CS 594 – Advanced Crypto (Spring 2026)

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Lecture 12

February 25, 2026

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- So when someone says they are using "zero-knowledge proofs," what do they mean?
- Usually, when people say "zero-knowledge proof," they mean the system at least has computational zero-knowledge and soundness, and has honest-verifier zero-knowledge.

Statistical Zero-knowledge Proofs

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$\subseteq$ IP

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  - A single message is sent first by the verifier, followed by the prover.

$L : L \in \mathrm{AM}$

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- Limits the usefulness of **SZK**: cannot prove 3SAT in **SZK**!
  - So we cannot use **SZK** as a general-purpose tool for arbitrary verifiable computation.
- In contrast, if one-way functions exist, then **NP** $\subseteq$ **CZK** (computational zero-knowledge), and in fact **CZK** = **IP**.
- We will still examine some **SZK** proofs to convey the power and some intuition about zero-knowledge.

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- Otherwise, we say that G_0 and G_1 are *not* isomorphic.
- There is no known poly-time algorithm for deciding if two graphs are isomorphic!
- We give an honest-verifier SZK protocol (actually PZK) for both Graph Isomorphism and Non-Isomorphism.

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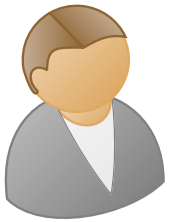

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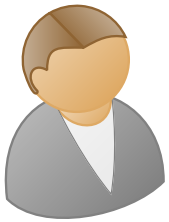

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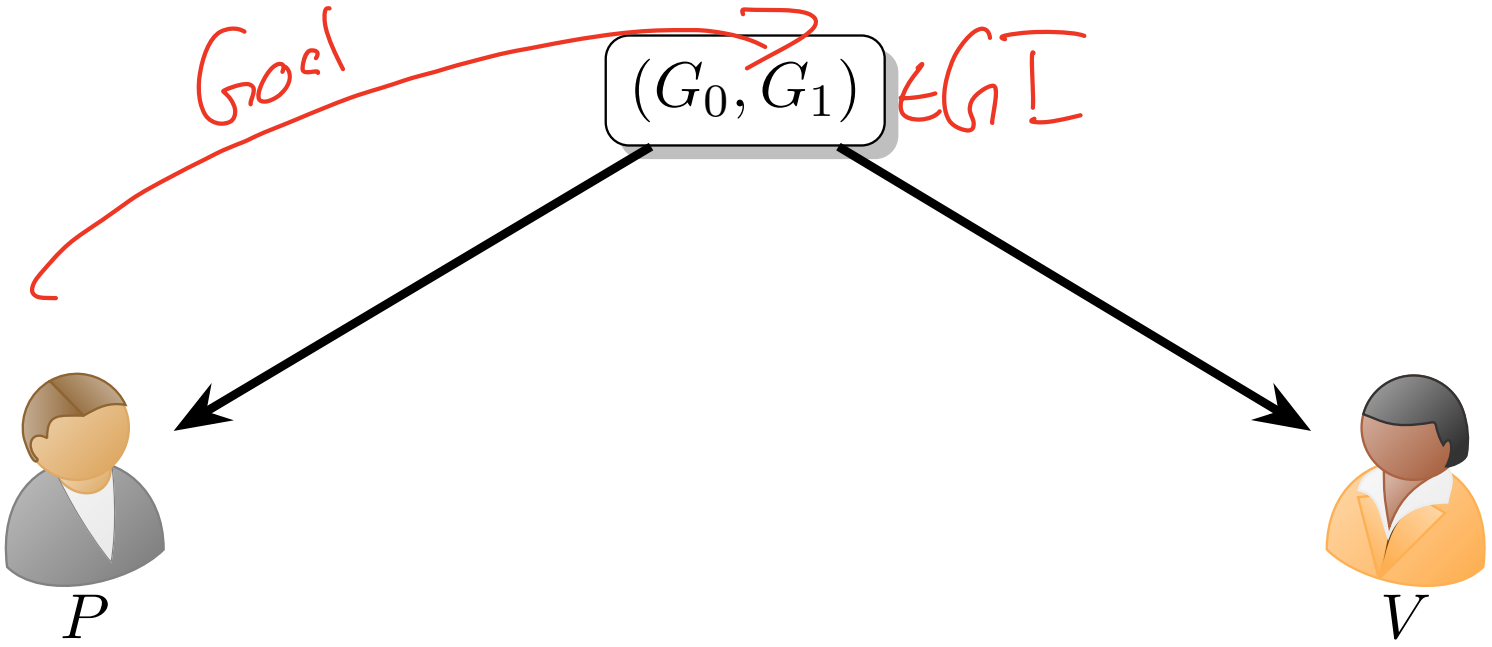

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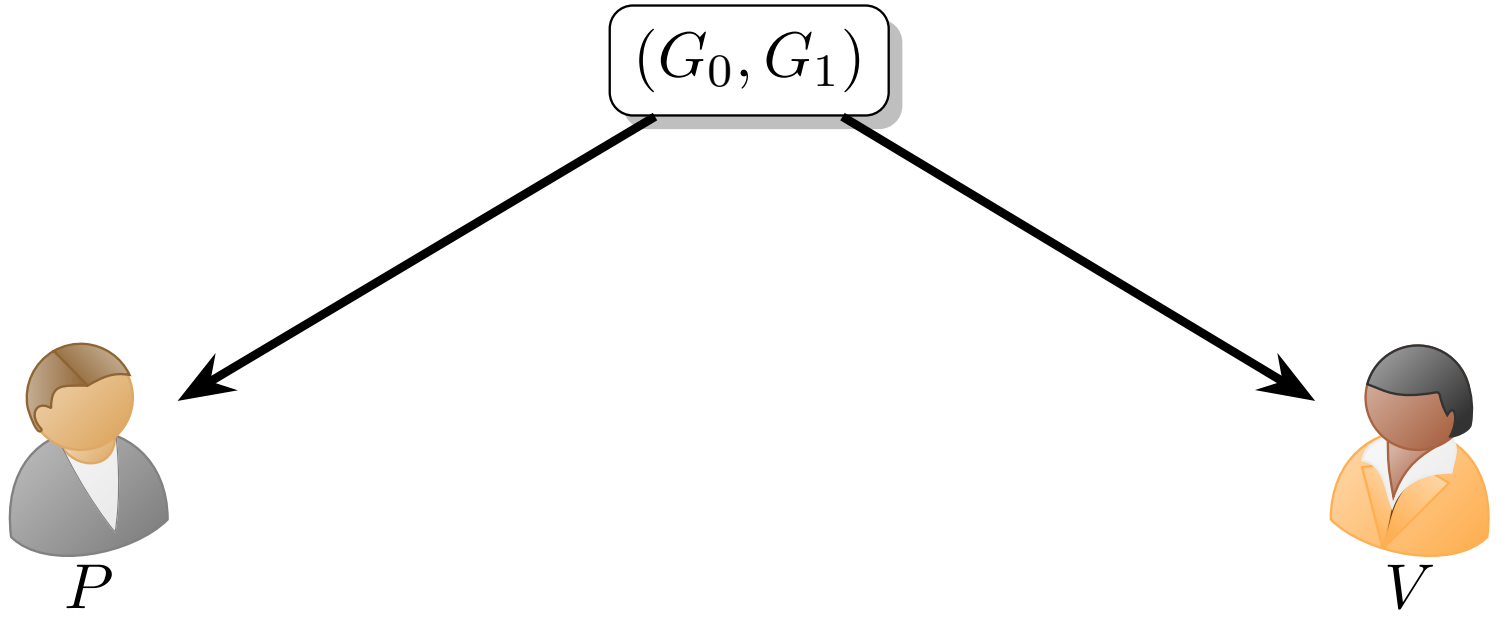

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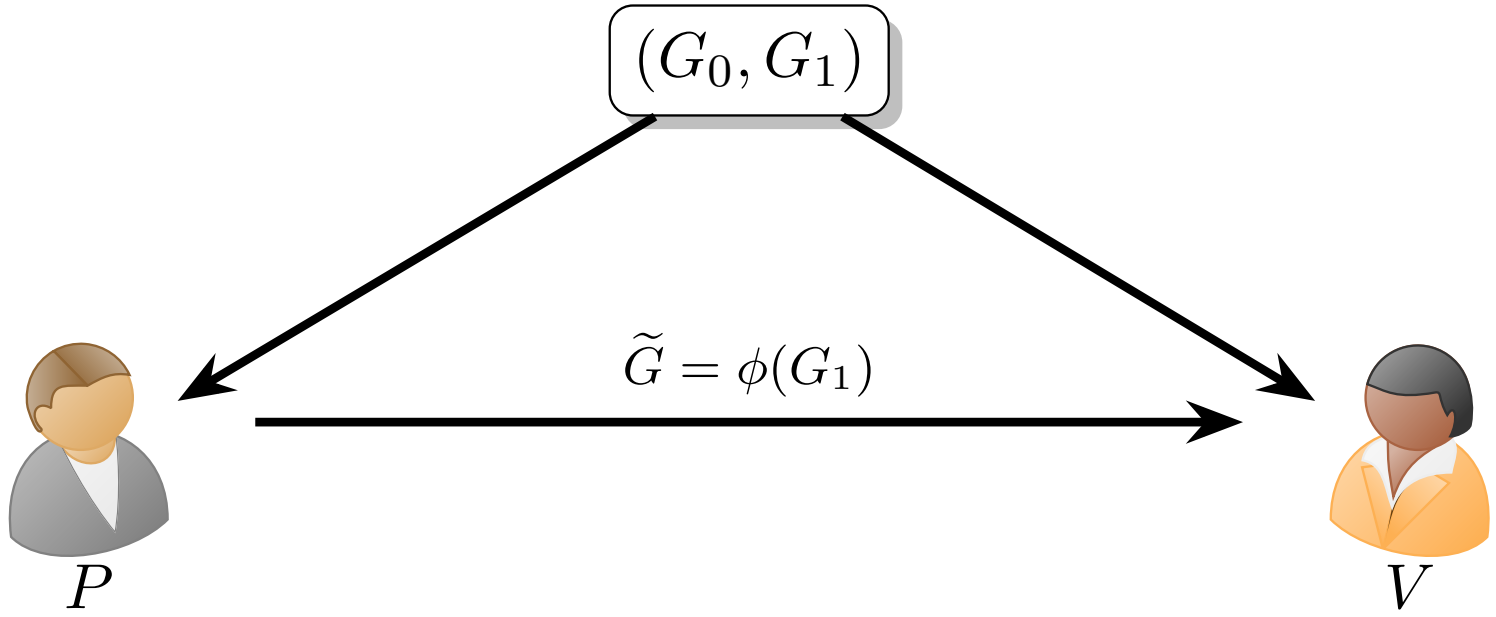

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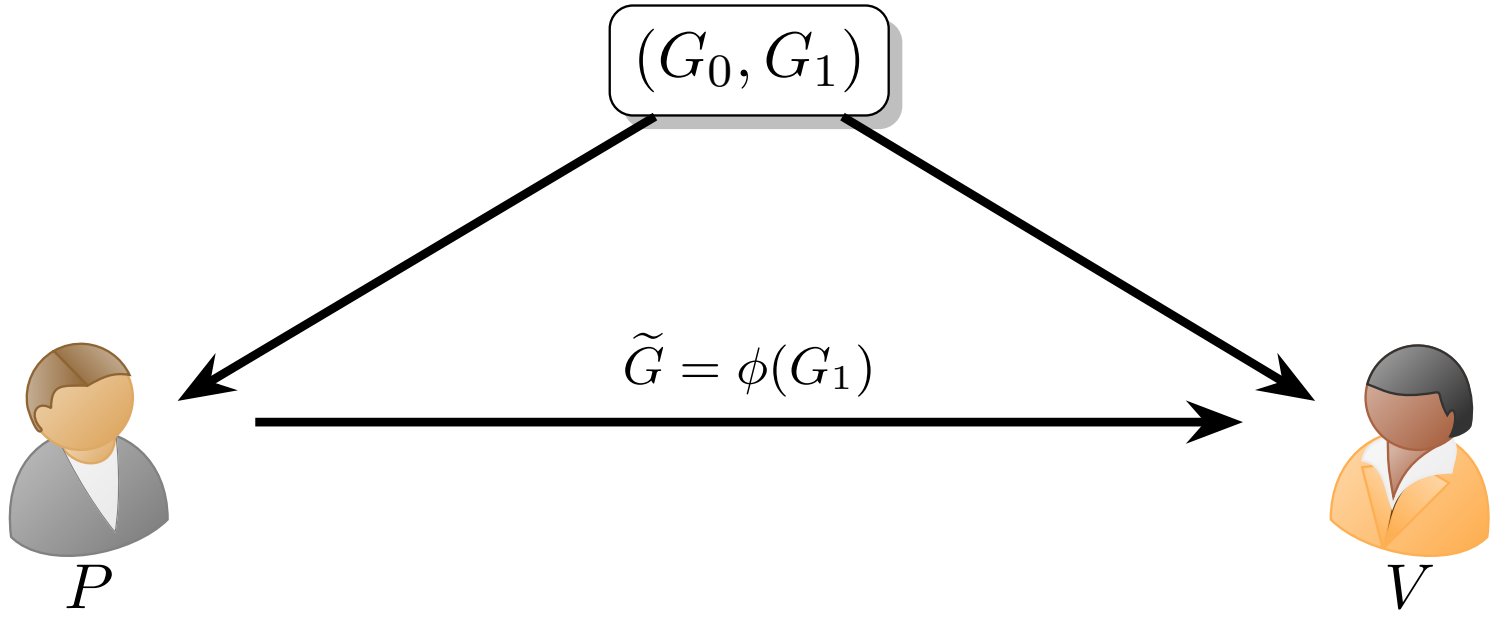

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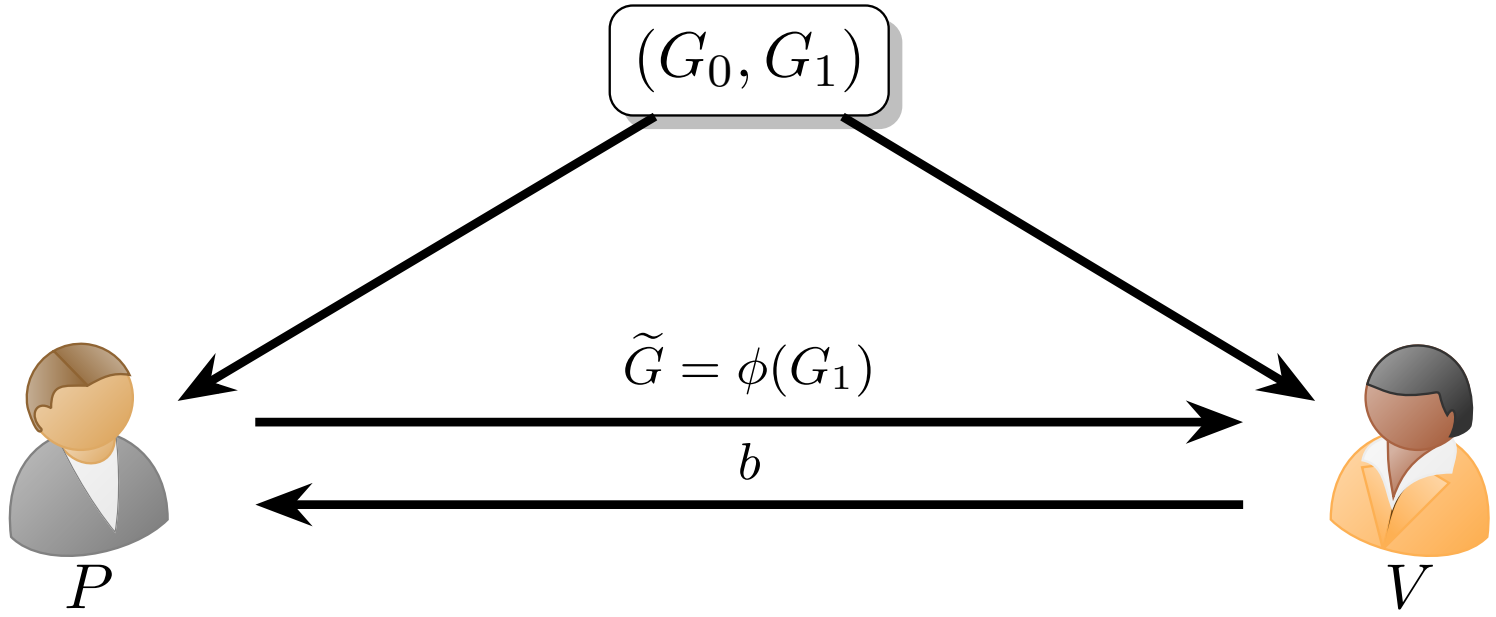

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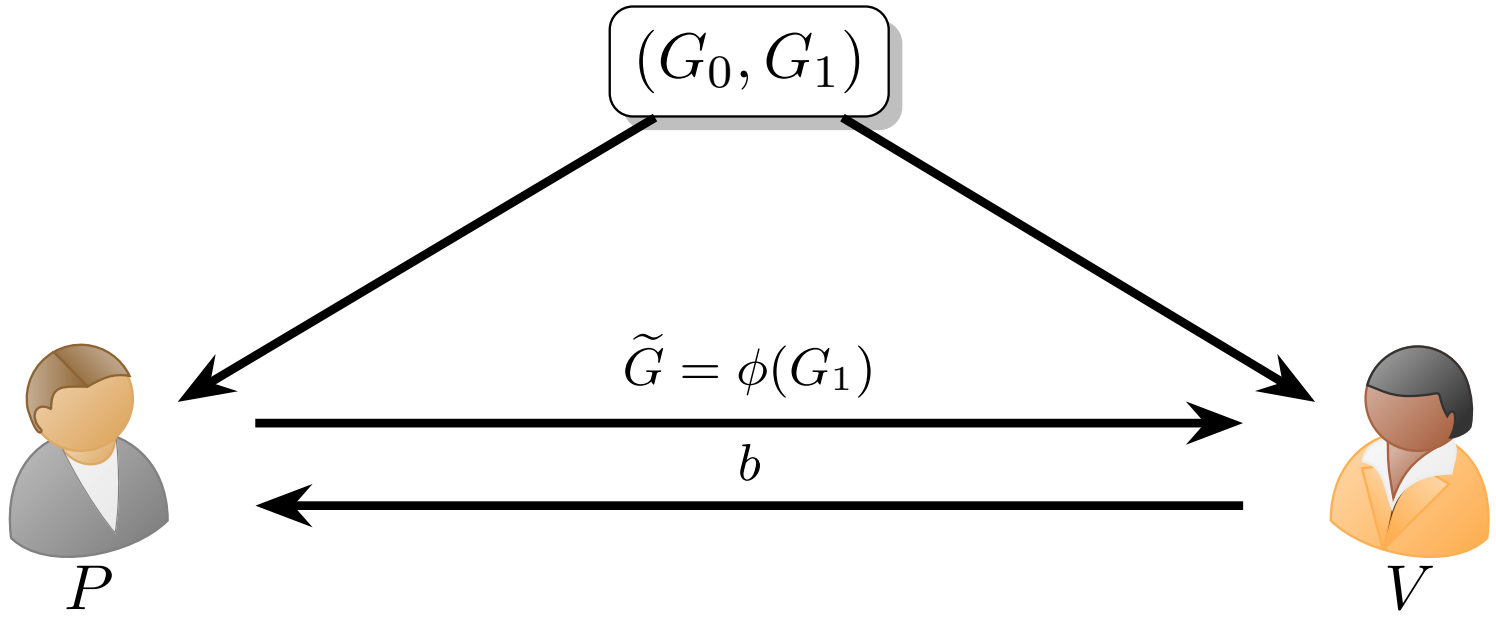

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(P2) Set $\sigma = \begin{cases} \phi & b = 1 \\ \phi \circ \pi & b = 0 \end{cases}$.

$(\phi \circ \pi)(x) = \phi(\pi(x))$

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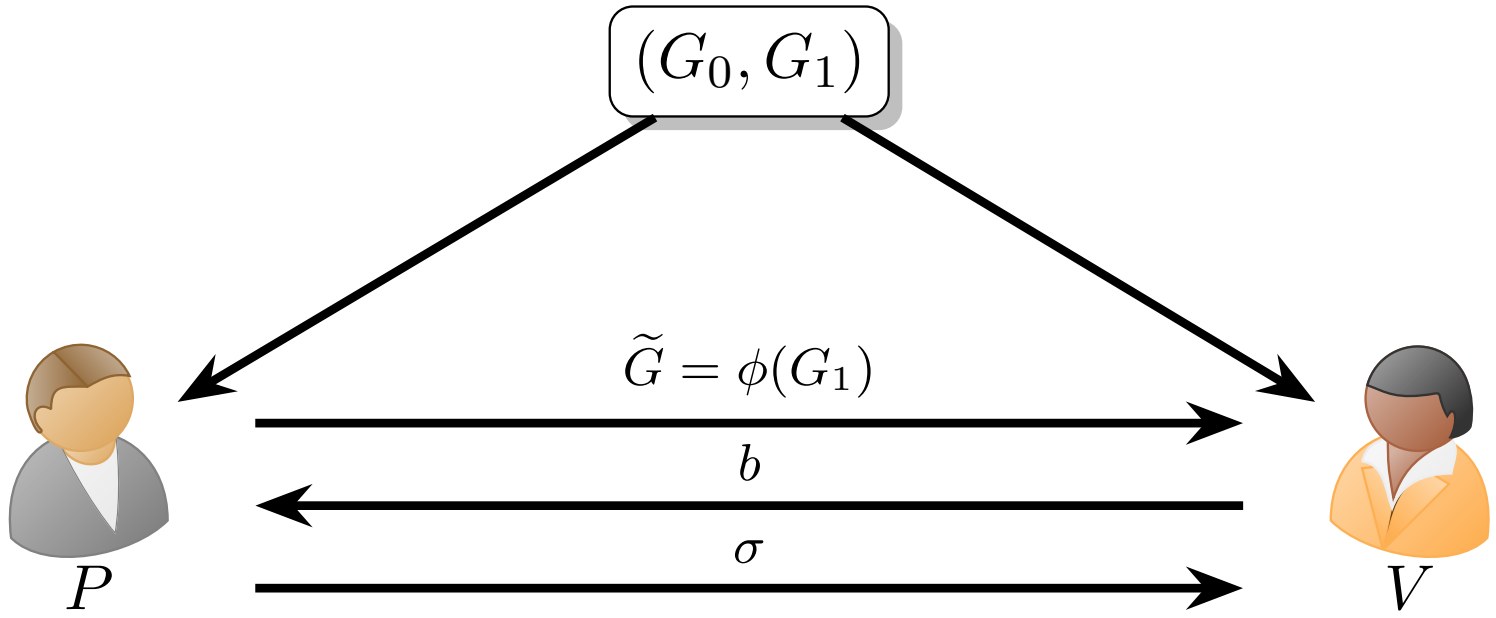

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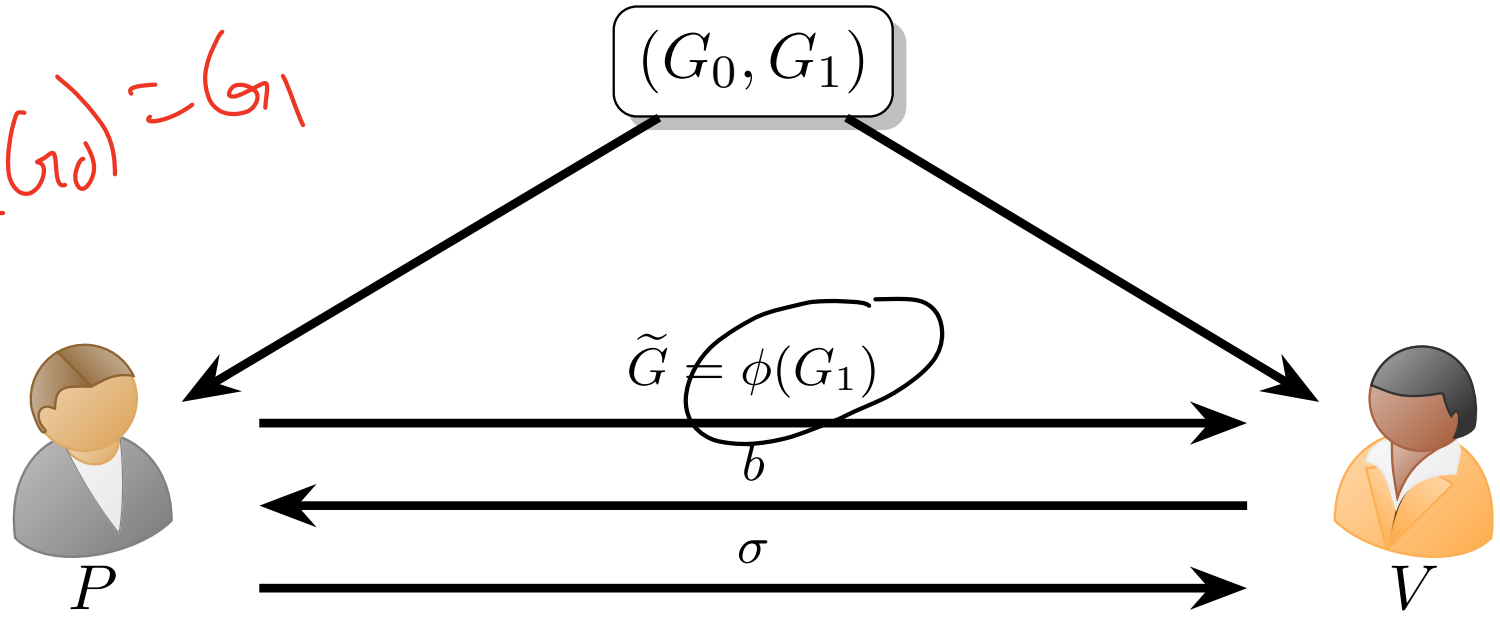

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(V2) Check if $\sigma(G_b) = \widetilde{G}$.

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$\tilde{G} = \phi(G_1)$

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  - Sequentially compose this protocol twice to obtain soundness error $1/4 < 1/3$; 2^{-k} in general with k sequential compositions.

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- Sample $c \xleftarrow{\$} \{0, 1\}$.
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- Set $H = \varphi(G_c)$.
- Output (H, c, φ). $(\tilde{G}, b, \sigma)$

HZPZK Proof for Graph Isomorphism

Proof.

□

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- Can extend to *malicious* perfect zero-knowledge by allowing the simulator to run in *expected* polynomial time.

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- Expected number of loops before halting: 2.

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- The simulator succeeds with probability exactly $1/2$ whenever G_0 and G_1 are isomorphic.
  - This is because b cannot depend on c since φ is random.
- Expected number of loops before halting: 2.
- Implies expected poly-time (running V^* is poly-time).

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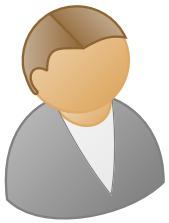

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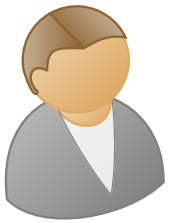

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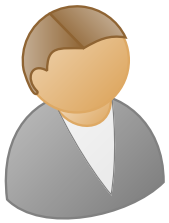

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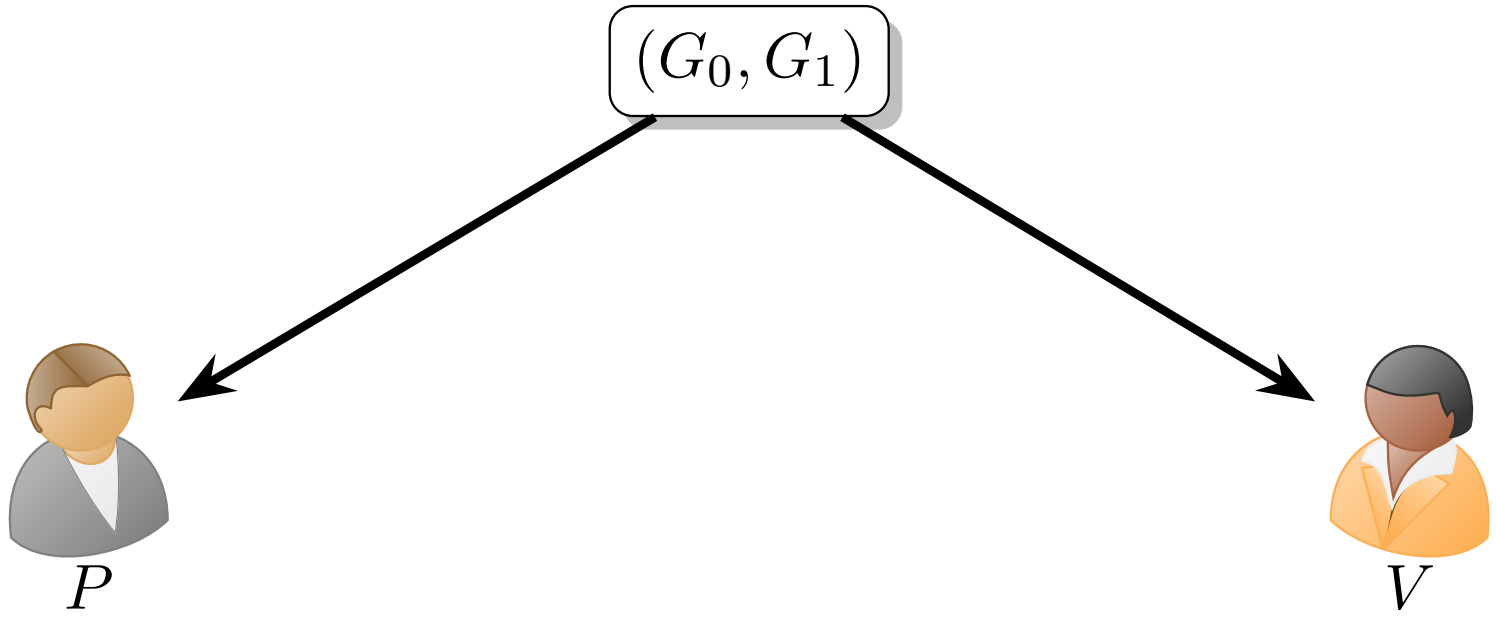

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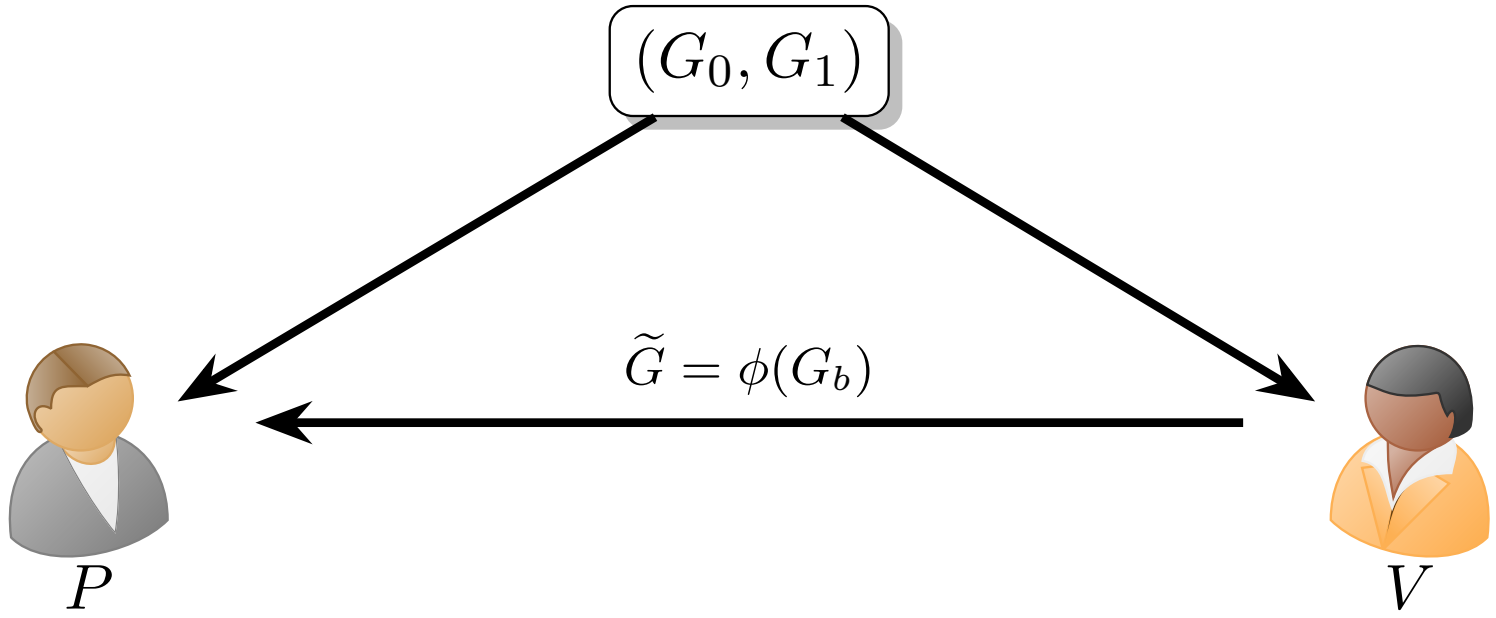

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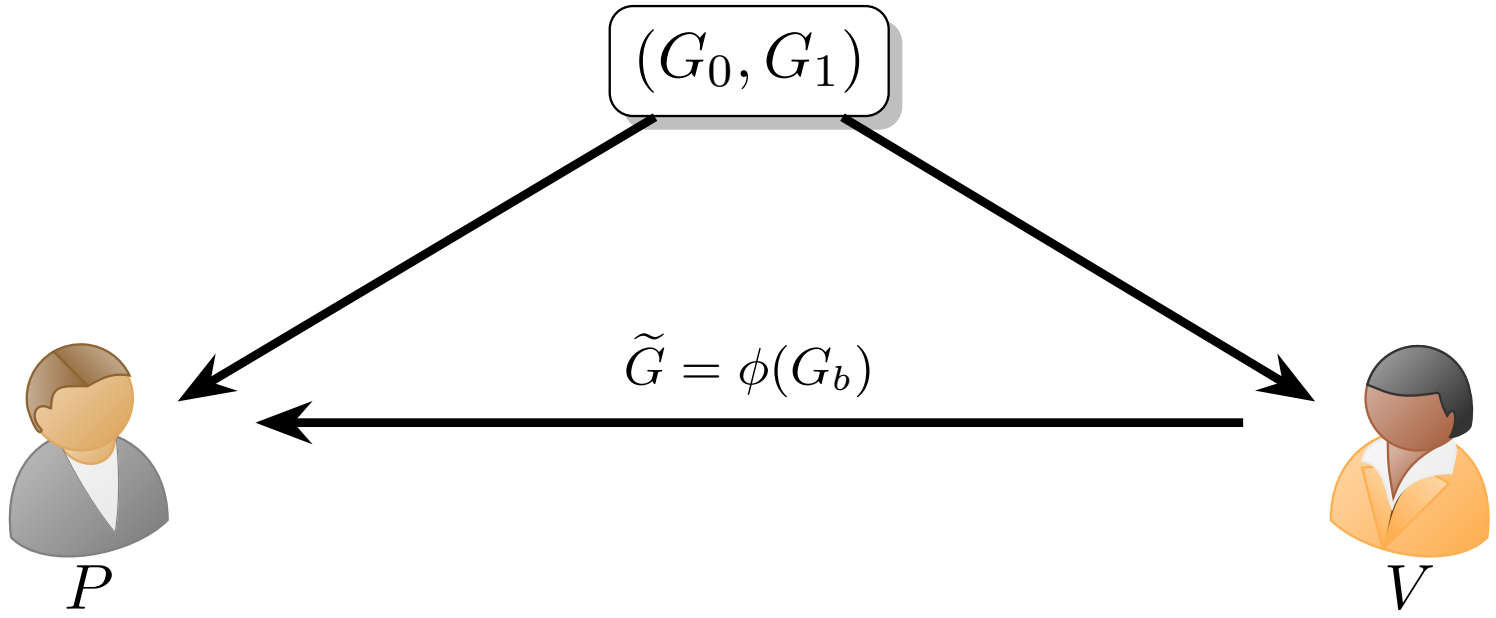

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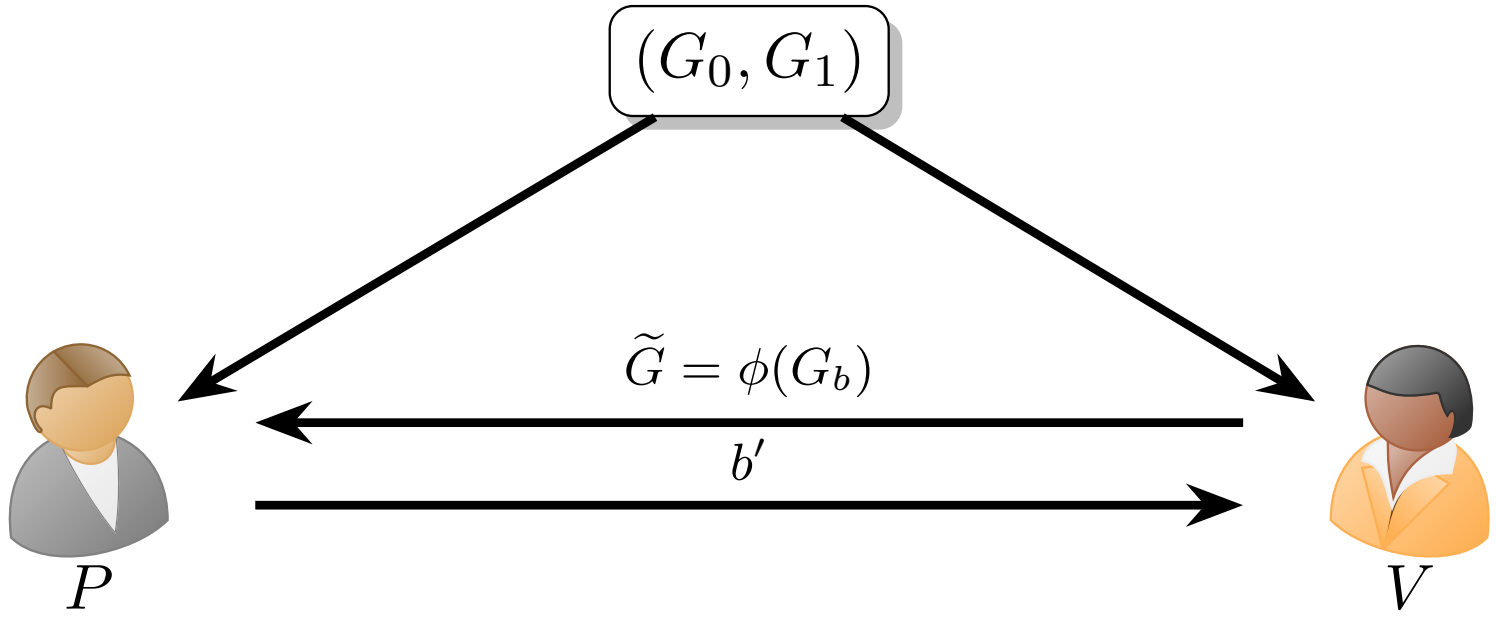

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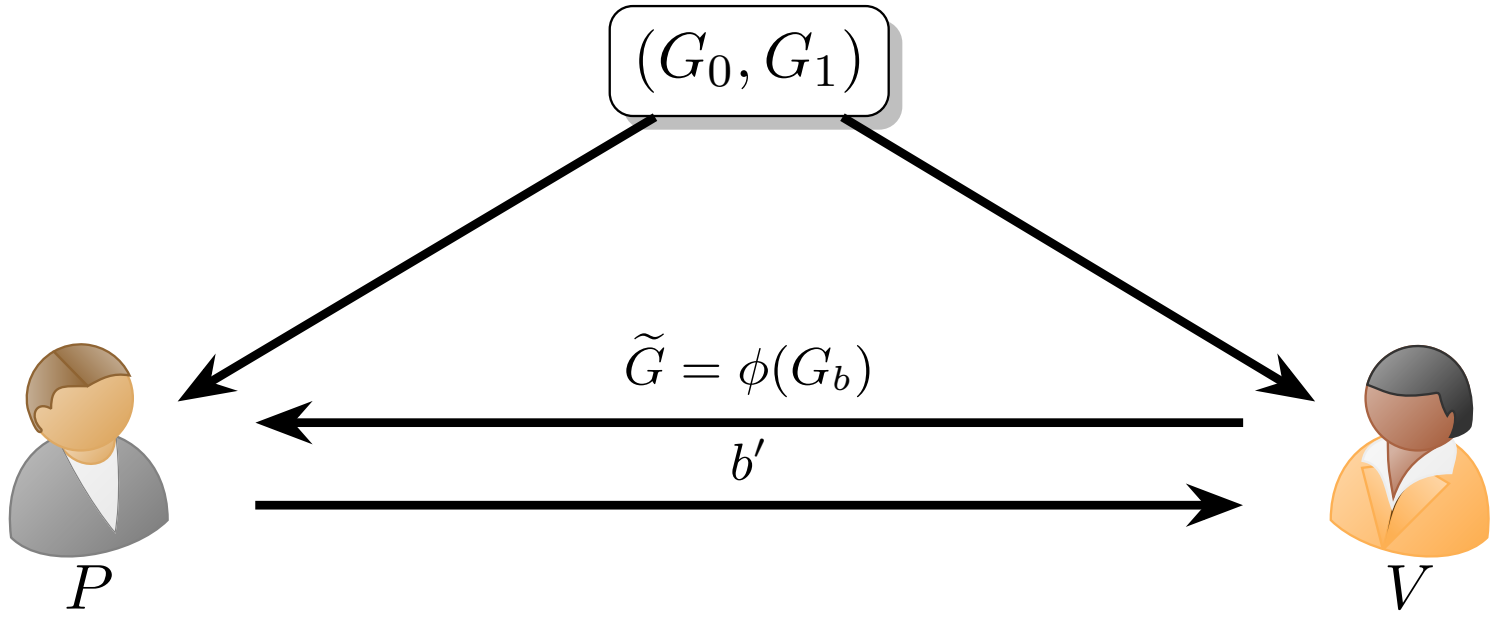

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  - Sequentially compose this protocol twice to obtain soundness error $1/4 < 1/3$; 2^{-k} in general with k sequential compositions.

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- Moreover, if the graphs are not isomorphic, an honest prover always outputs $b' = b$. □

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- High-level fix: have the verifier *prove* it knows which graph $\widetilde{G}$ is isomorphic to, without revealing sampled bit b.

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- Schnorr's Identification Protocol allows a prover to convince a verifier it knows the discrete-log of some public element $A \in \mathbb{G}$.
- Formally, the protocol is an interactive argument for the language $L_{\mathsf{Sch}} = \{(\mathbb{G}, p, g, A) \mid \exists a \colon g^a = A\}$.

Schnorr Identification Protocol

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Private

$\in R_{Sch}$

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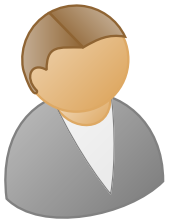

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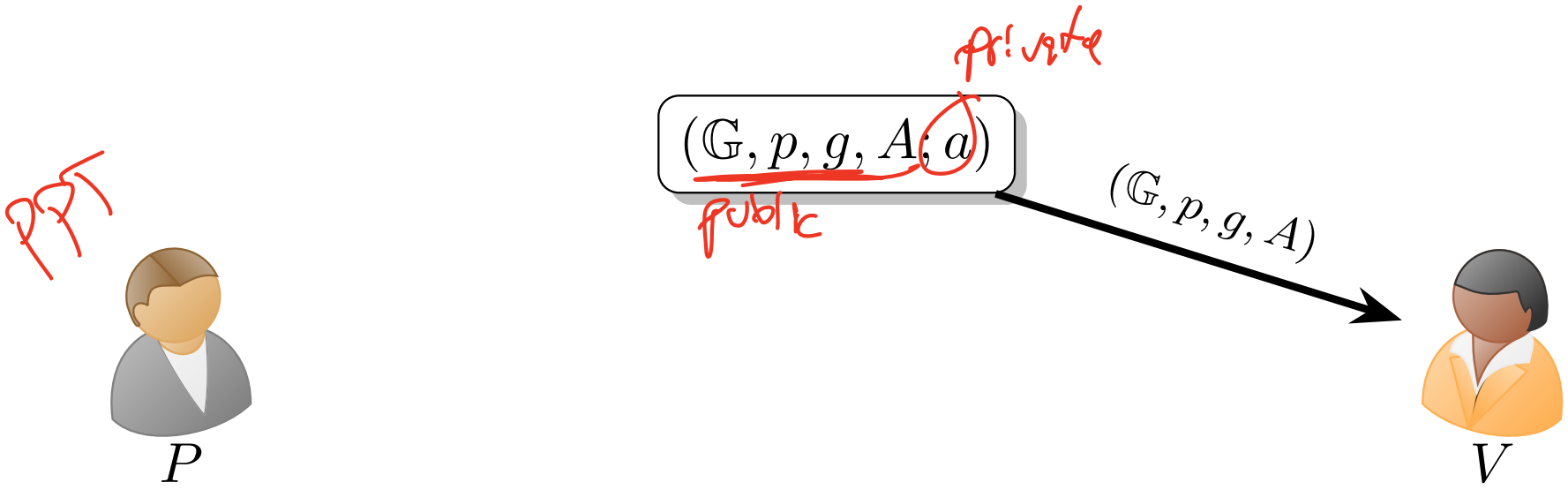

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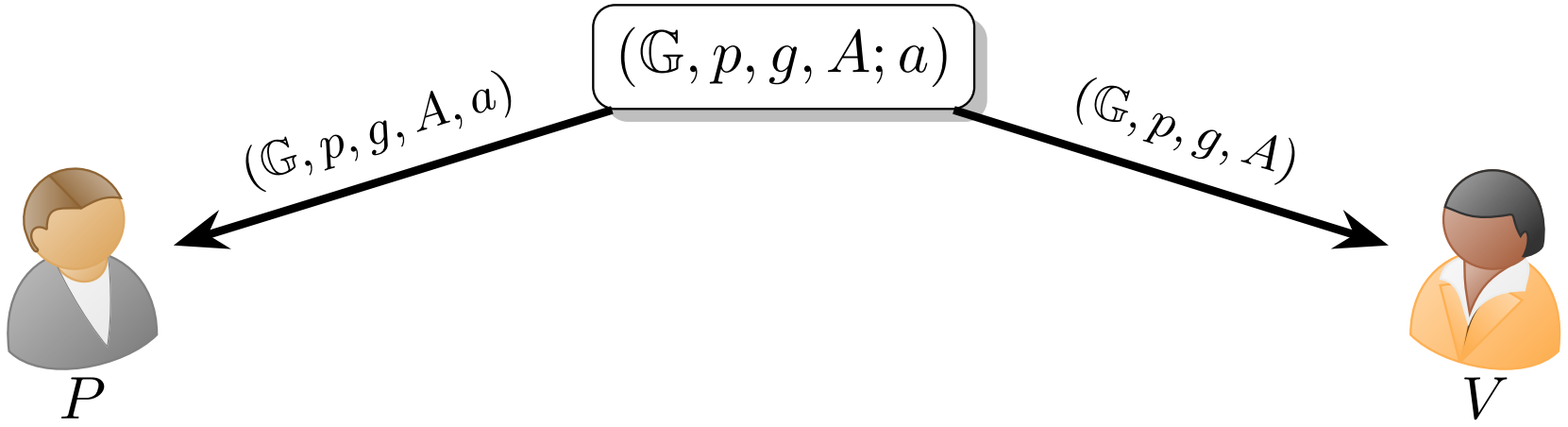

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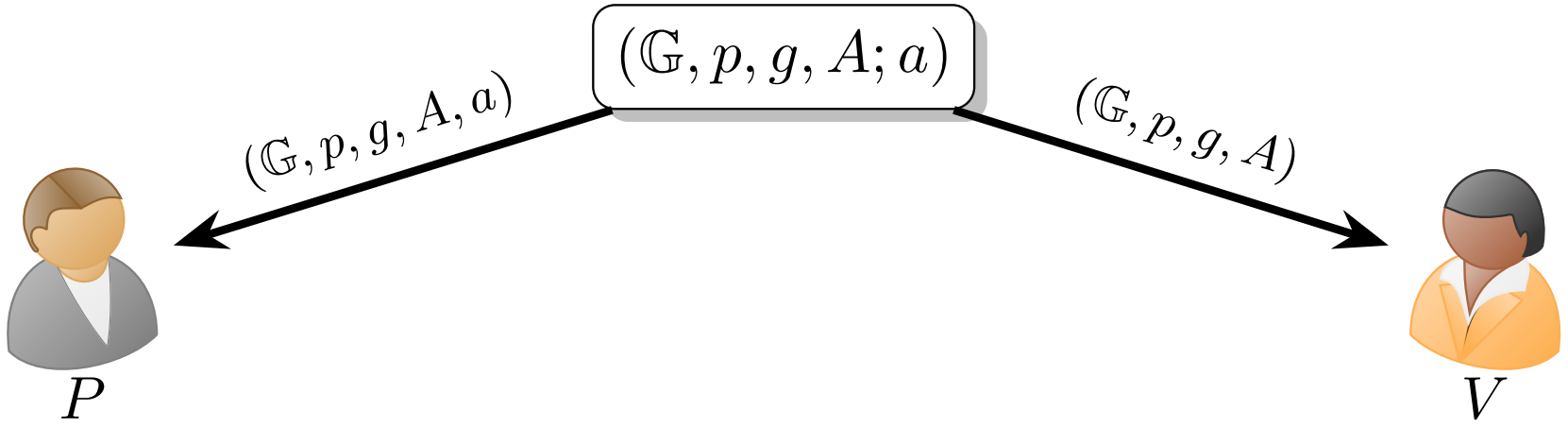

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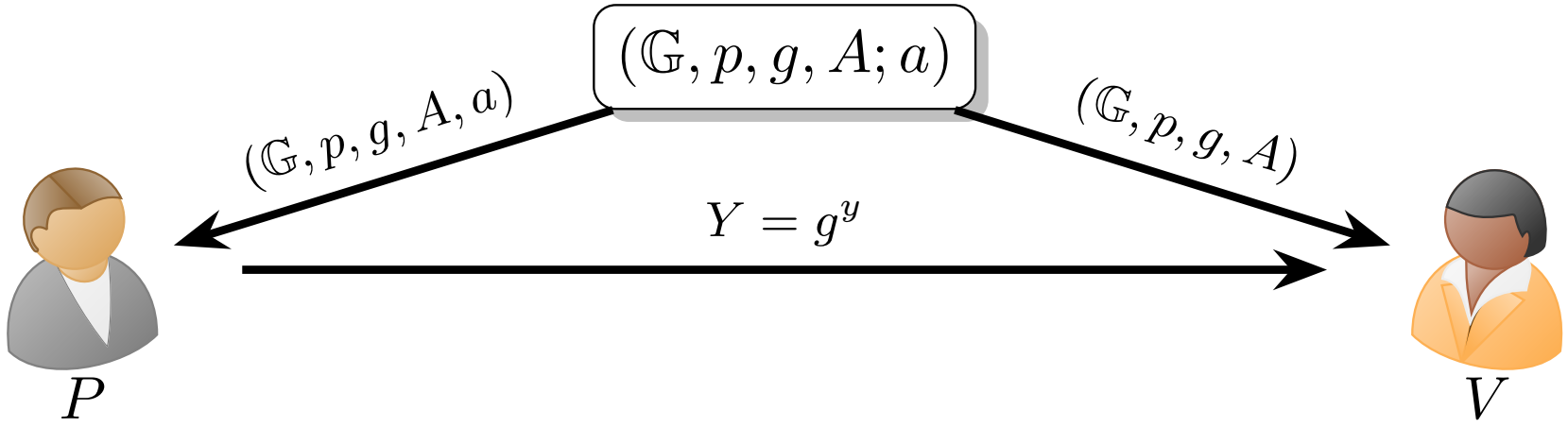

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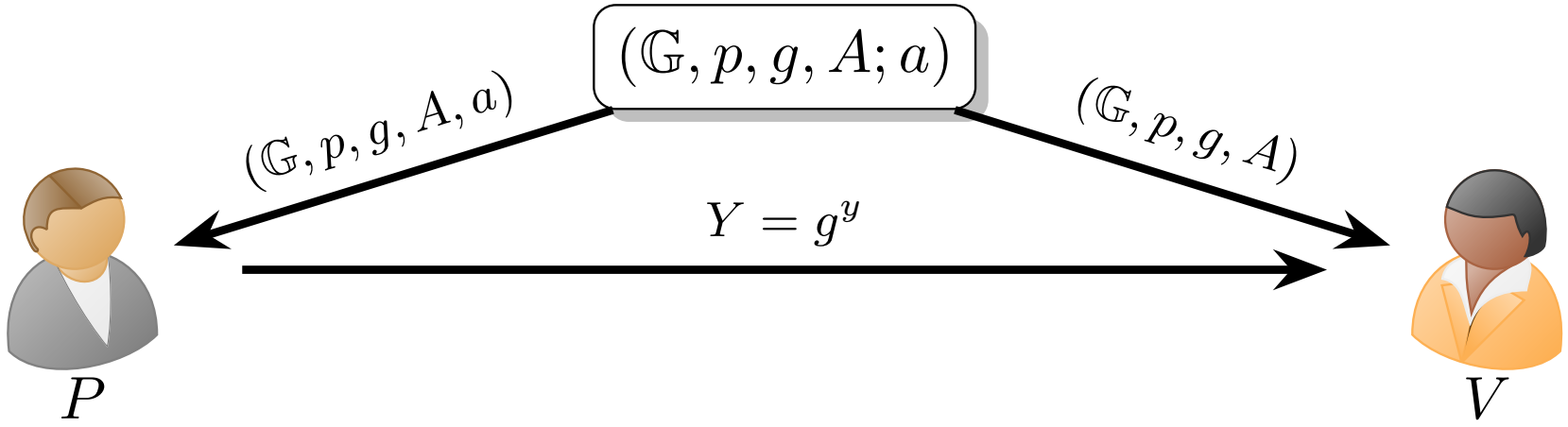

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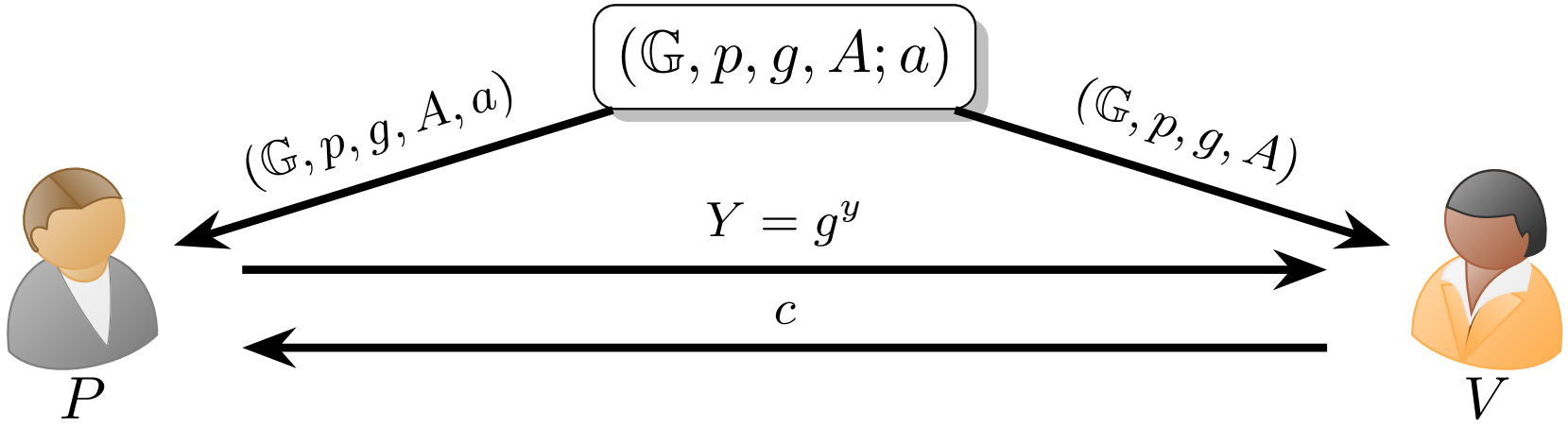

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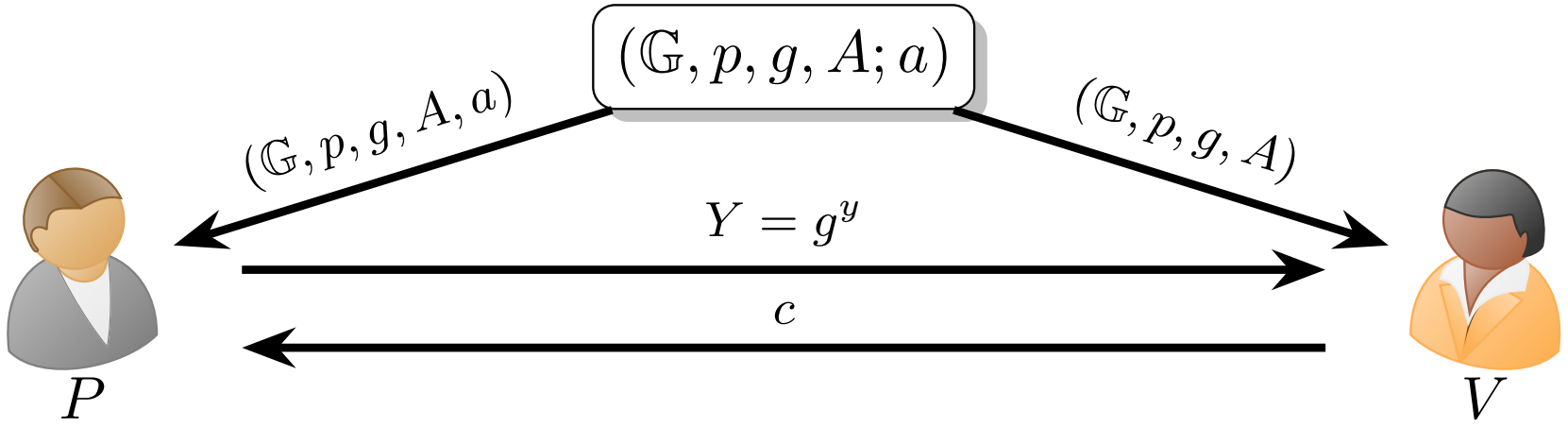

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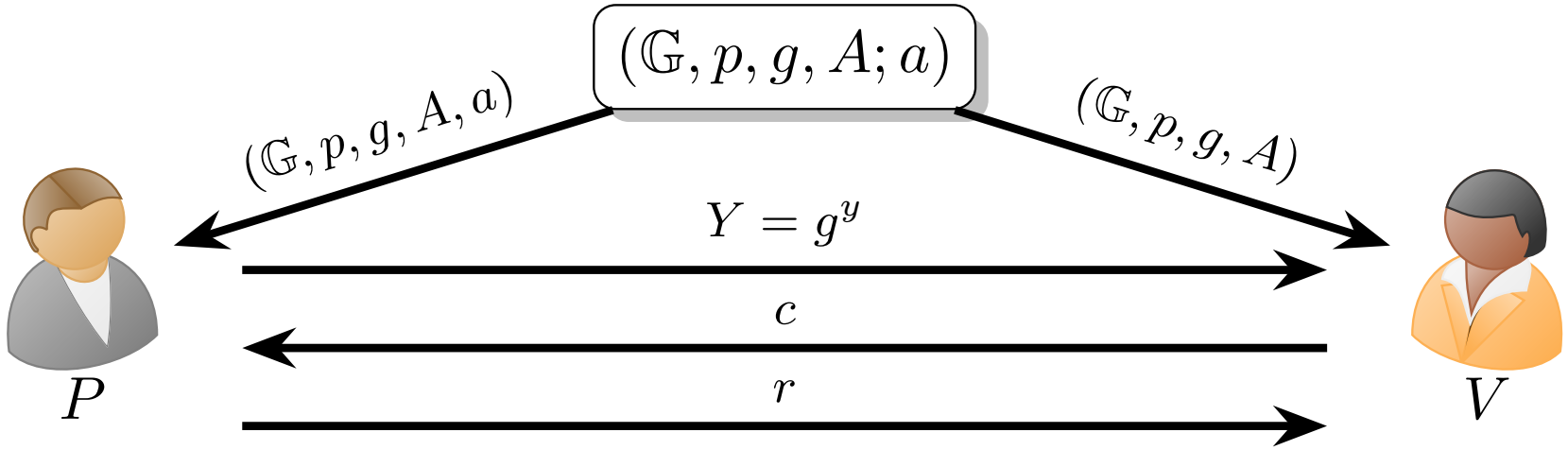

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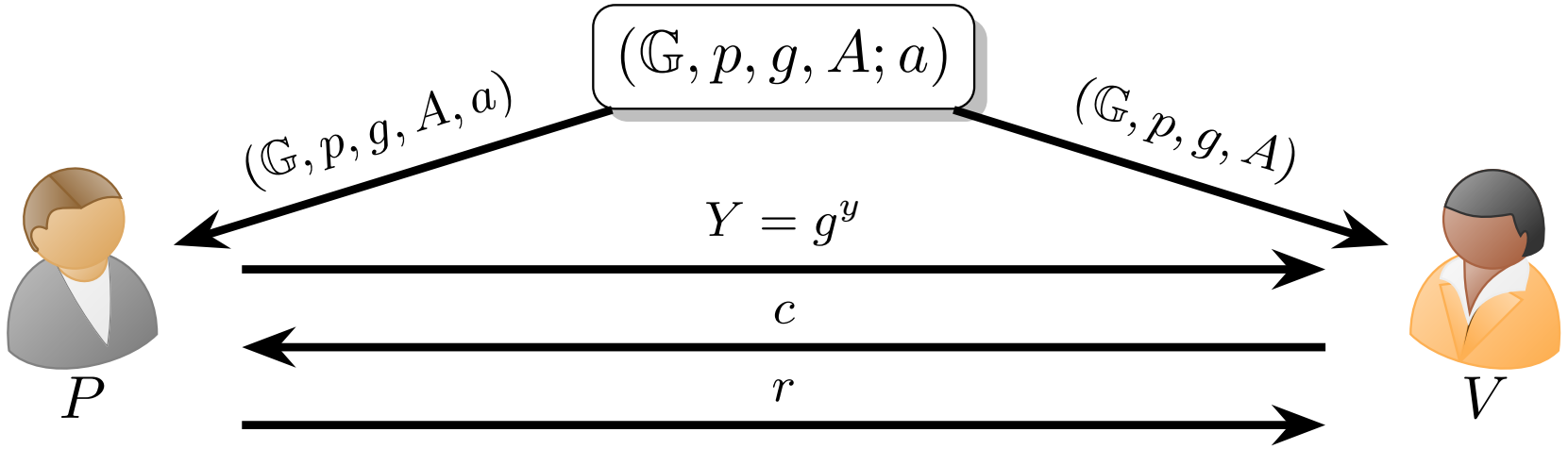

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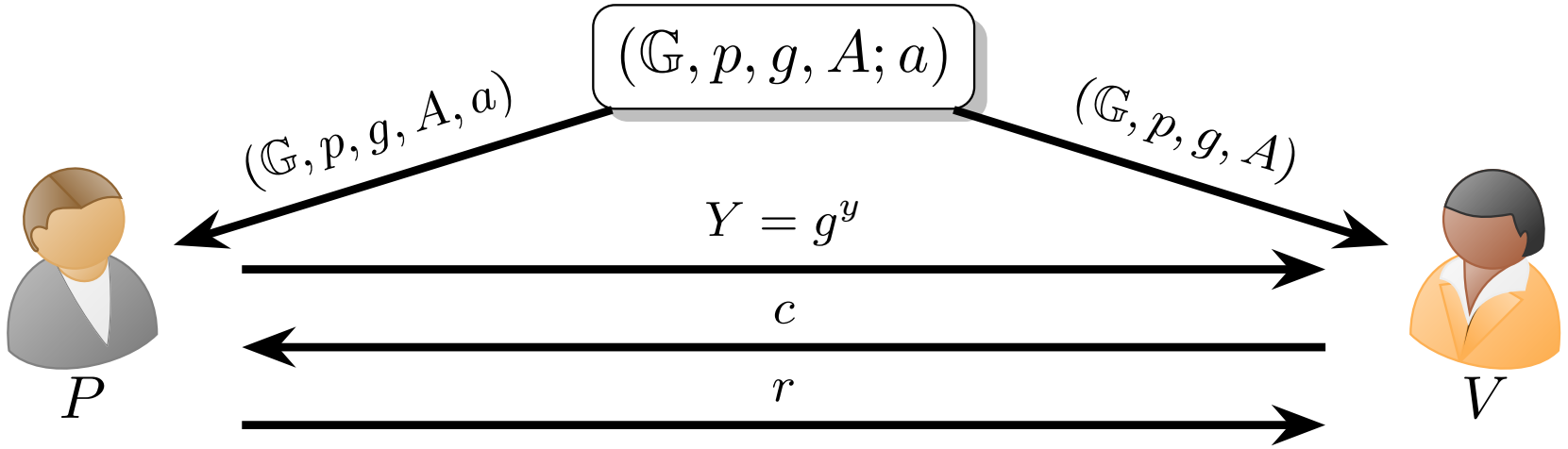

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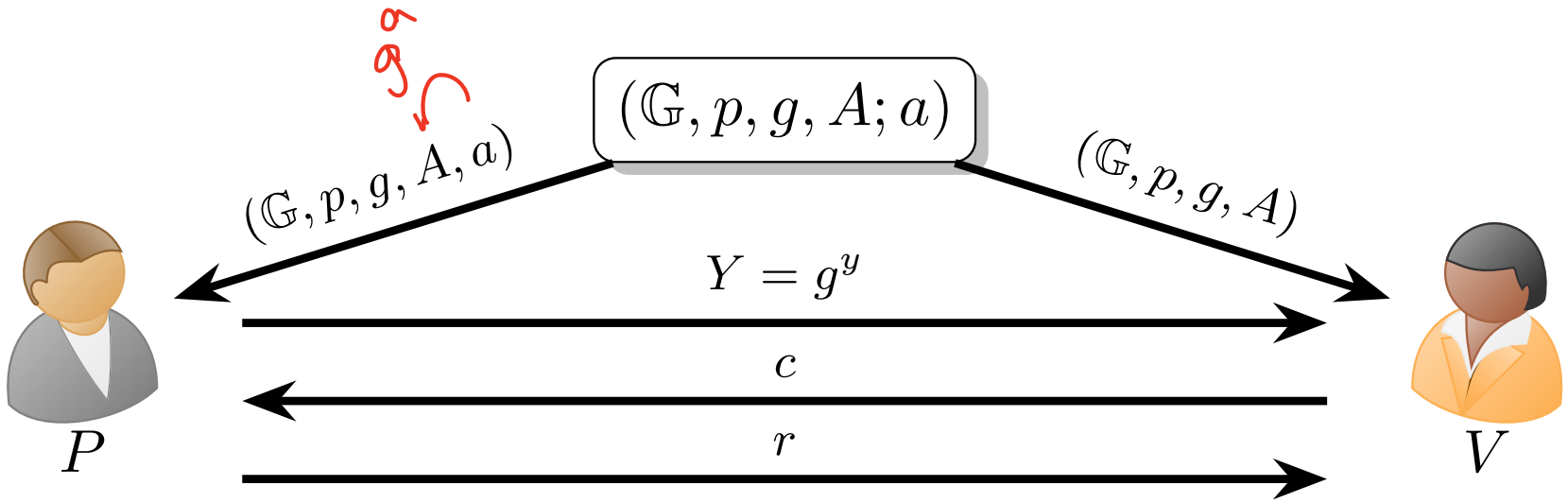

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  - If P could produce these transcripts *without knowing* a, then we'd be able to break discrete-log in polynomial-time!

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$$A^c \cdot Y = g^r$$

$$A^c (g^r \cdot A^{-c}) = g^r$$

$$\Rightarrow \quad g^r = g^r$$

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- So the two output distributions are identical.

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  - p is chosen to be cryptographically large (i.e., $\log(p) = \lambda$), so the probability of getting the same first message Y in two independent executions of the protocol is $2^{-\lambda}$.

Next Time: Zero-knowledge for all of NP