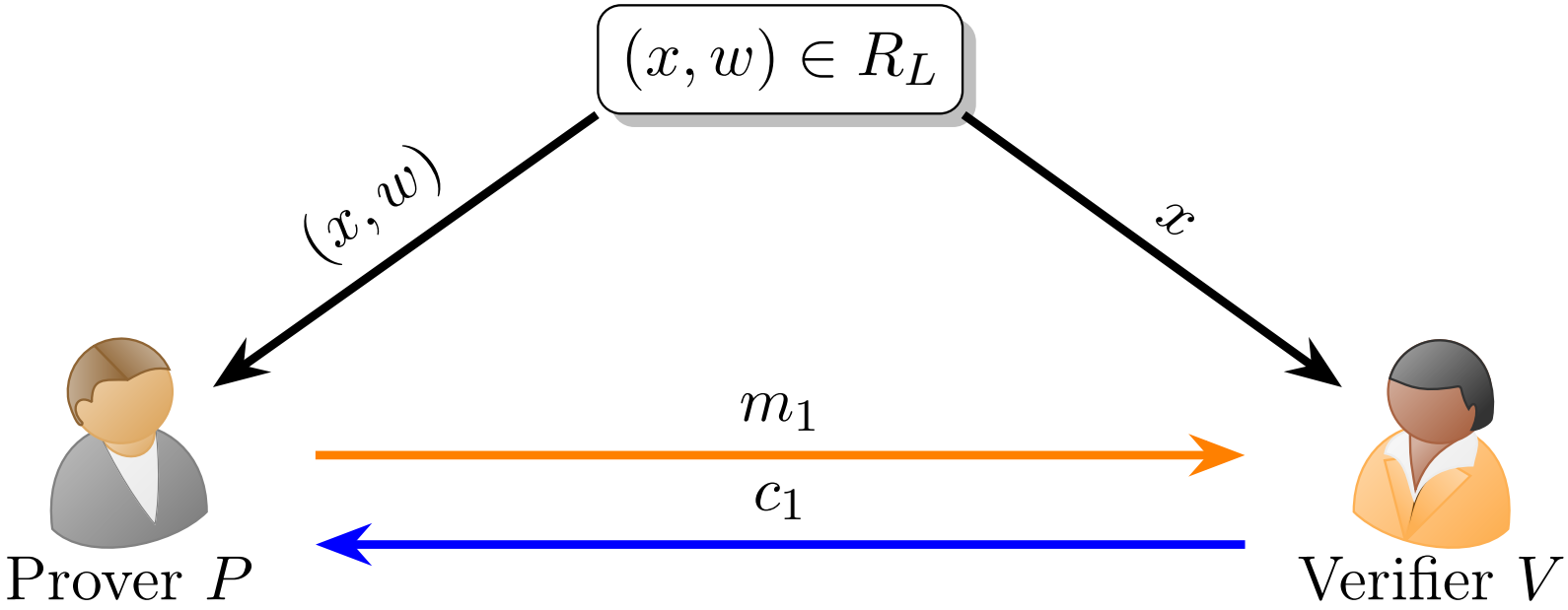

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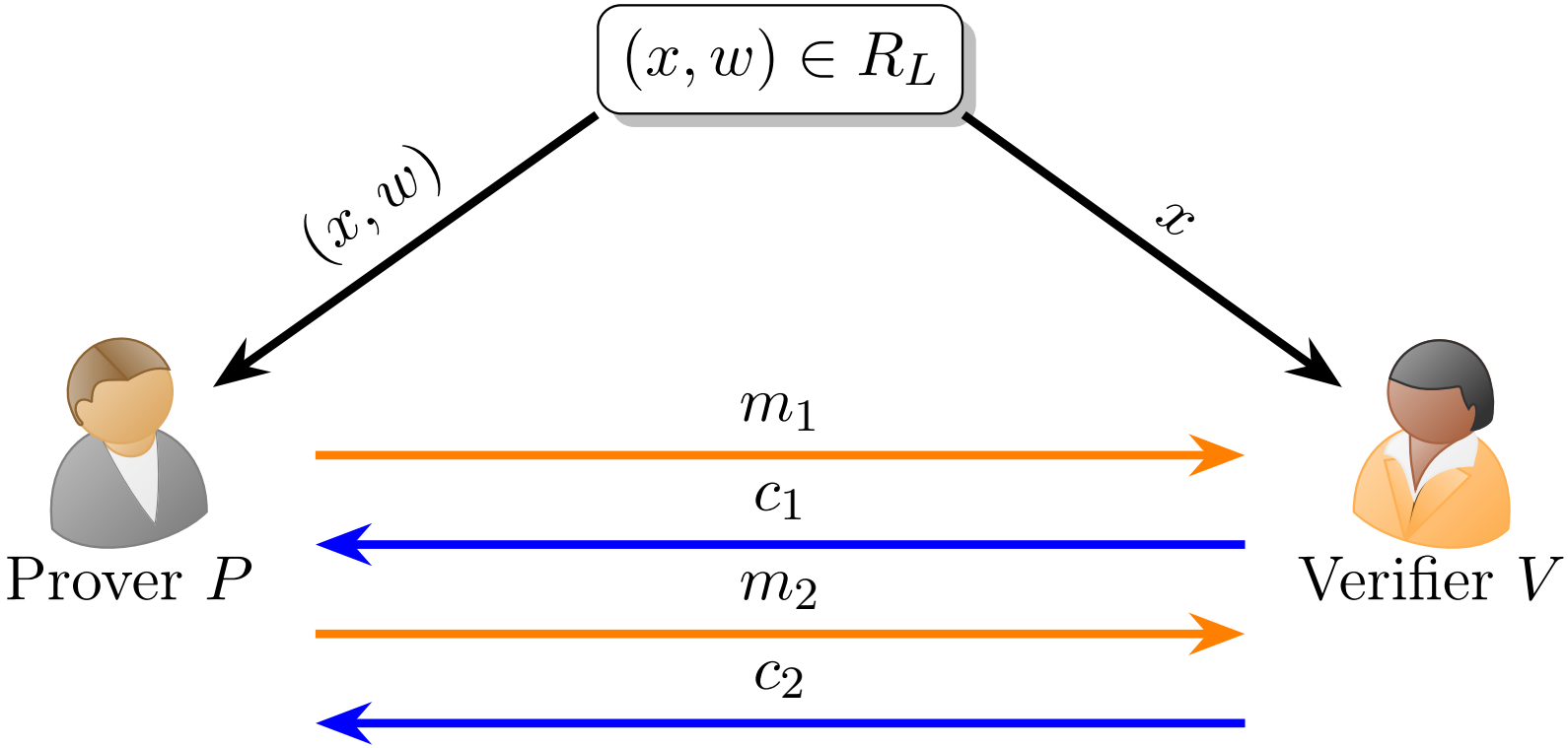

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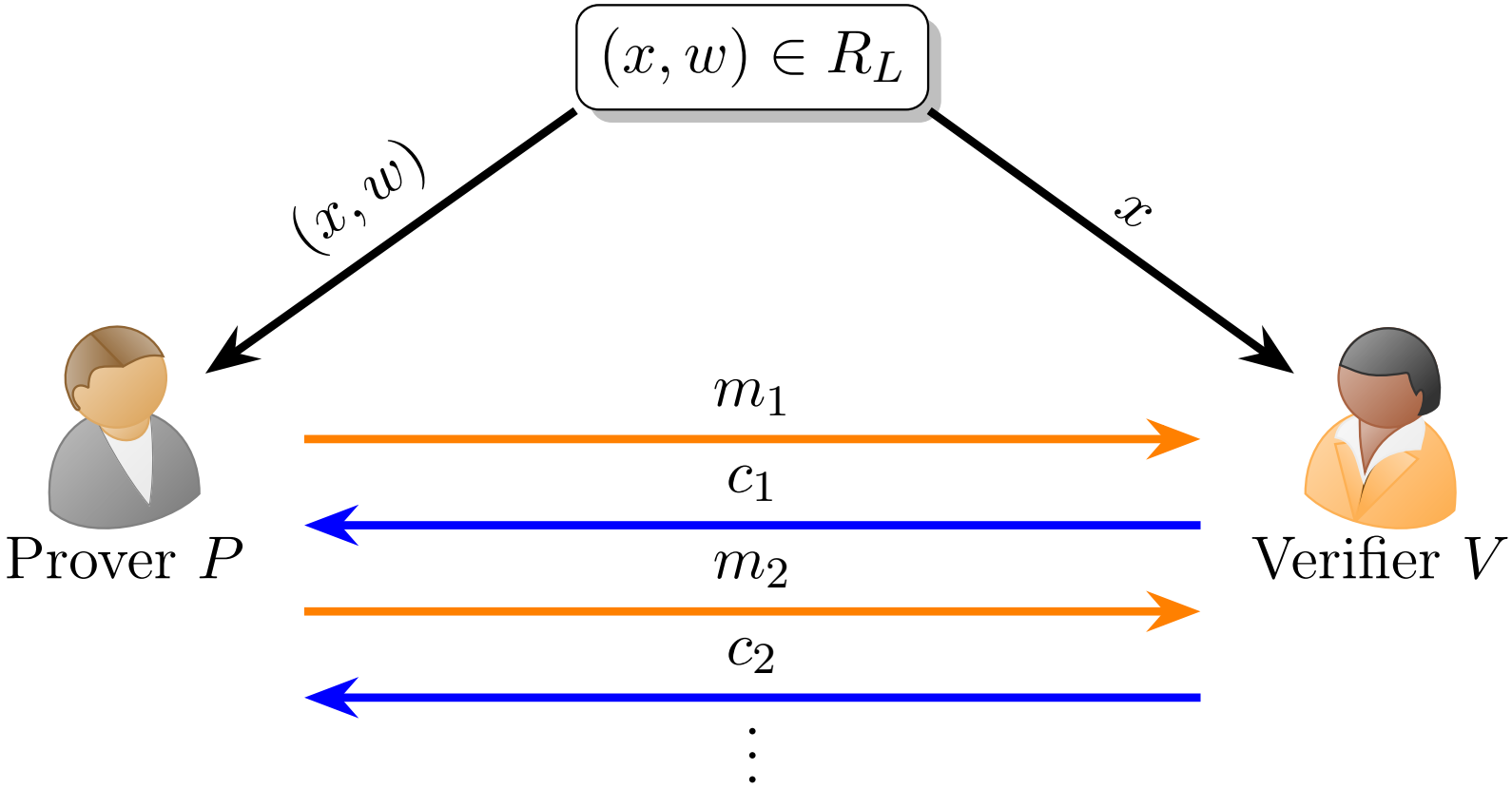

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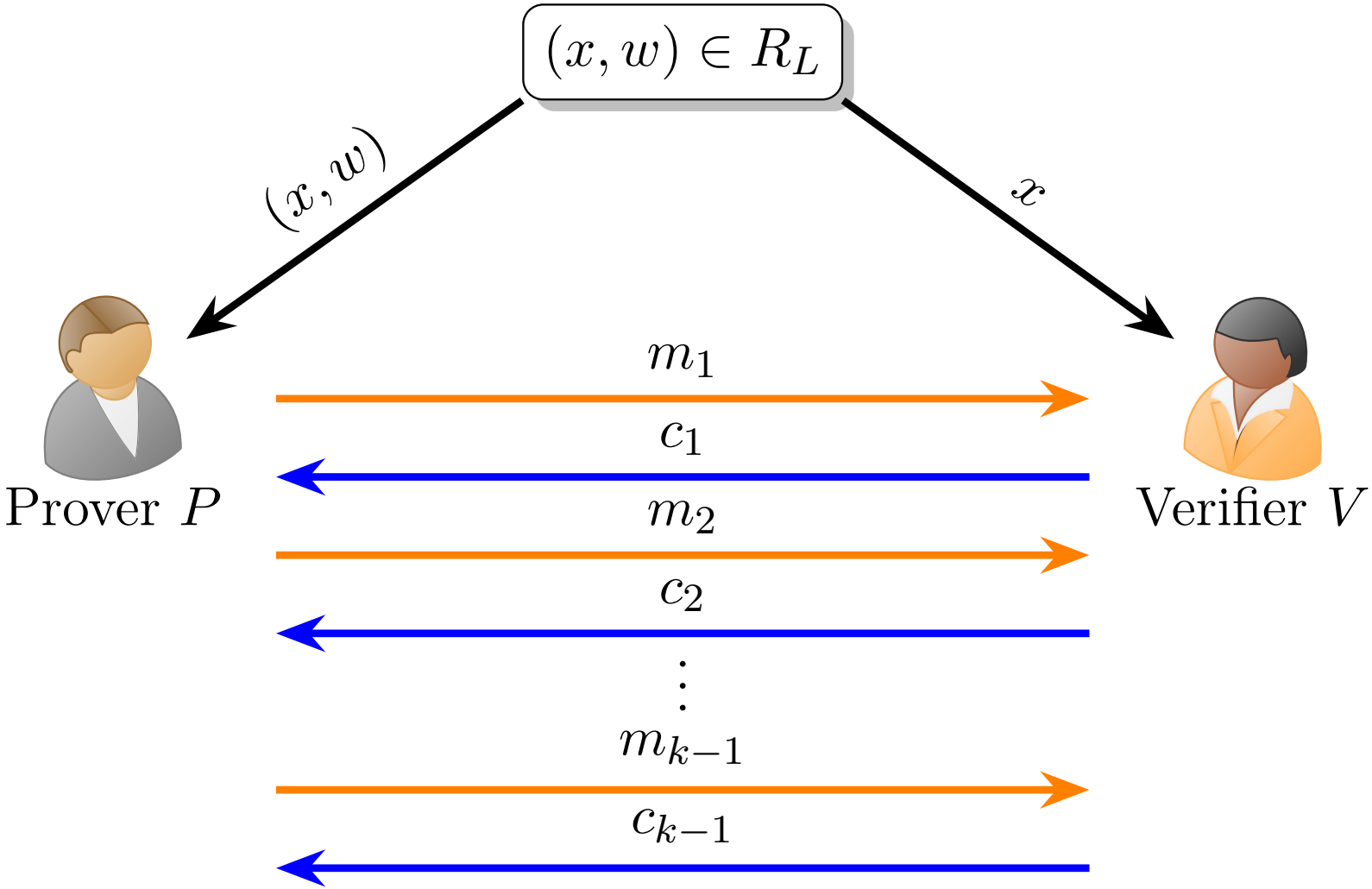

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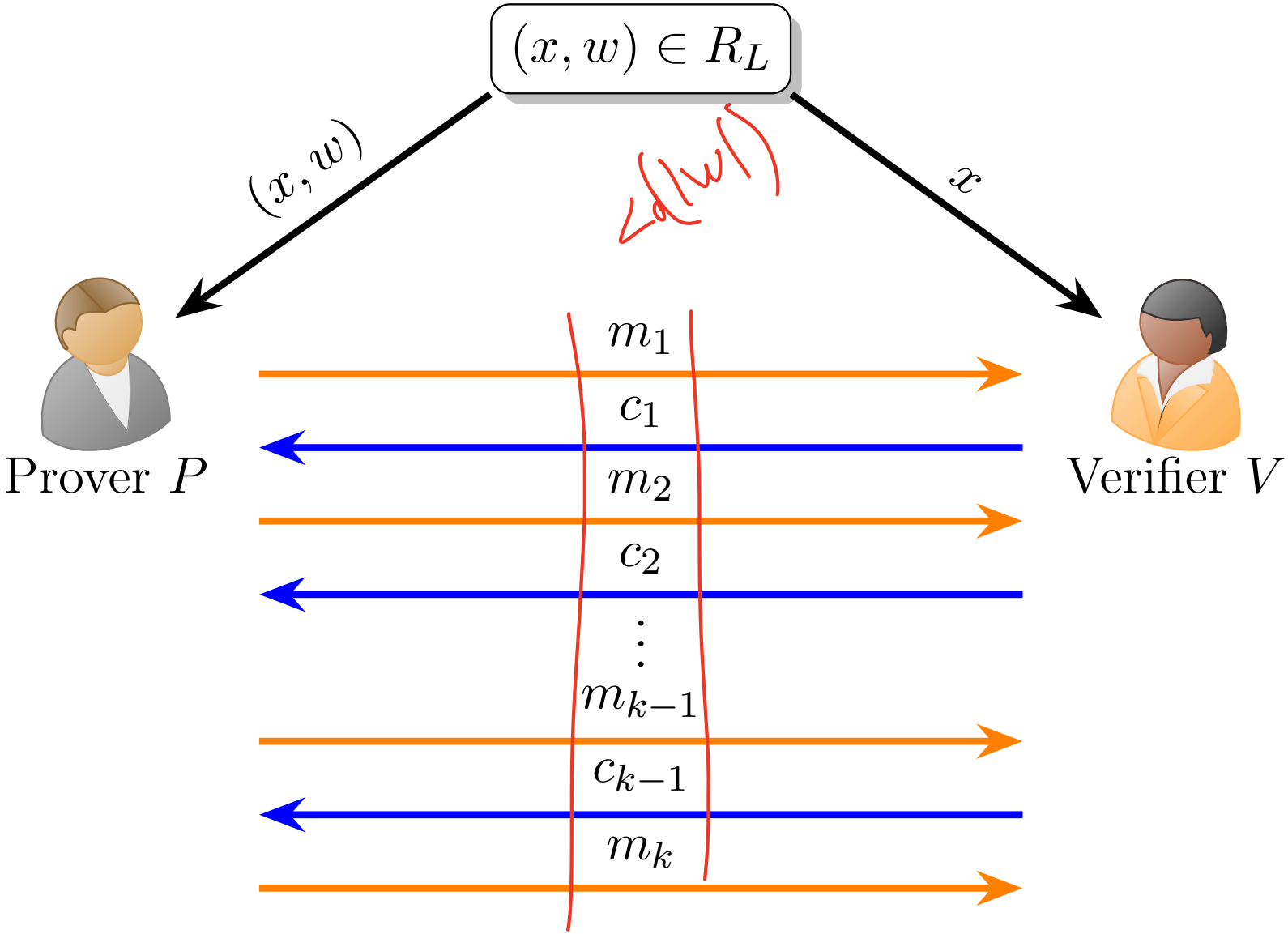

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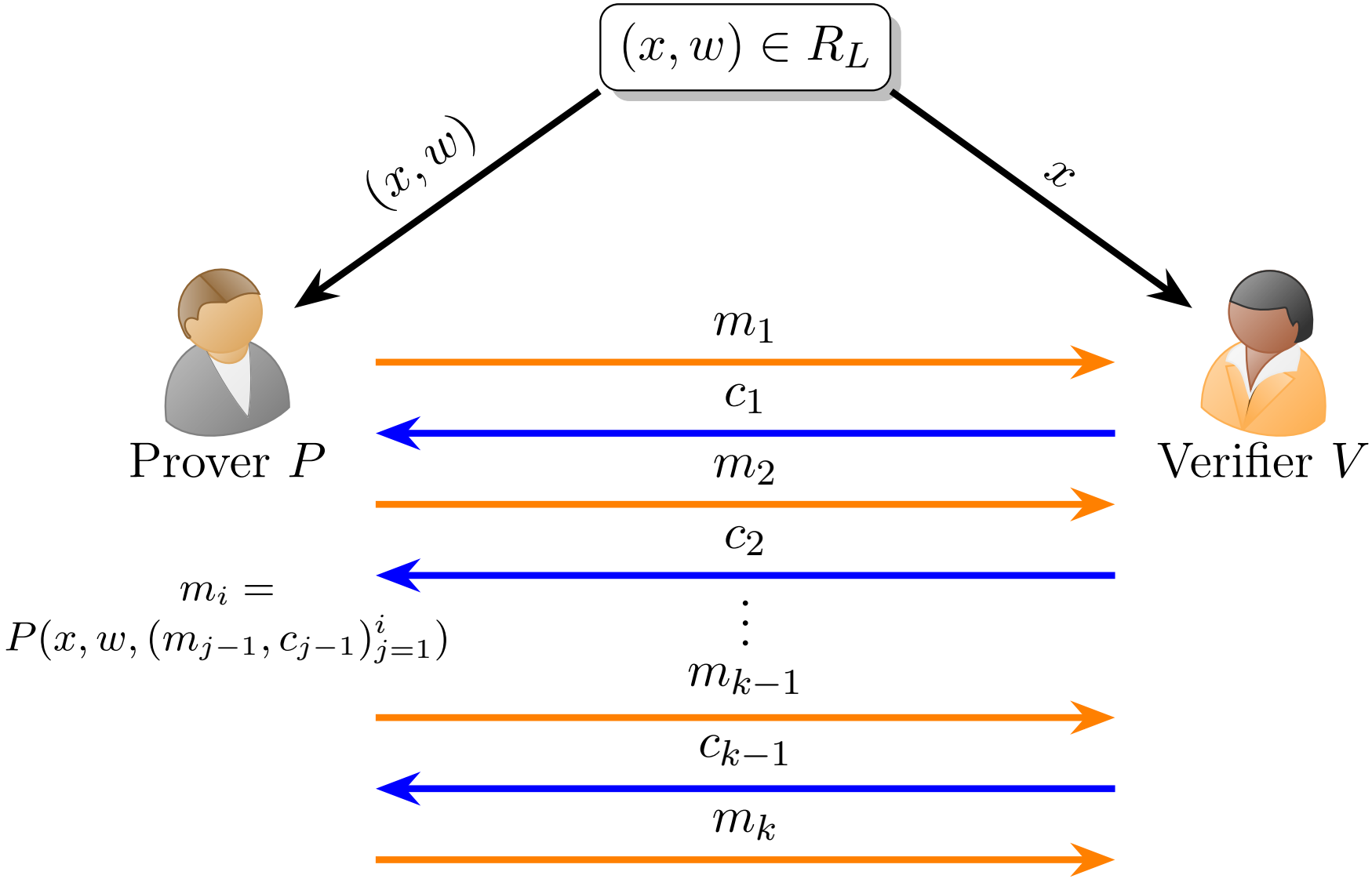

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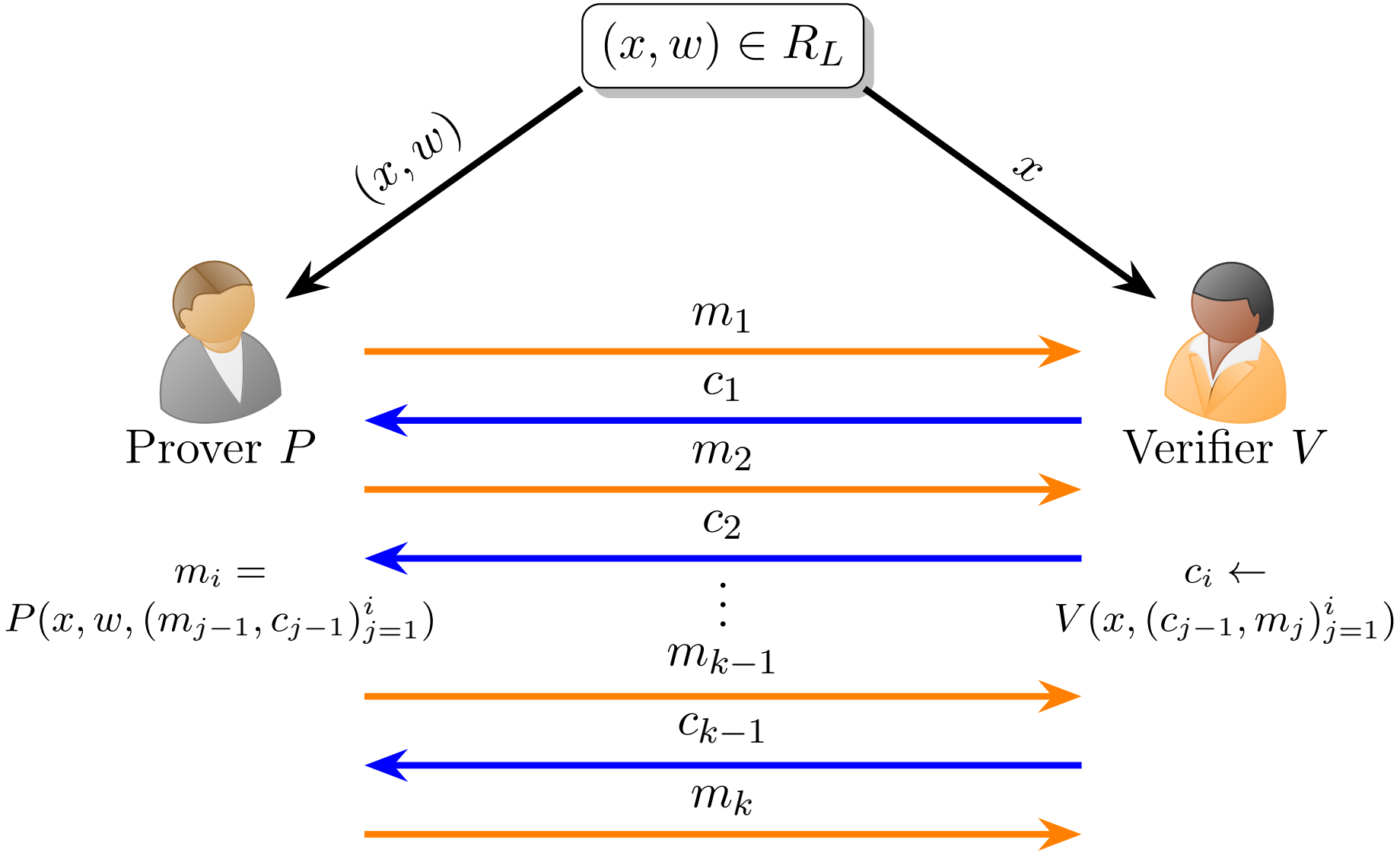

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  - x is the common input to both parties; w is private to P and z is private to V.

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- π is *publicly verifiable* if any other verifier V' can examine π and output accept/reject with the same guarantees.

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- $|\pi|$ is the *proof size*, and we want $|\pi| = o(|w|)$, where $(x, w) \in R_L$.

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poly # rounds in length of $|x|$

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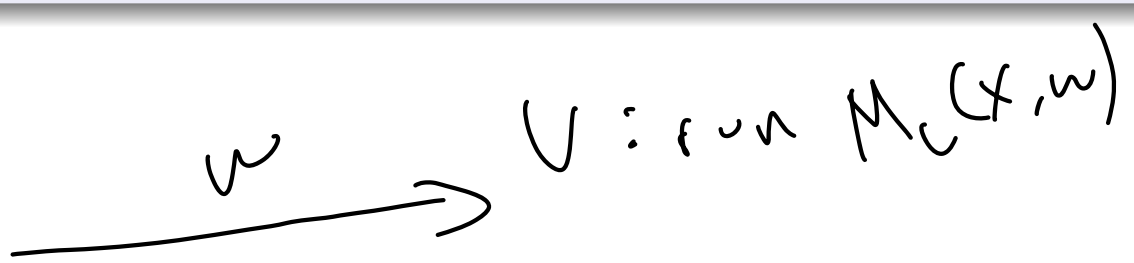
NP ⊆ dIP



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Computational!

Zero-knowledge Proofs

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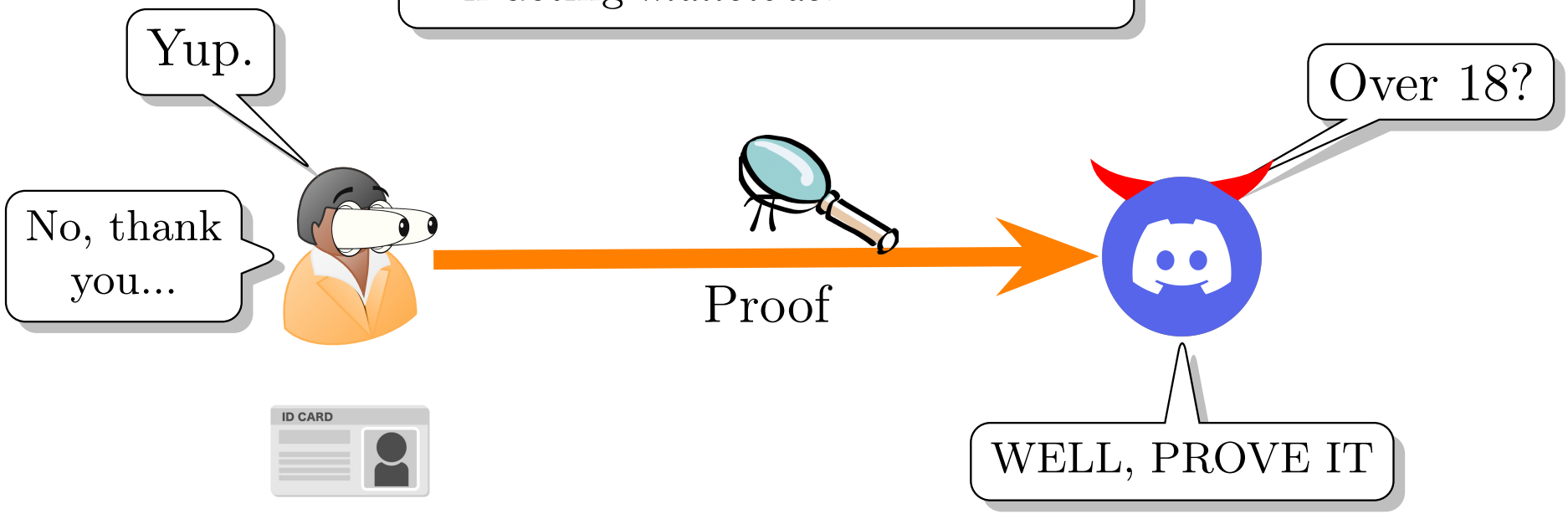
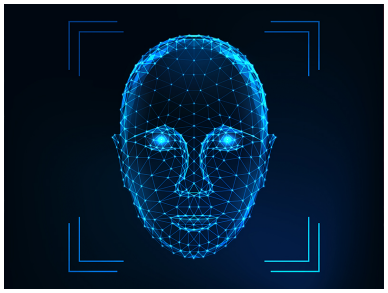
- *Zero-knowledge* attempts to handle the following threat scenario:

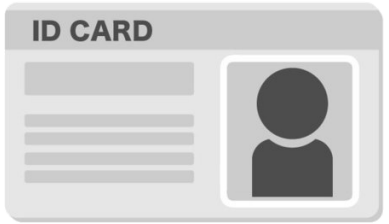
> What if (honest) P's witness contains *sensitive* information?

Example from Before

1. Proves you are 18+
2. Discord learns *nothing else*, even if acting *malicious*.







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Let (P, V) be a k-round interactive proof for a language L. Then, we say that the proof system has *zero-knowledge* if for every PPT verifier algorithm V^*, there exists a PPT algorithm S (the *simulator*), which can depend on V^*, such that for all $x \in L$, the distribution $S(x)$ is "indistinguishable" from $\text{View}_{V^*}(\langle P, V^* \rangle(x))$. Here, $\text{View}_{V^*}(\langle P, V^* \rangle(x))$ denotes the distribution (with respect to V^*) over proofs/transcripts generated by the interaction between P and V^*.

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  - If $x \in L$, V *learns nothing from interacting with* P beyond what V could have computed efficiently herself.
  - This is because given x, V cannot distinguish between a transcript generated from interacting with P and one generated by $S(x)$.
  - Another (bad) view: if $x \in L$, V could just compute $S(x)$ herself and be convinced $x \in L$.

no guarantees if $x \notin L$

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  - Zero-knowledge is defined the same for both.
- We now have 6 different meanings of "zero-knowledge proof;" we'll see more.

Next Time: Examples of Zero-knowledge Proofs