CS 594 – Advanced Crypto (Spring 2026)

Alex Block

Lecture 13

March 02, 2026

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Theorem 2 (Blum (86))

If statistically-binding, computationally hiding commitments exist, then $\mathbf{HAM} \in \mathbf{CZK}$, *where*

$$\mathbf{HAM} = \{G = (V, E) \colon G \text{ has a Hamiltonian cycle.}\}.$$

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3. The output of the experiment is 1 if $f(x') = y$ and 0 otherwise.

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Commitment Schemes, Intuitively

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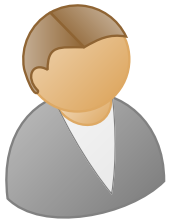

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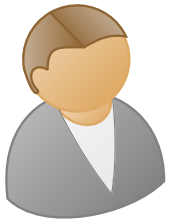
Commitment Schemes, Intuitively



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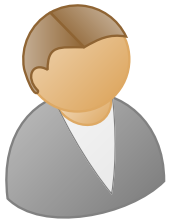

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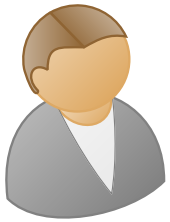

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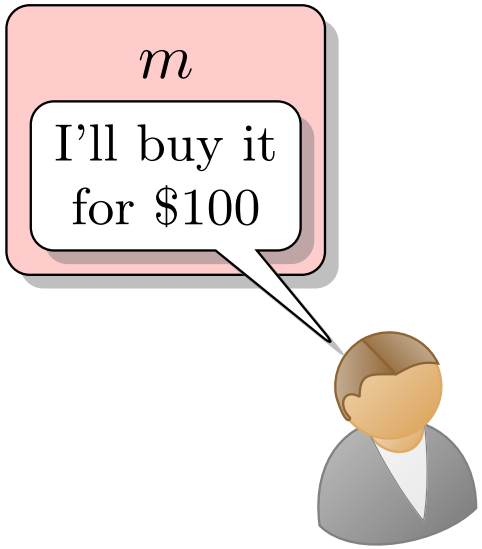

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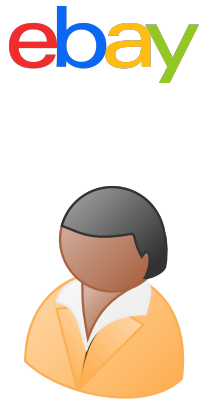

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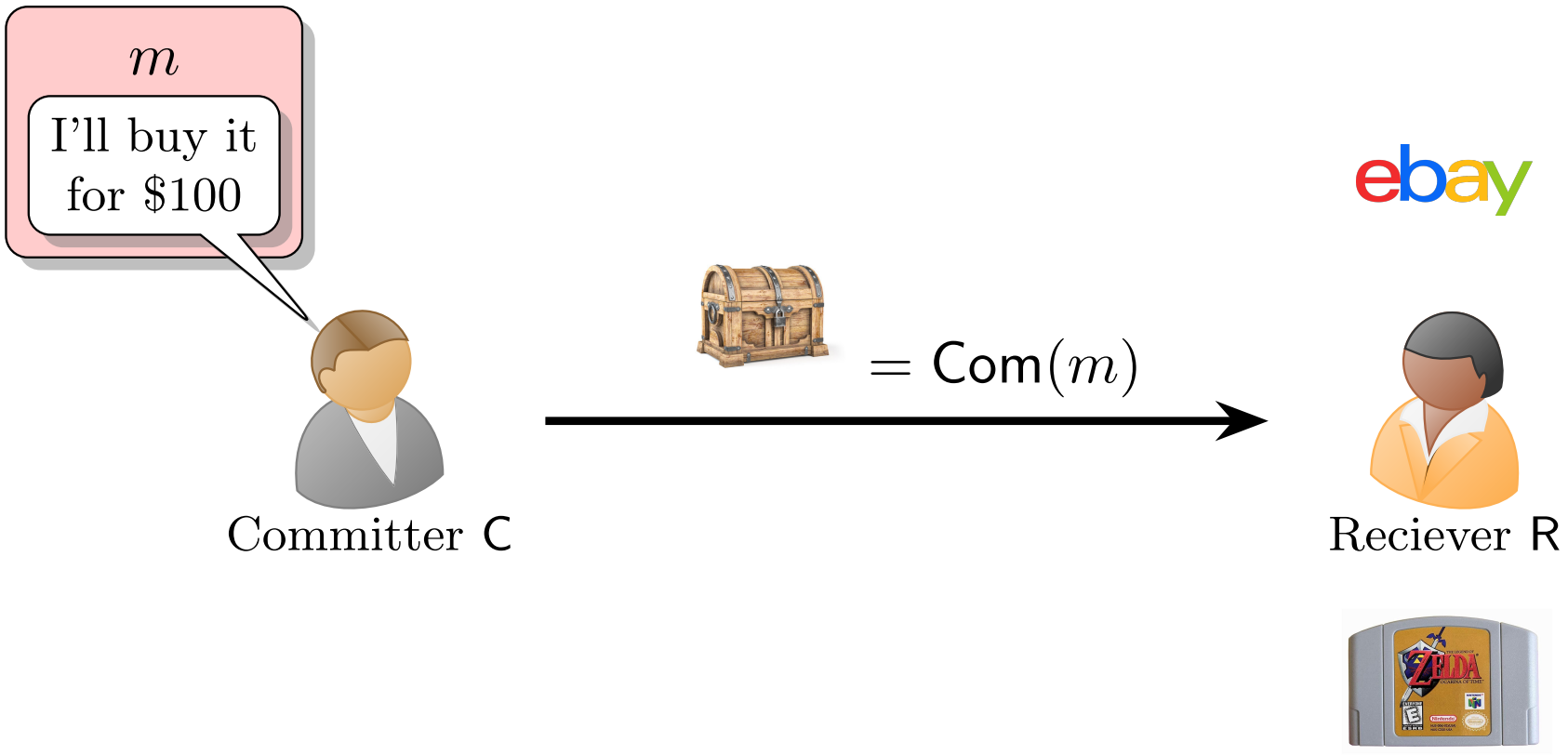

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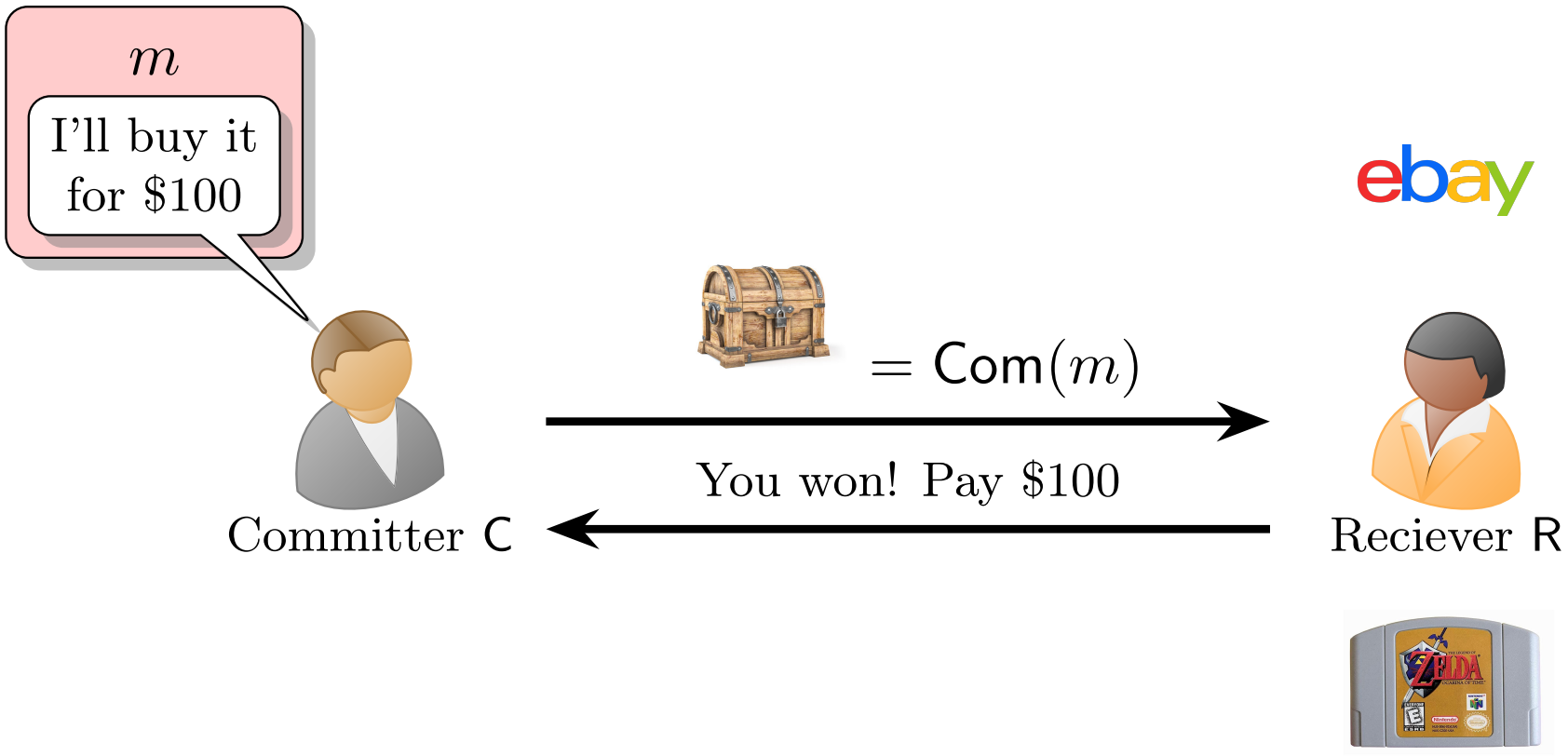

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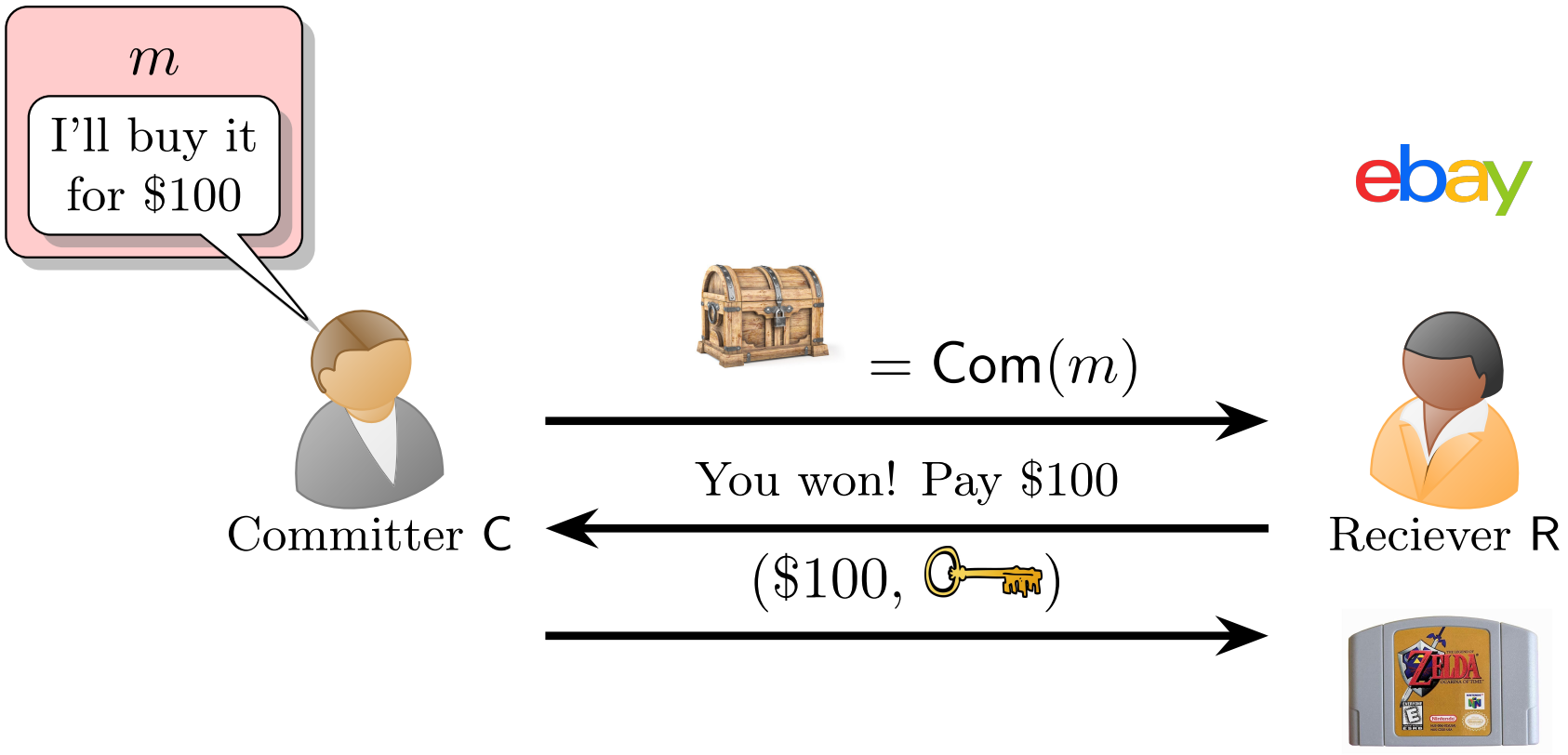

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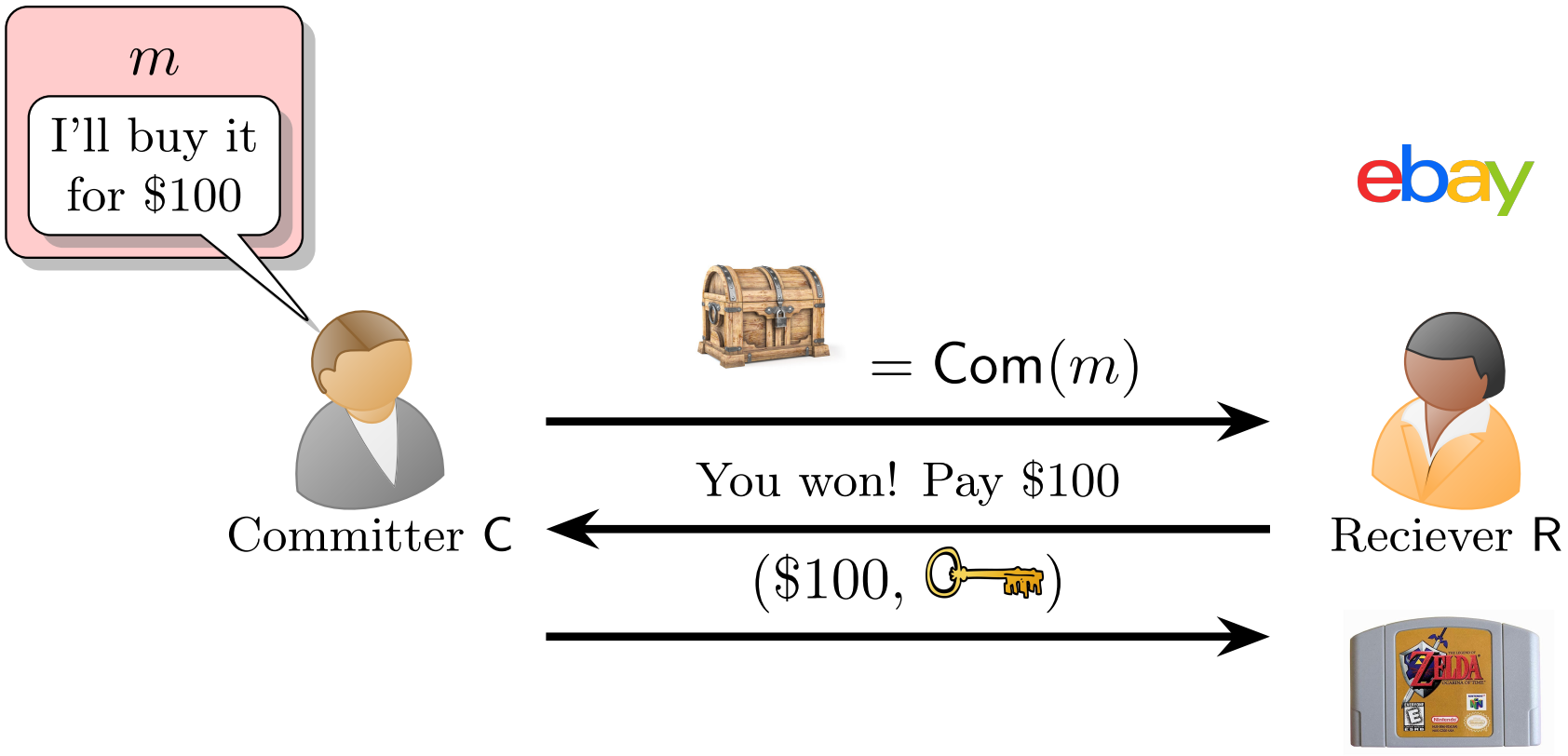

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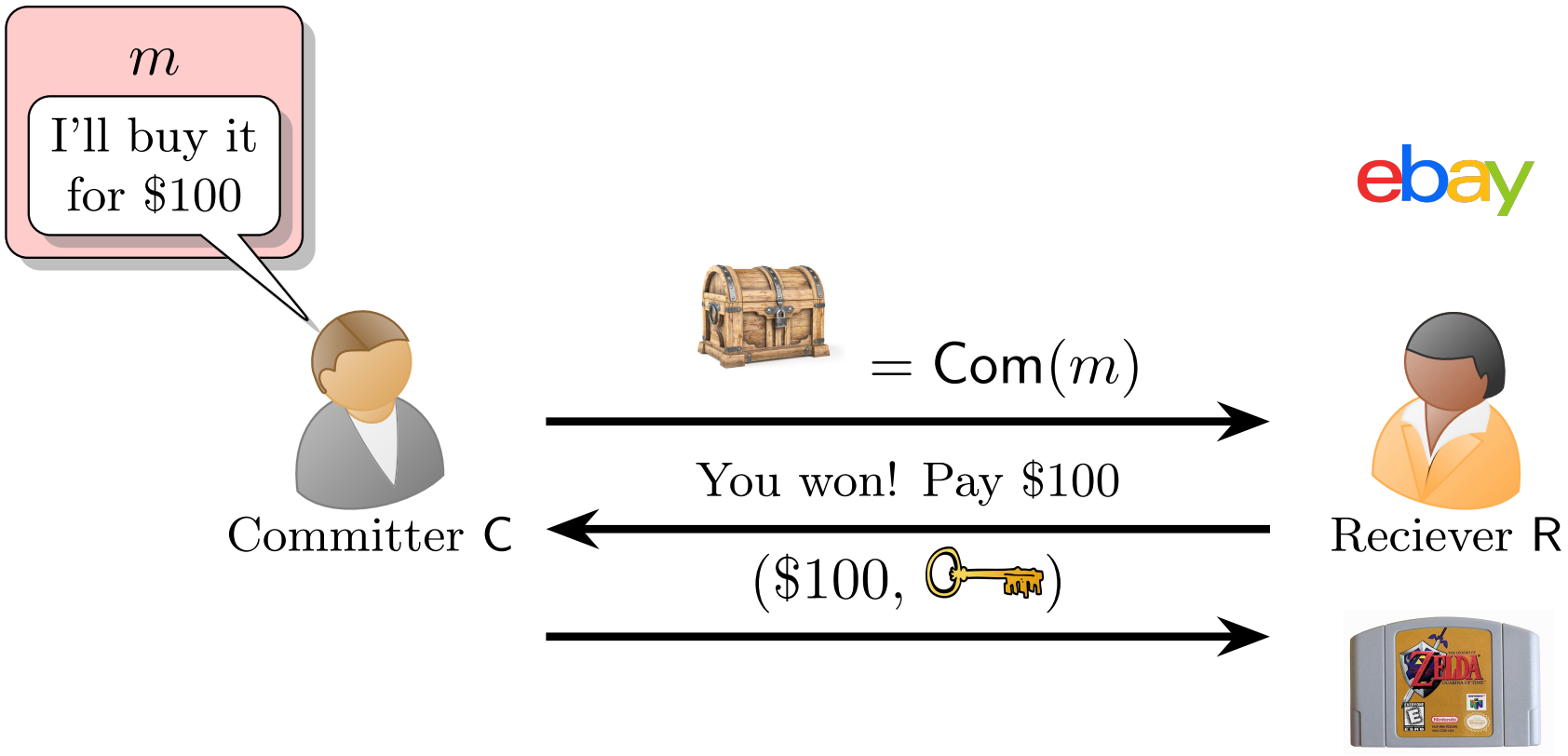

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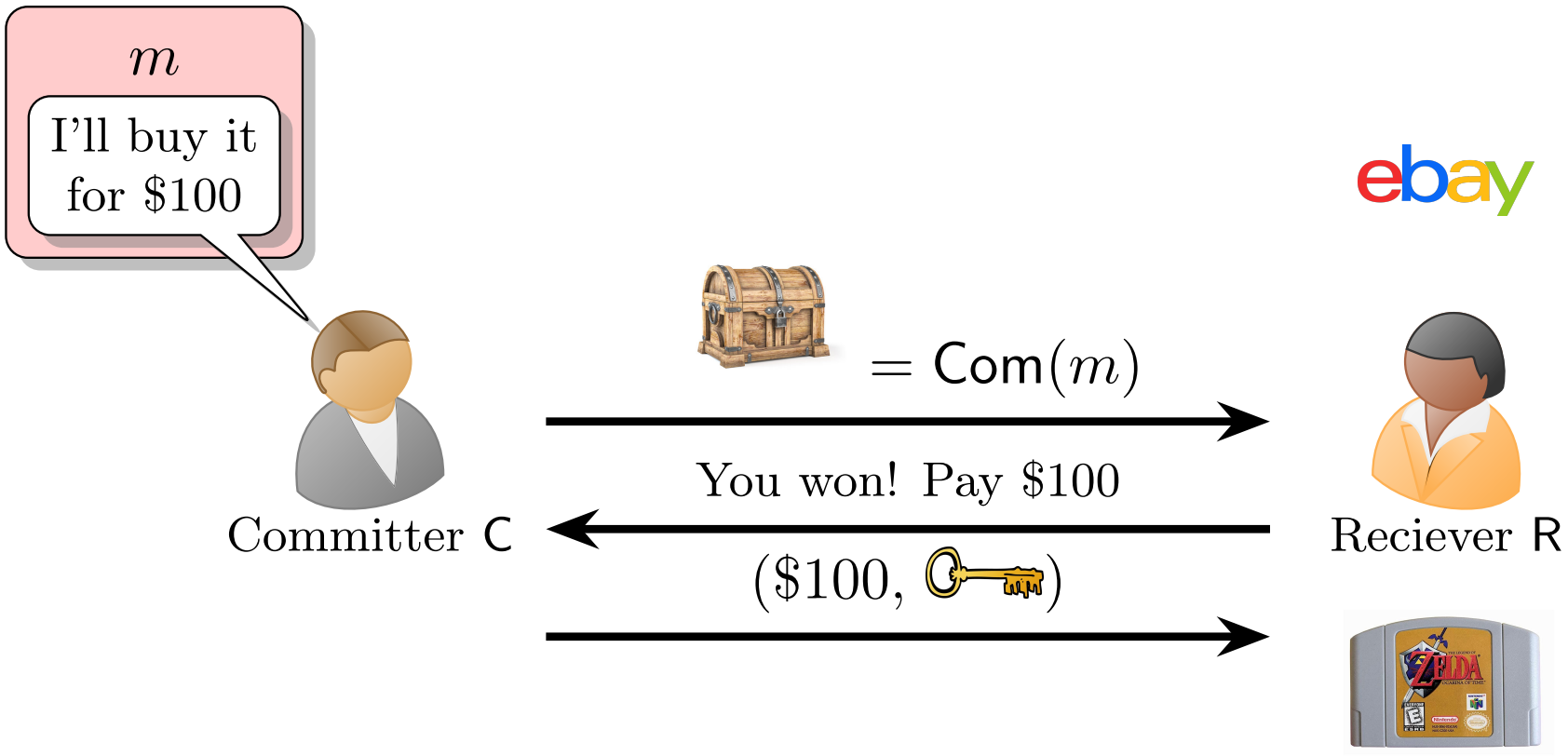

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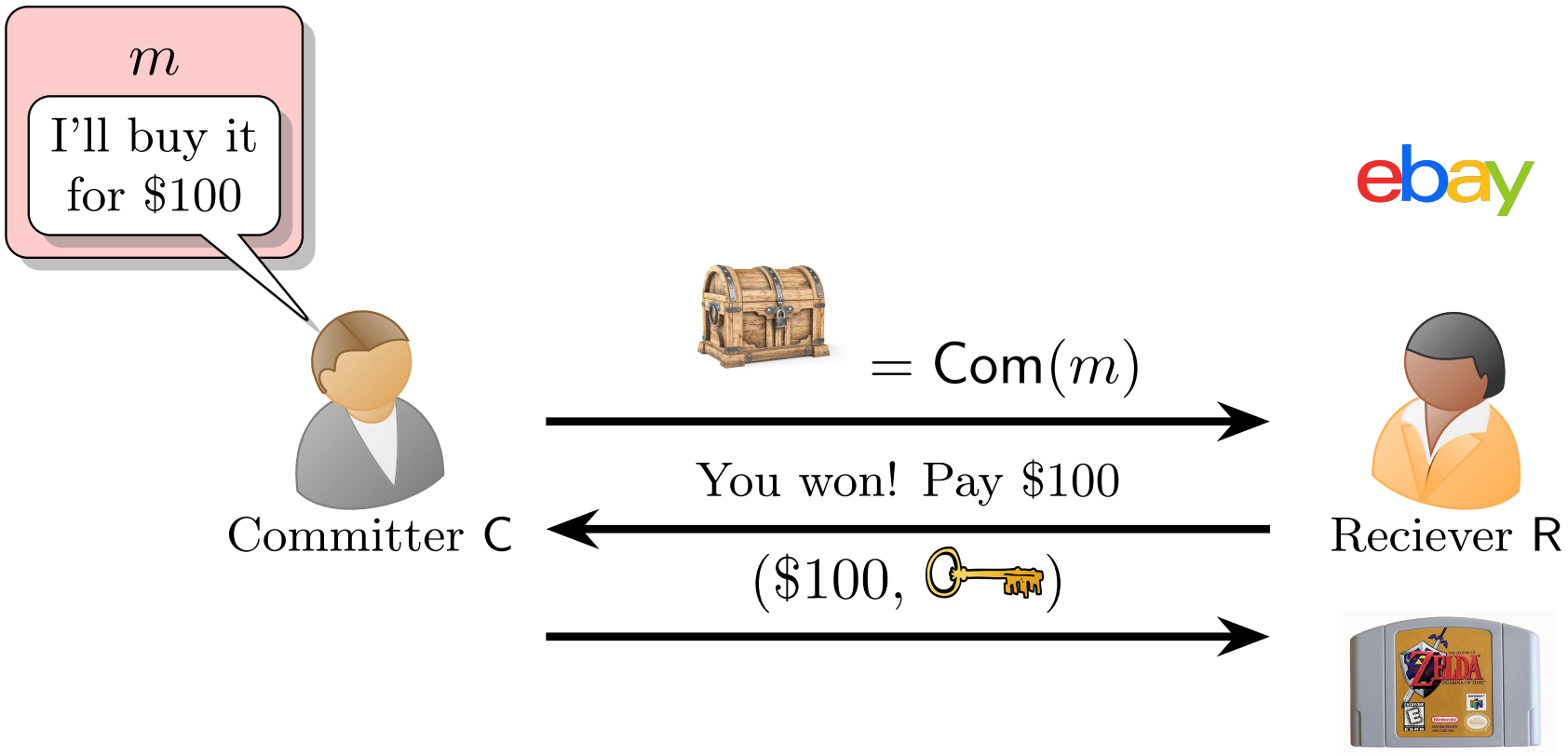

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- Usually, d is just the randomness used by Com.

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- Probability above is also taken over the randomness of Com.

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- R^* outputs $b' \in \{0,1\}$ (R^* *wins if* $b = b'$).

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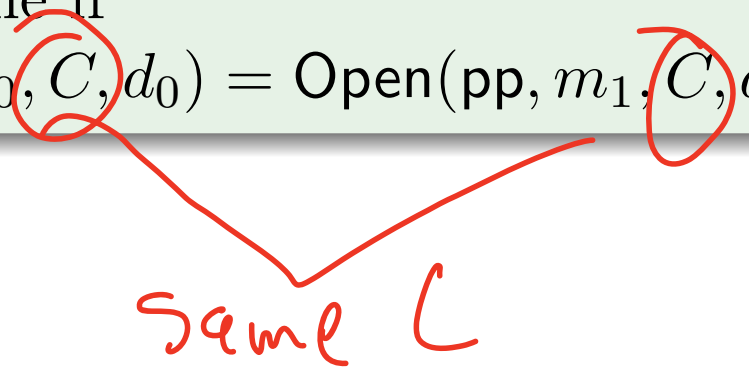

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- **Note**: We say "statistically-binding" commitments to mean they are both statistically binding and computationally hiding.

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ZK for all of NP

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$(c_{i,j}, d_{i,j})$

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(Gen, Com, Open)

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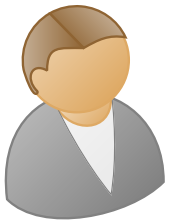

P

ZK for all of NP

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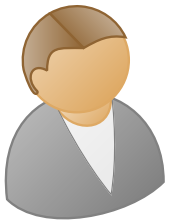

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V

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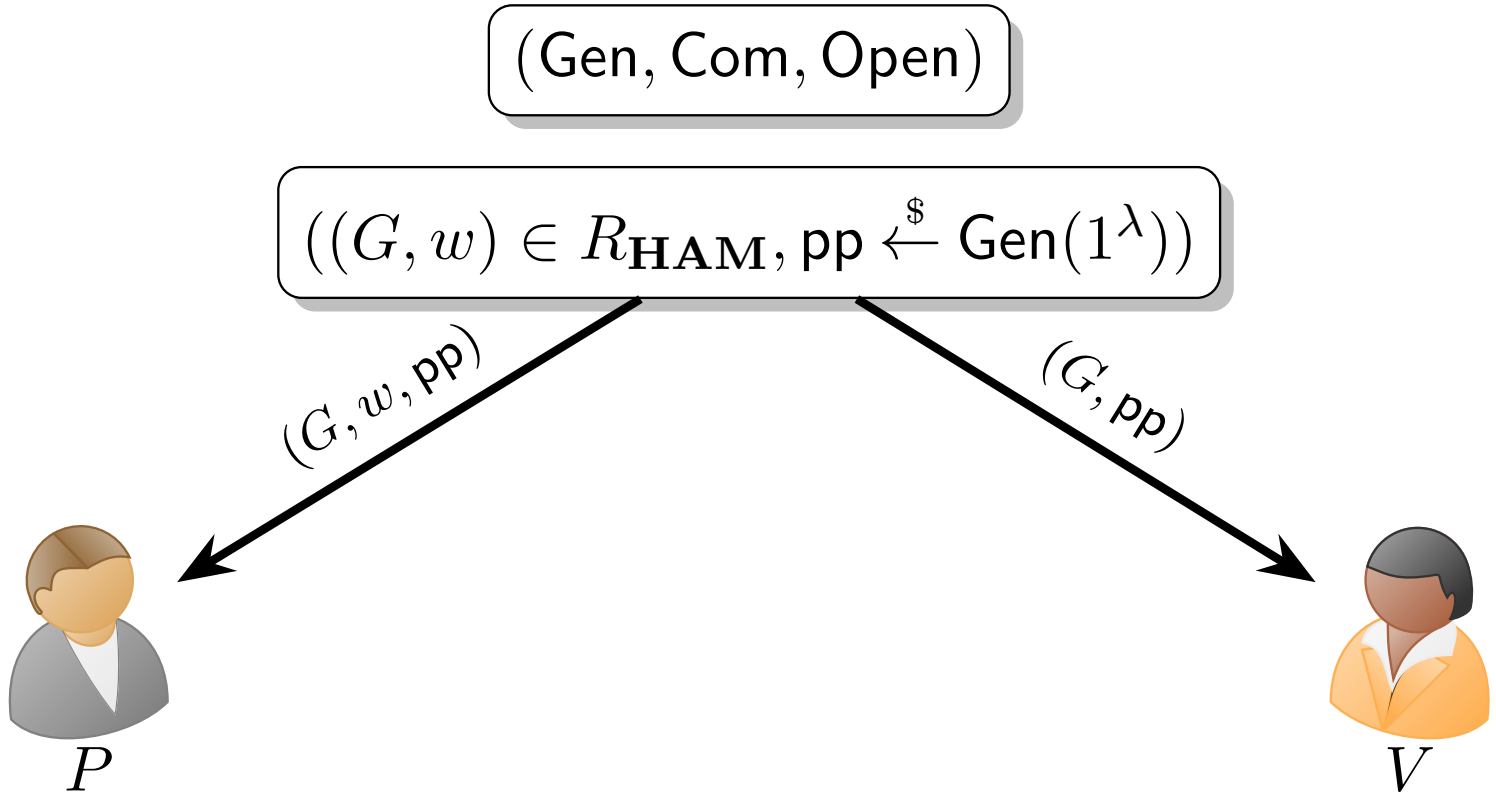

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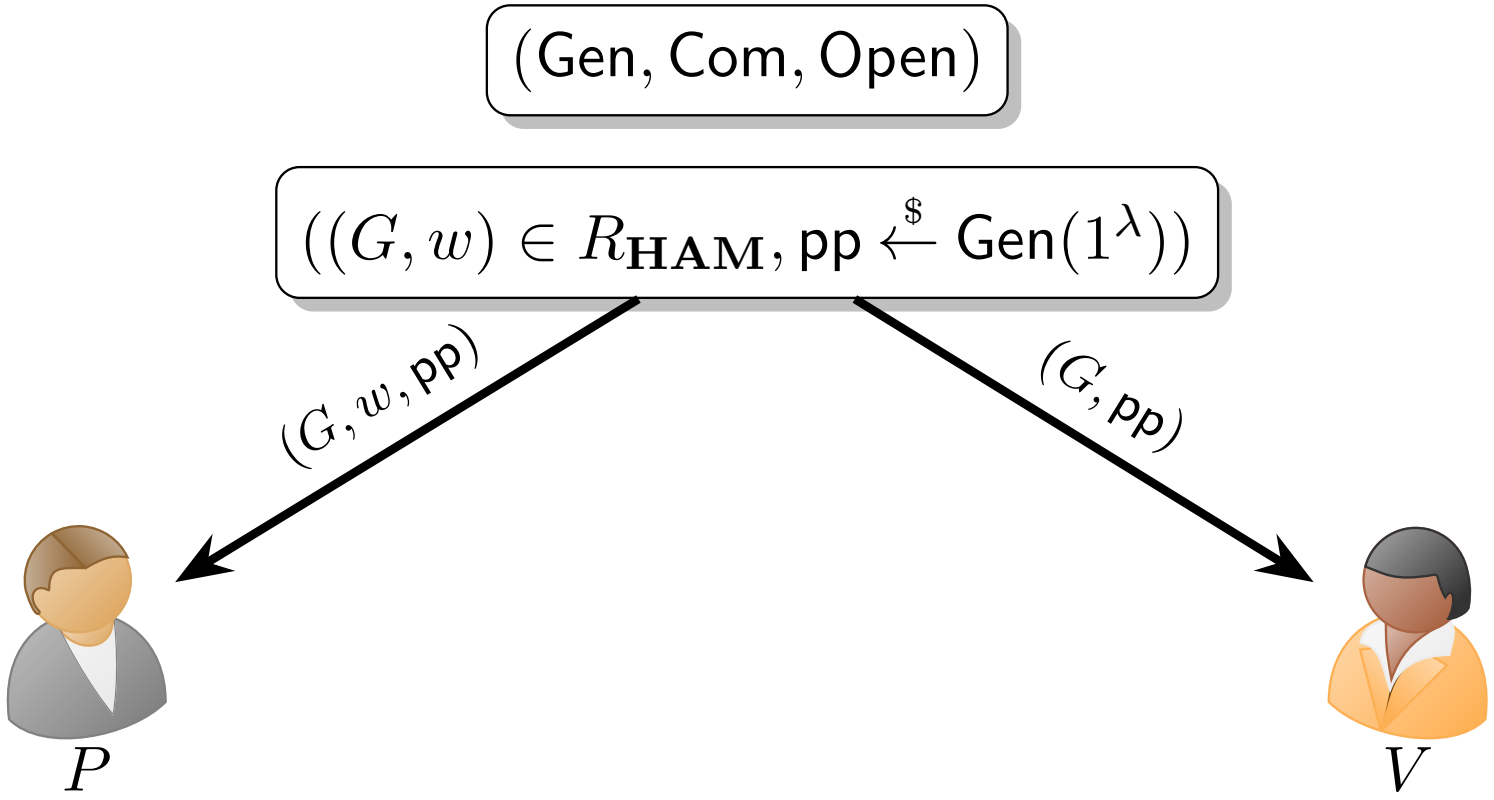

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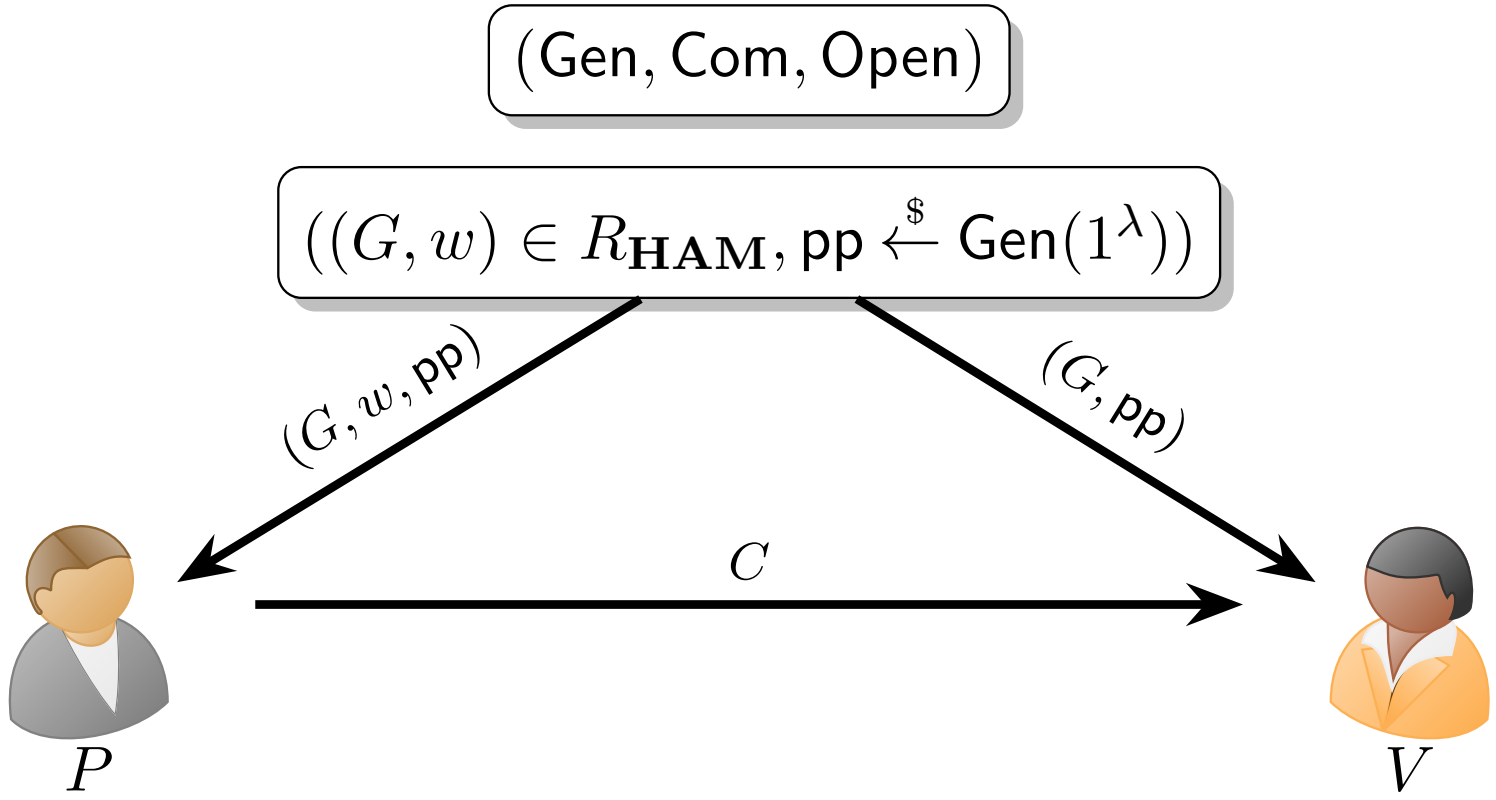

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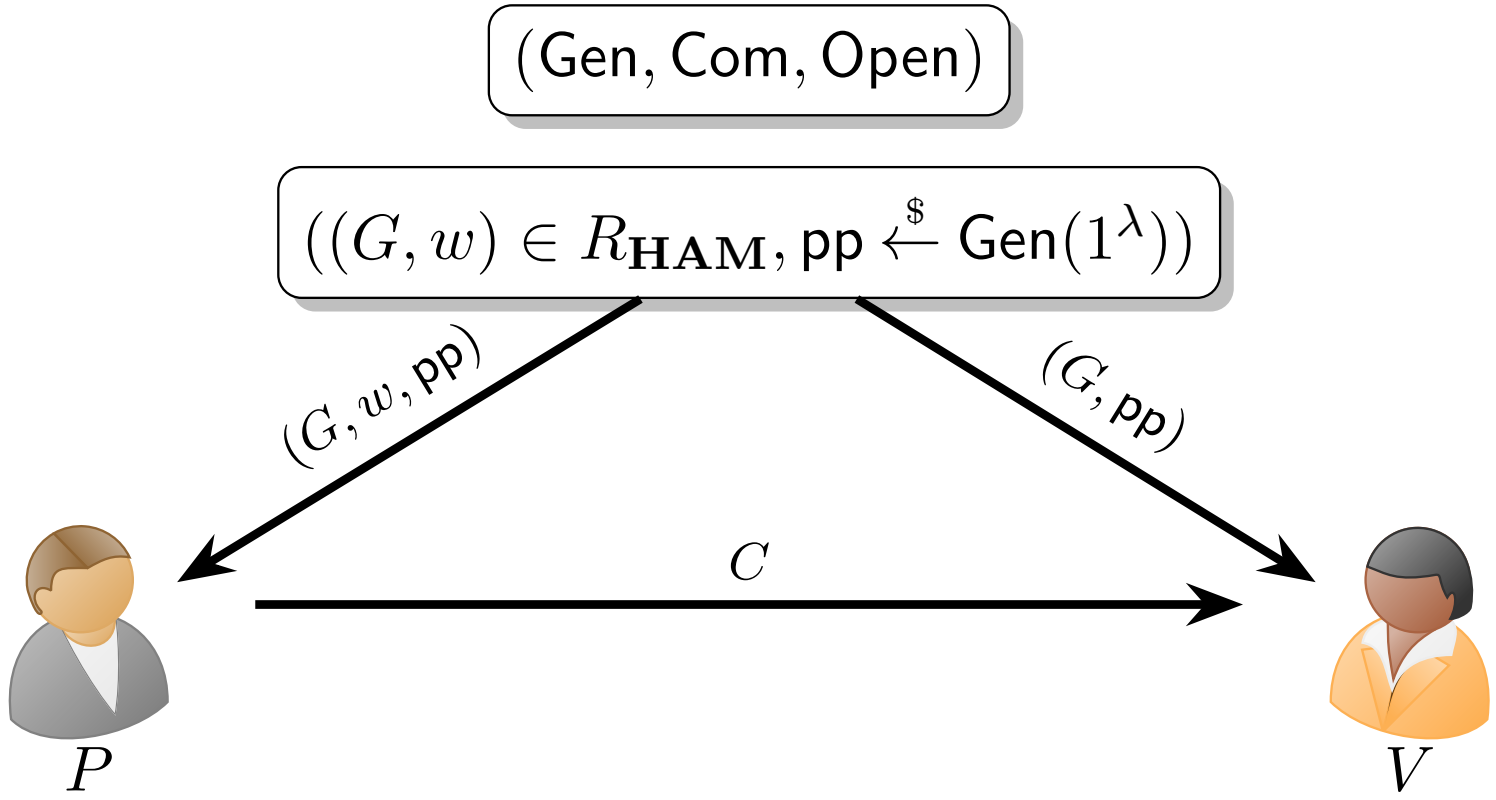

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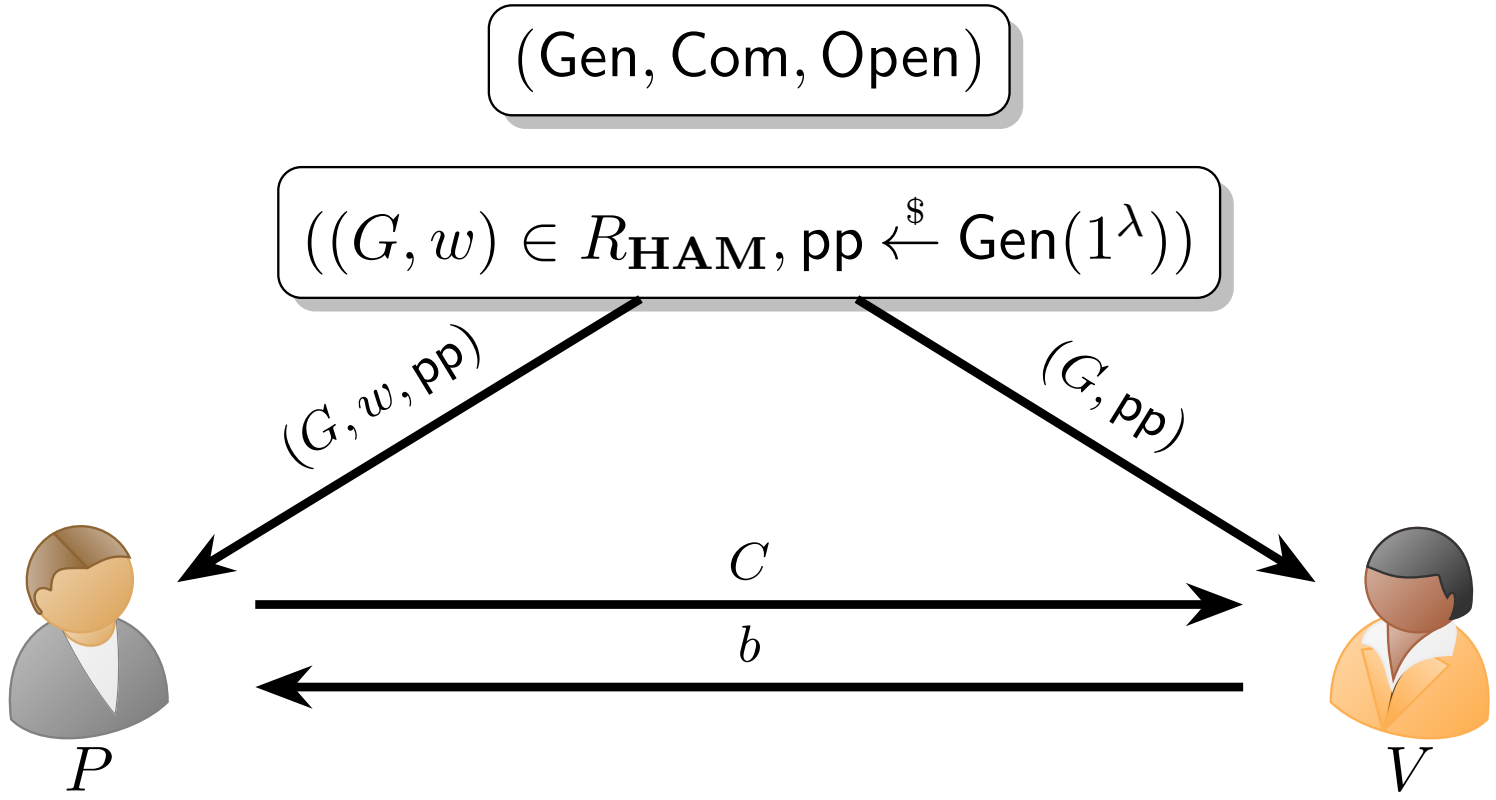

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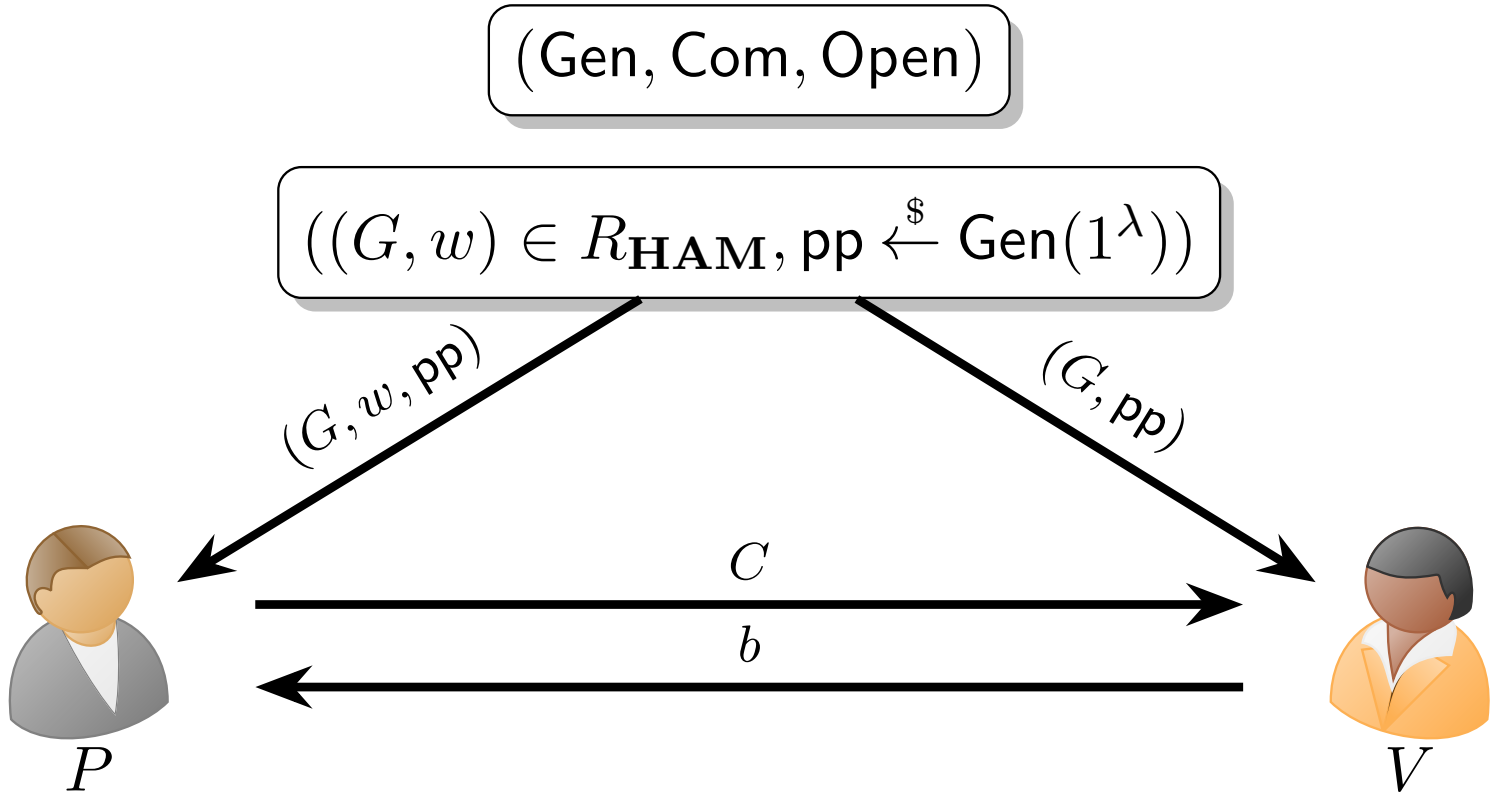

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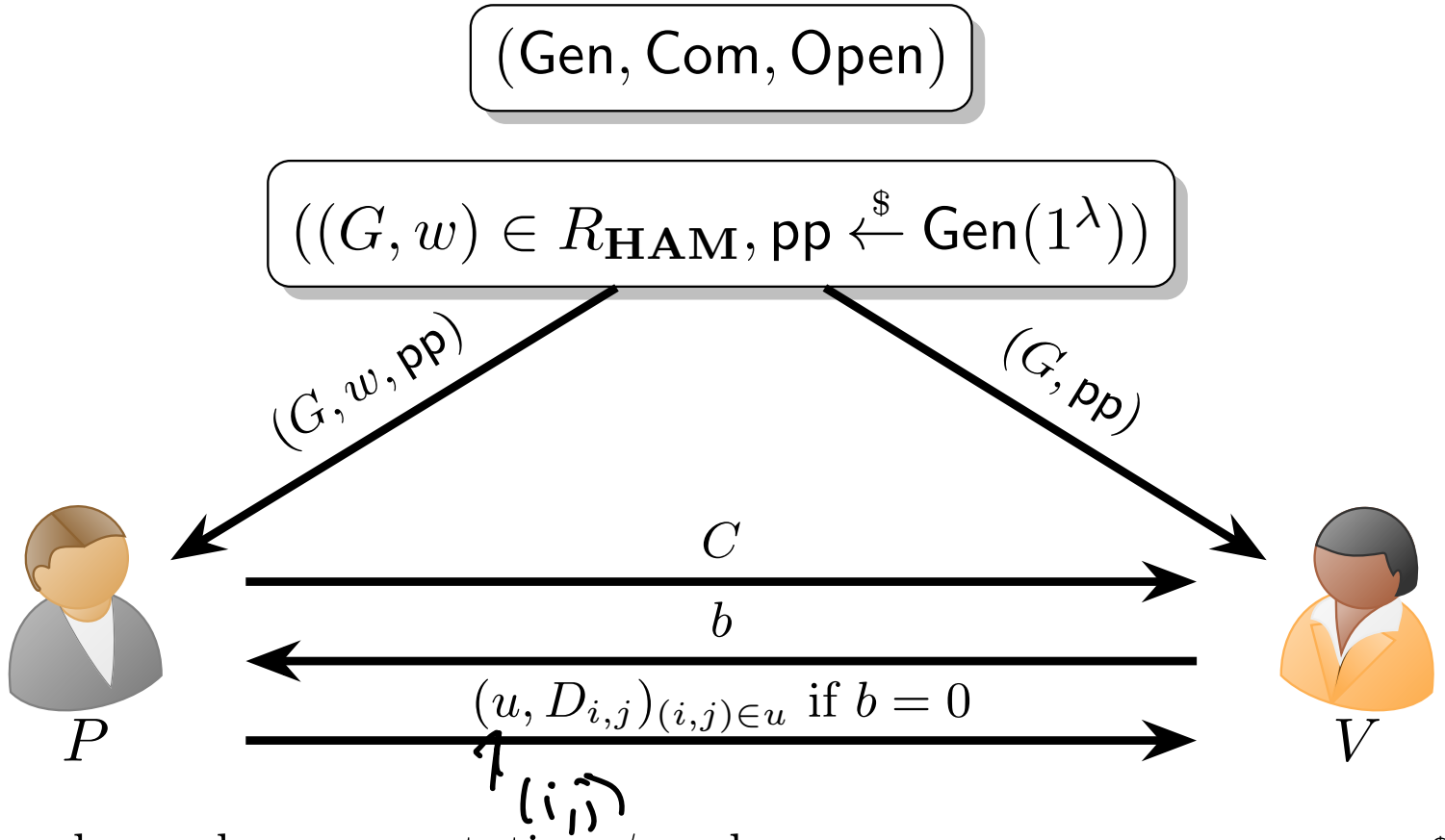

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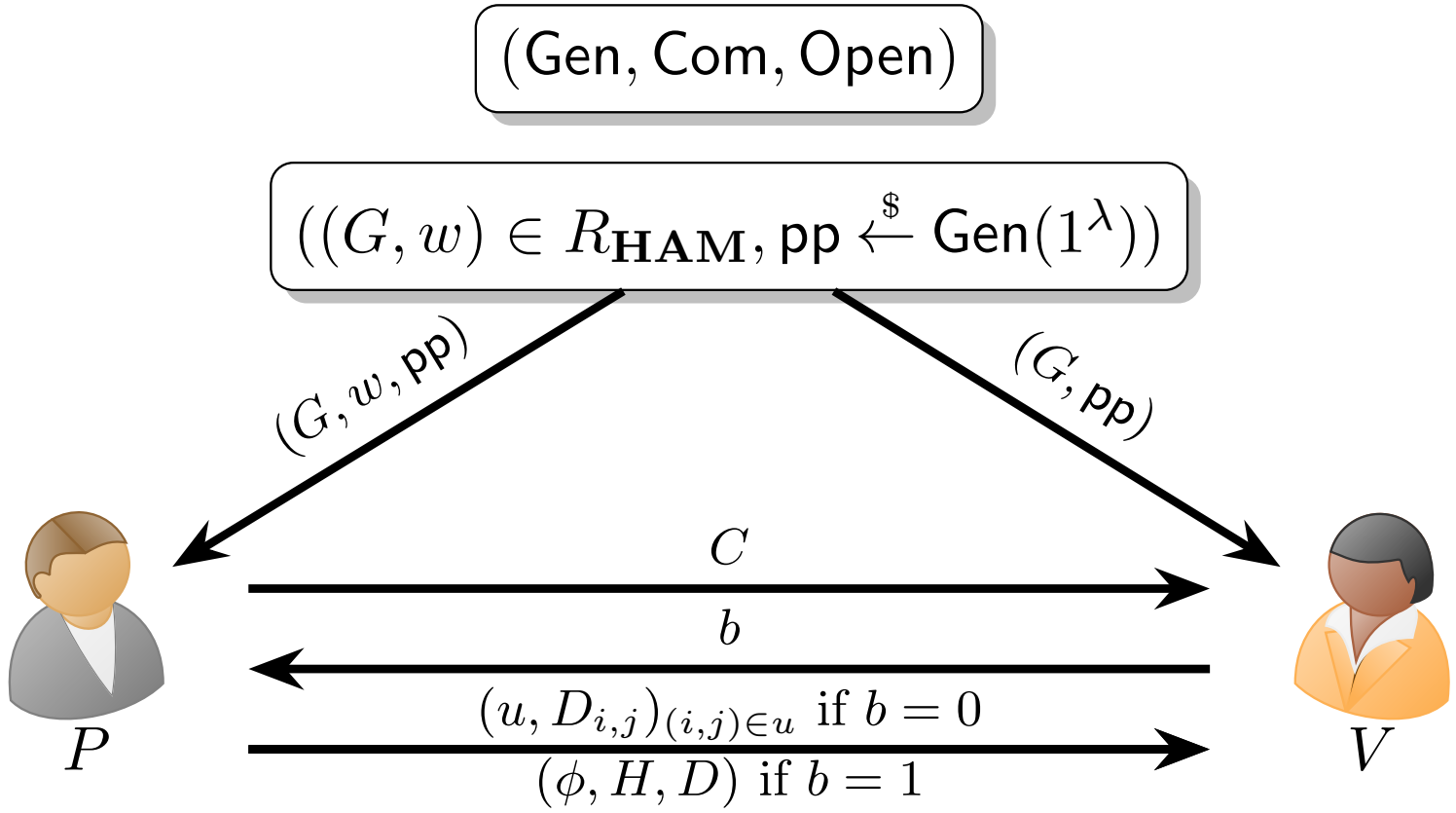

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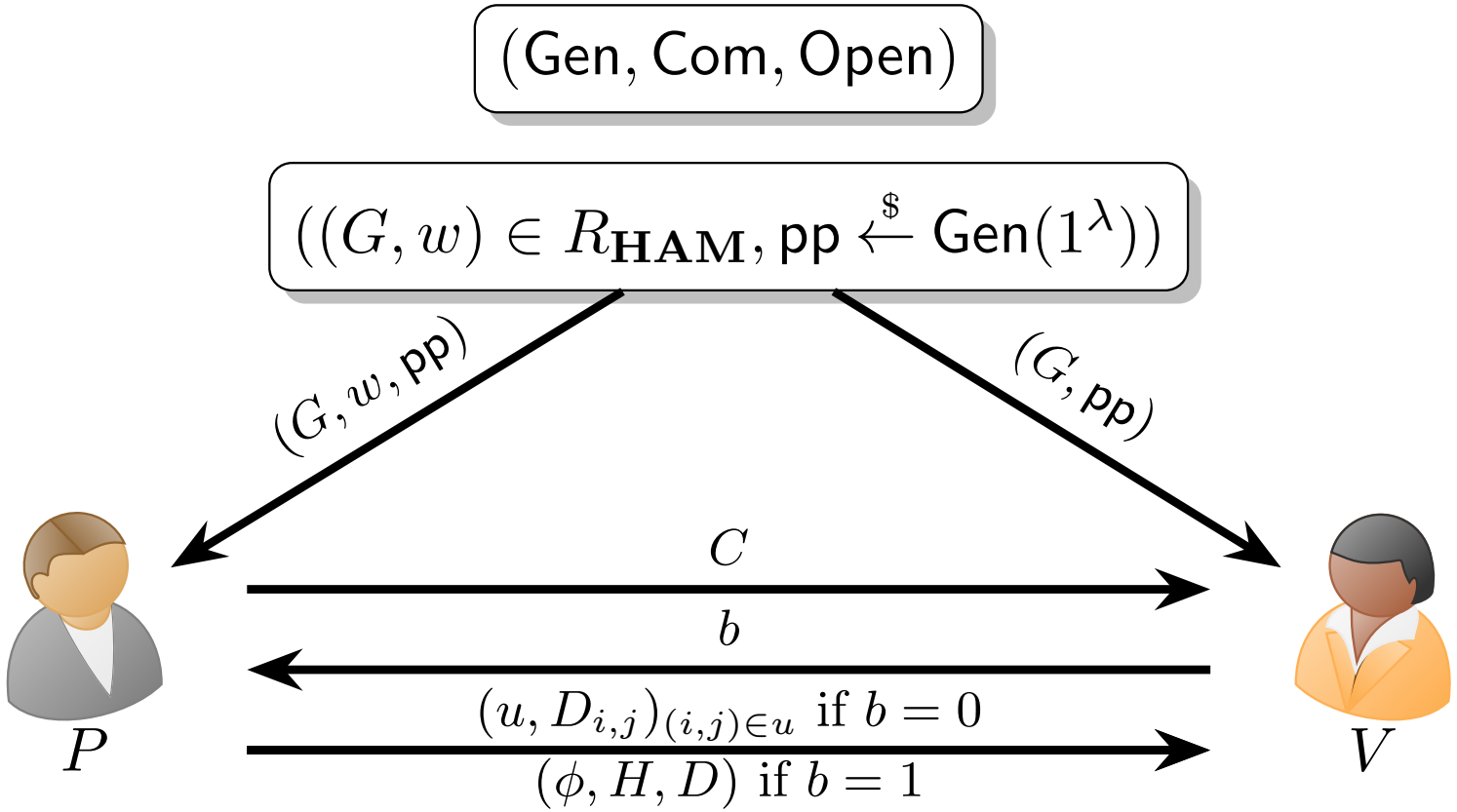

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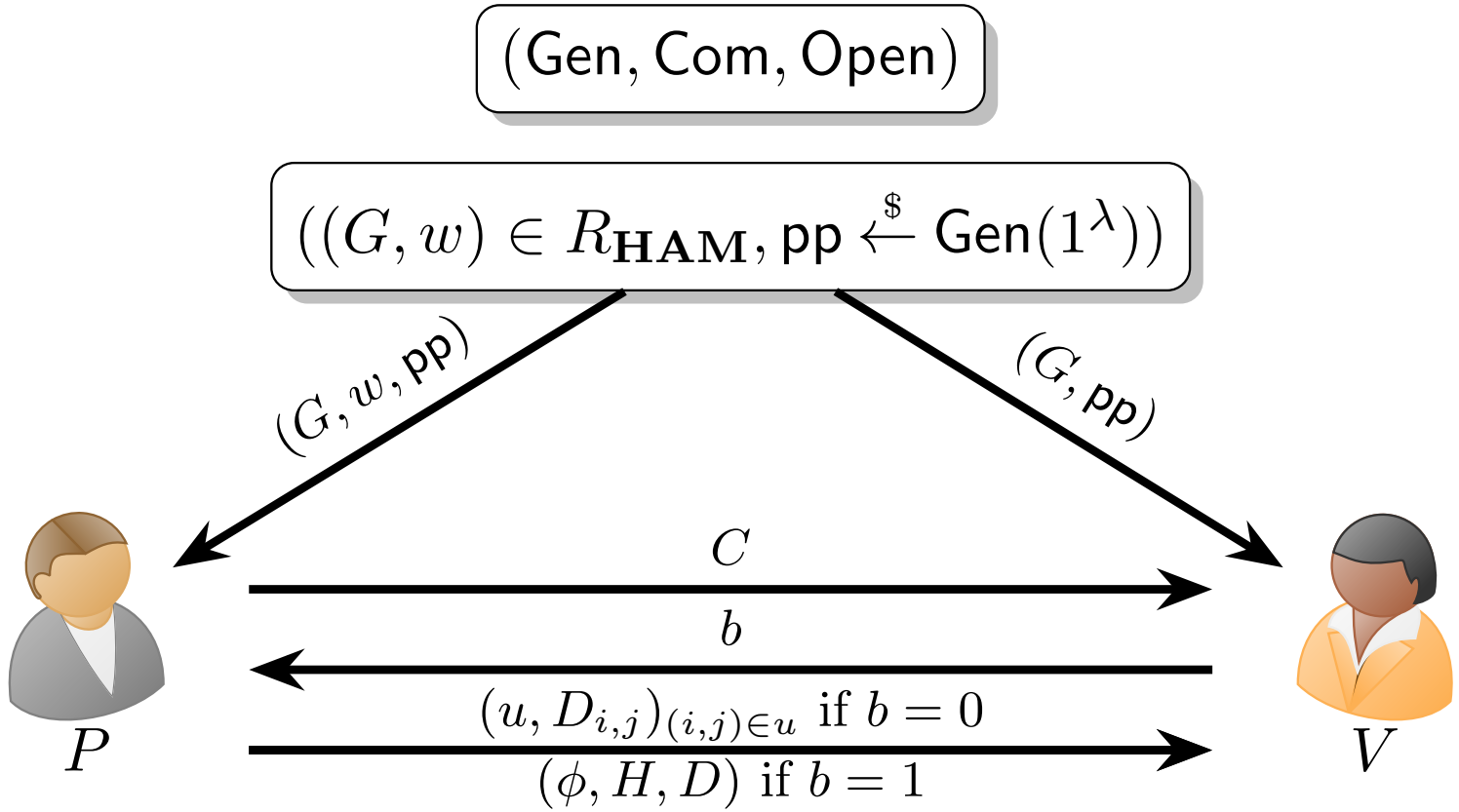

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Let $G \notin \mathbf{HAM}$. Suppose that $\Pr_b[\langle P^*, V\rangle(G, \mathbf{pp}) = 1] > 1/2$ for any prover strategy P^*. This implies that (1) u is a cycle, and (2) $H = \pi(G)$. Note that if this were *not true*, then the prover only convinces the verifier with probability $\leqslant 1/2$.

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ZK for all of NP: Soundness

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$\pi^{-1}(u)$ cannot be a cycle, u cannot be a cycle

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5. Else, goto (1).

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- This implies that the simulator repeats approximately 2 times in expectation, so it is expected polynomial time.

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- Thus, Π has computational zero-knowledge! □

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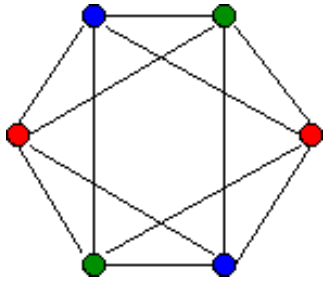

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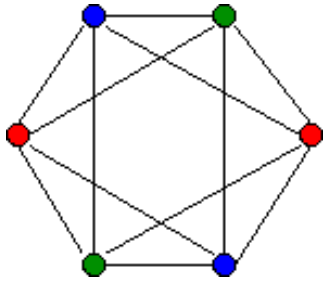

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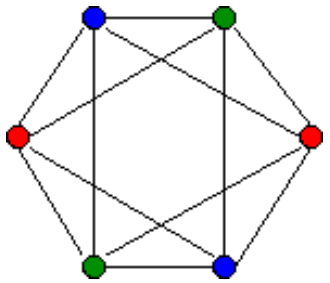

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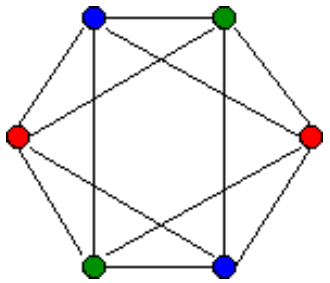

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Protocol for 3Col

(Gen, Com, Open)

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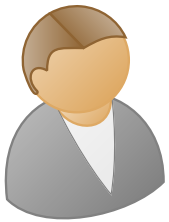

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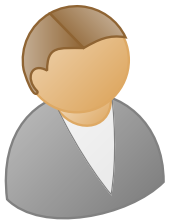

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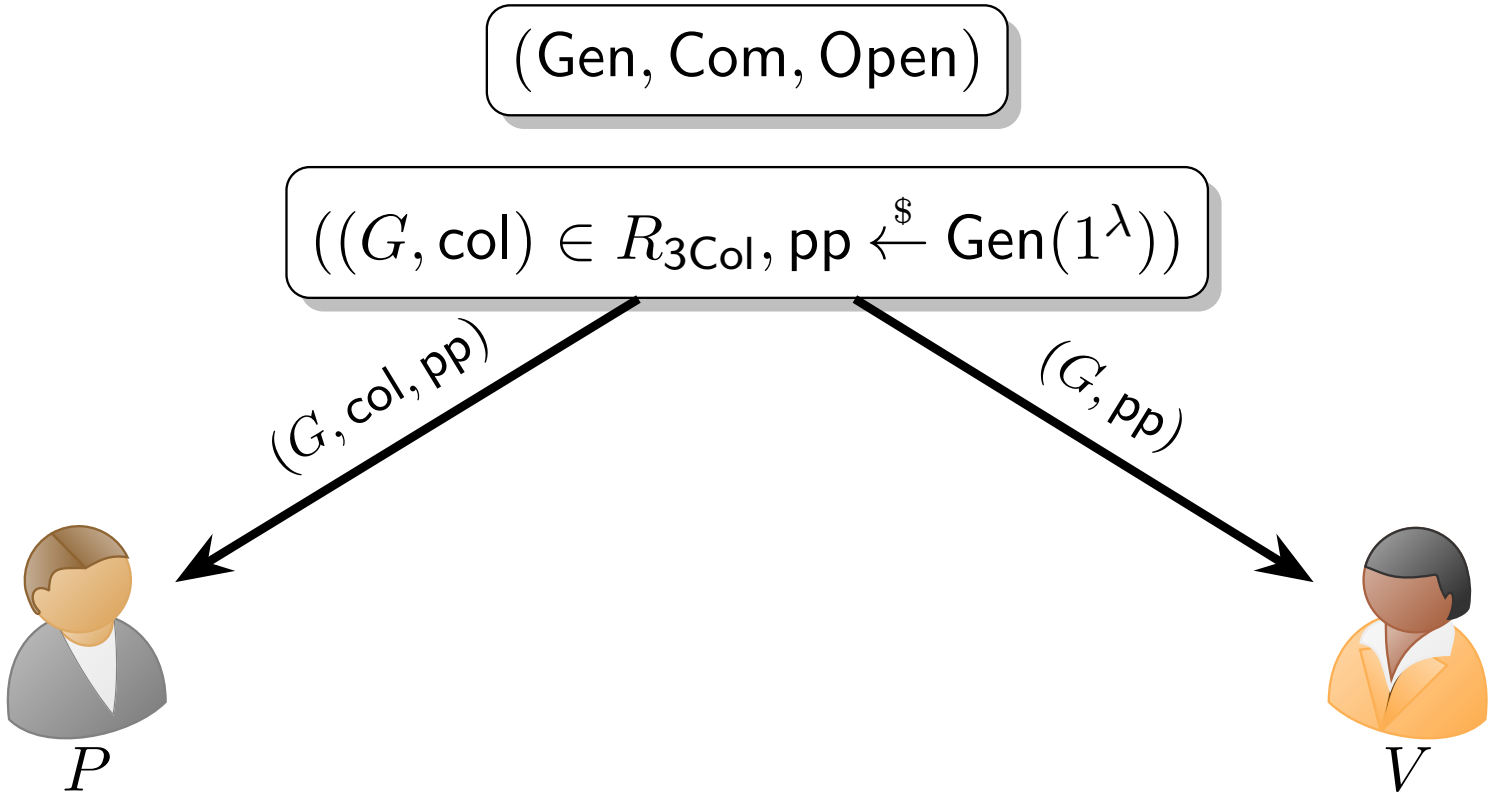
V

Protocol for 3Col



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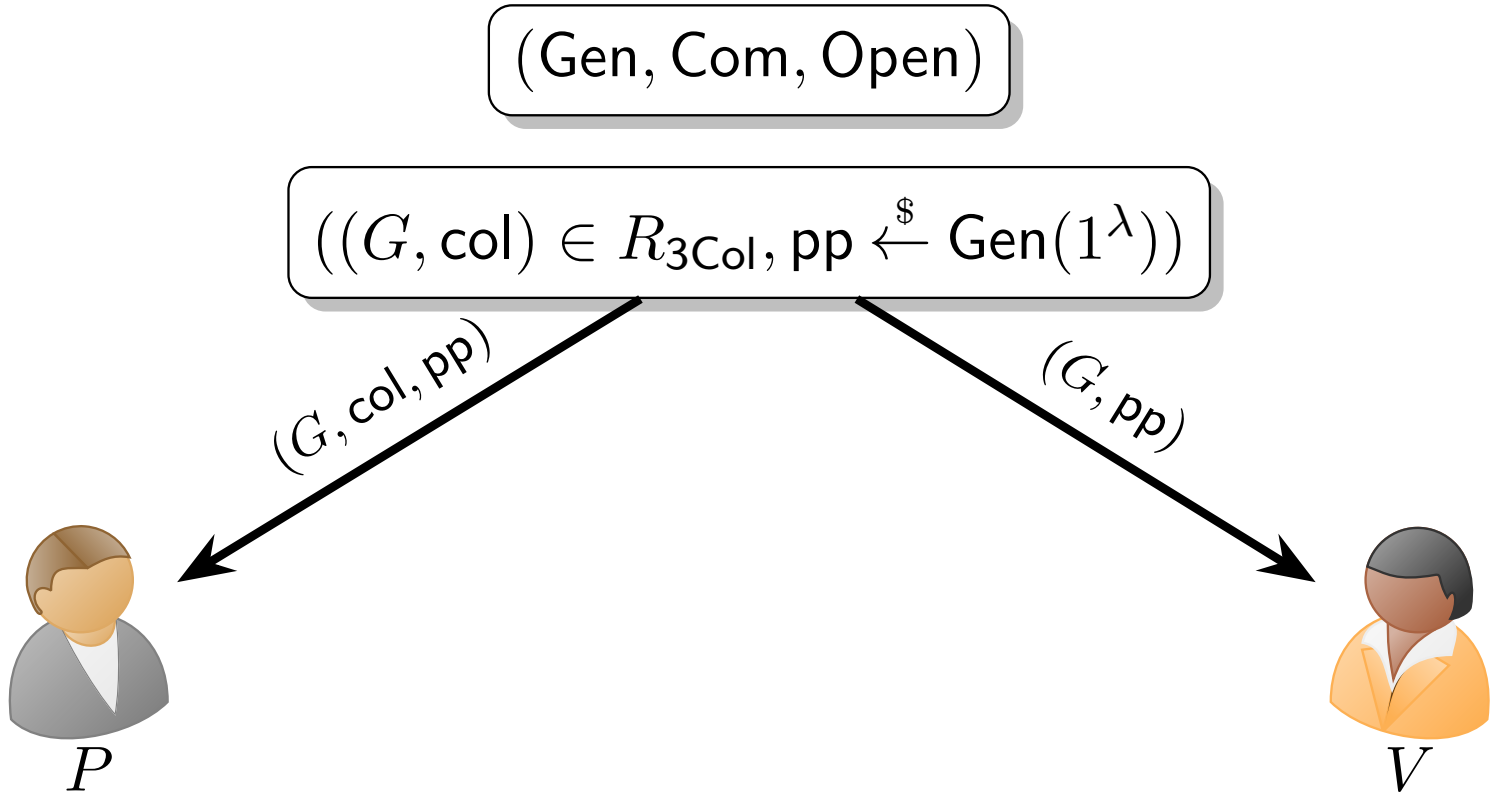

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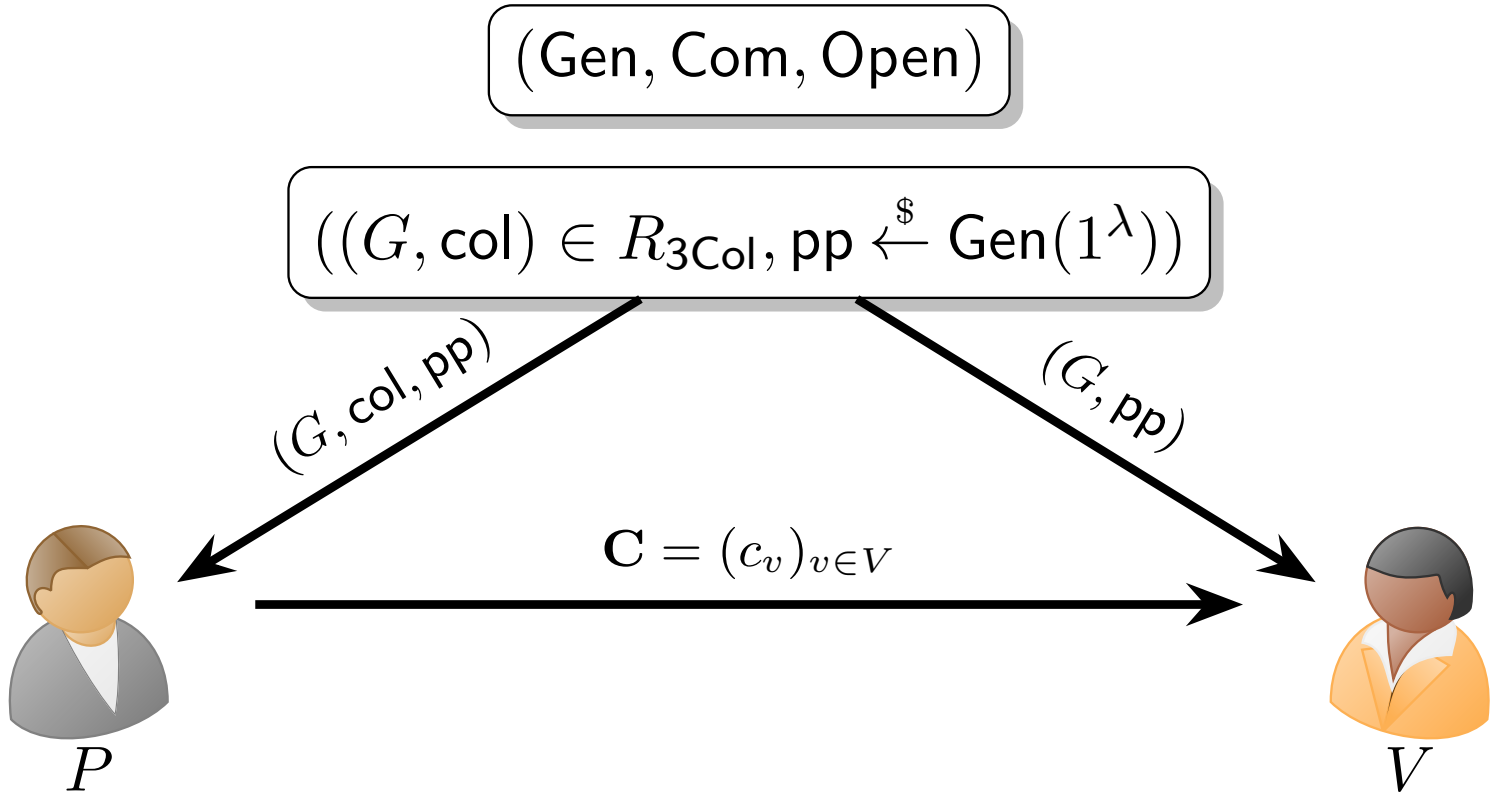

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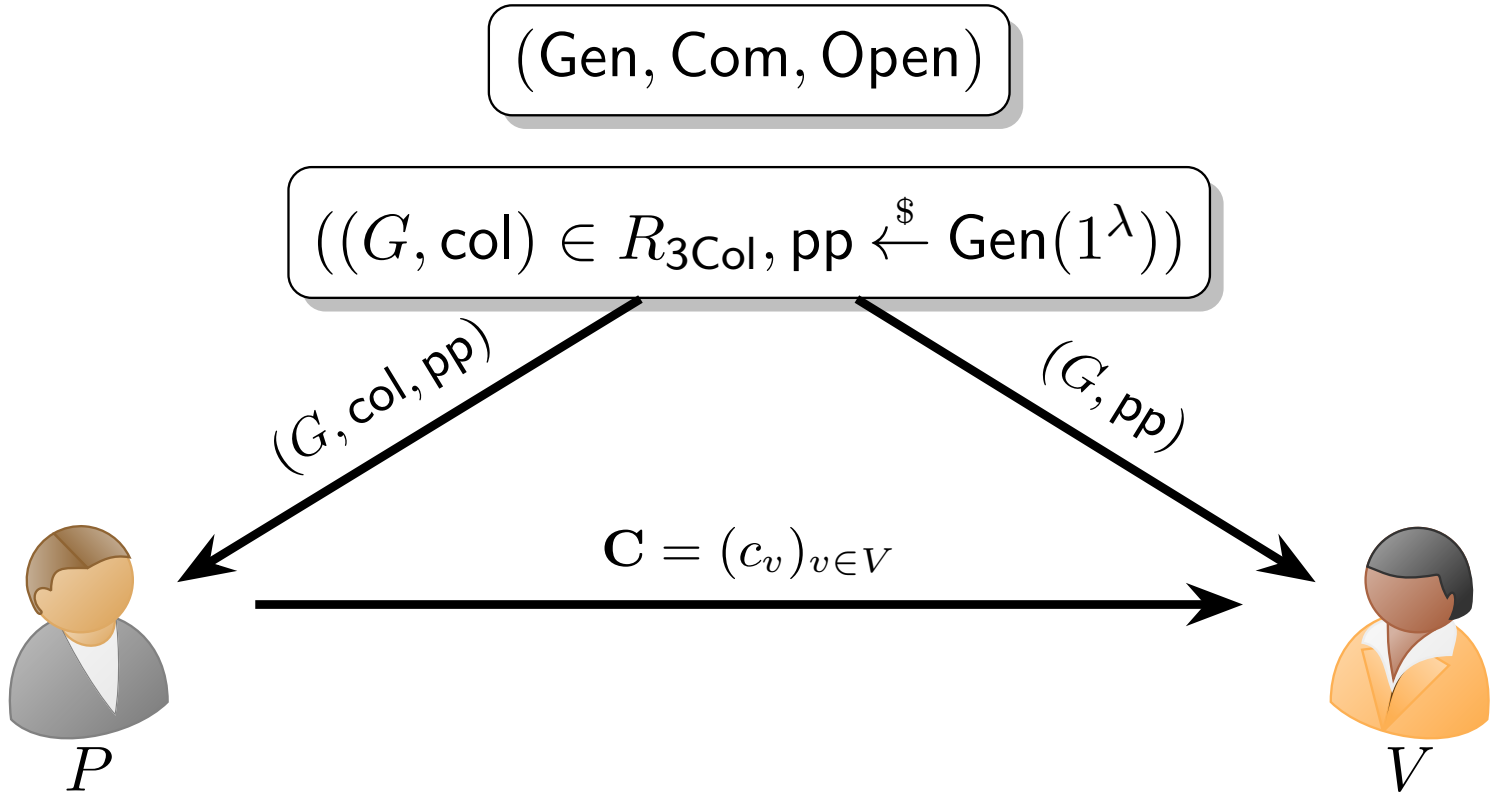

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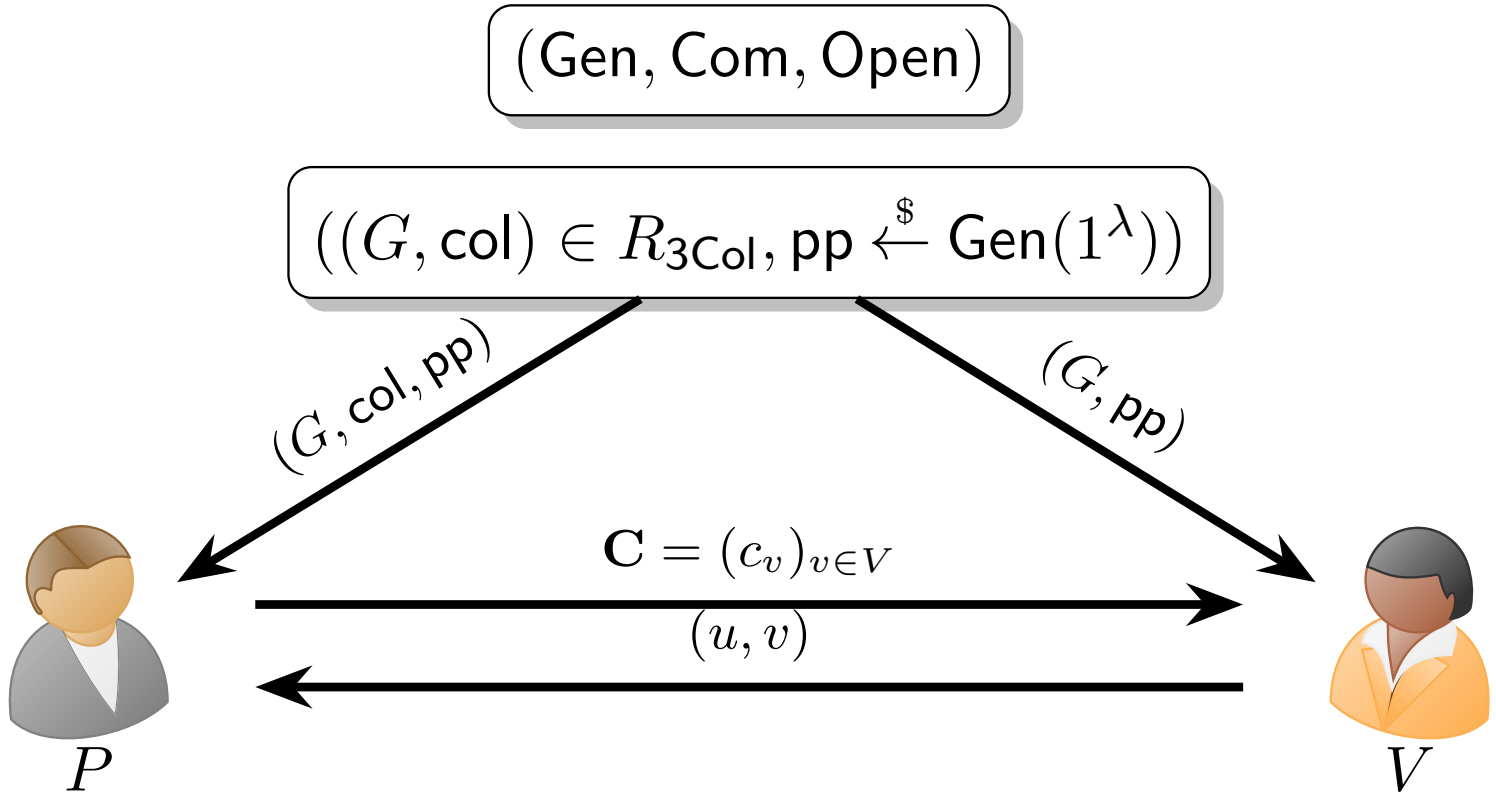

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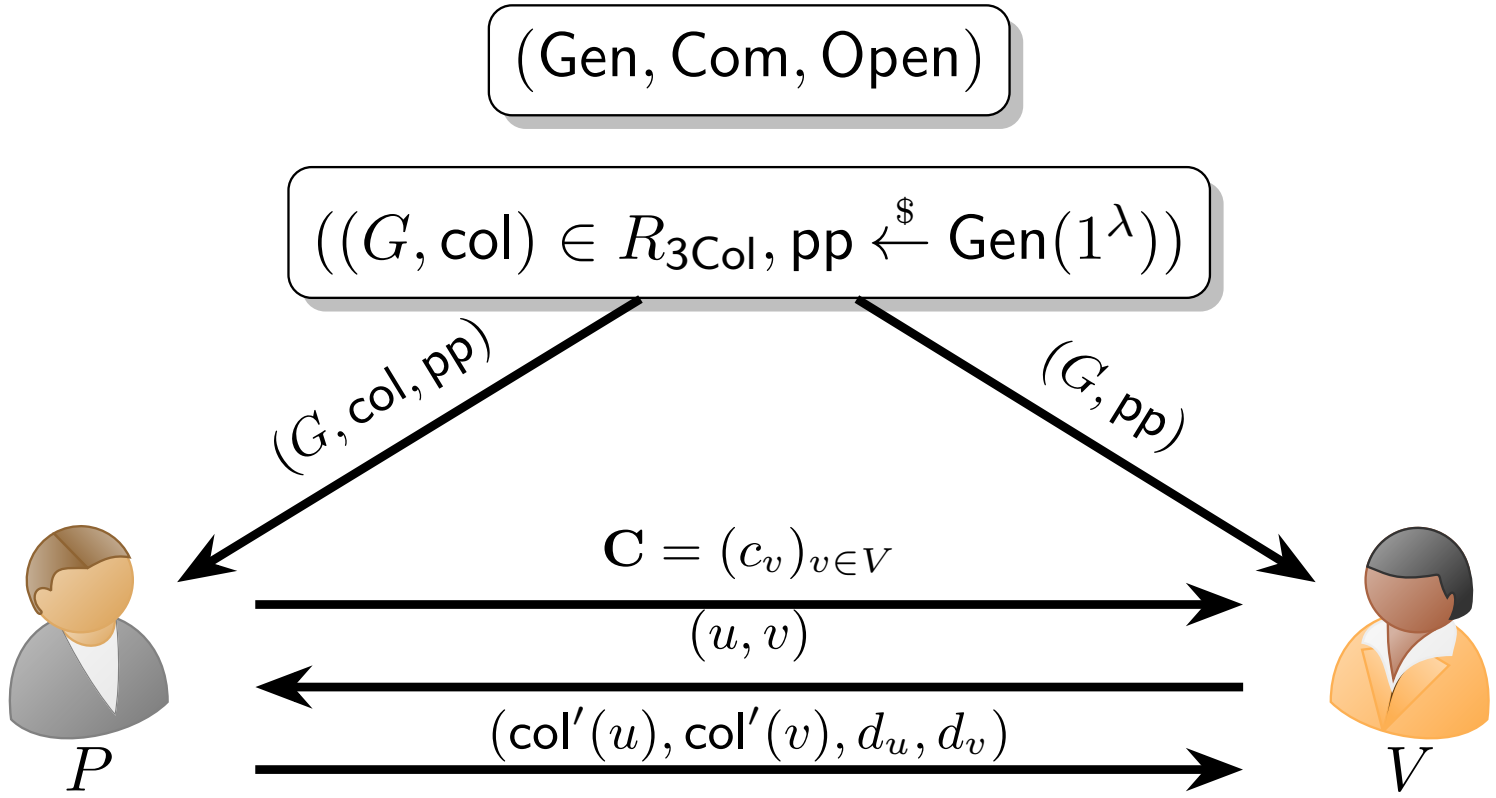

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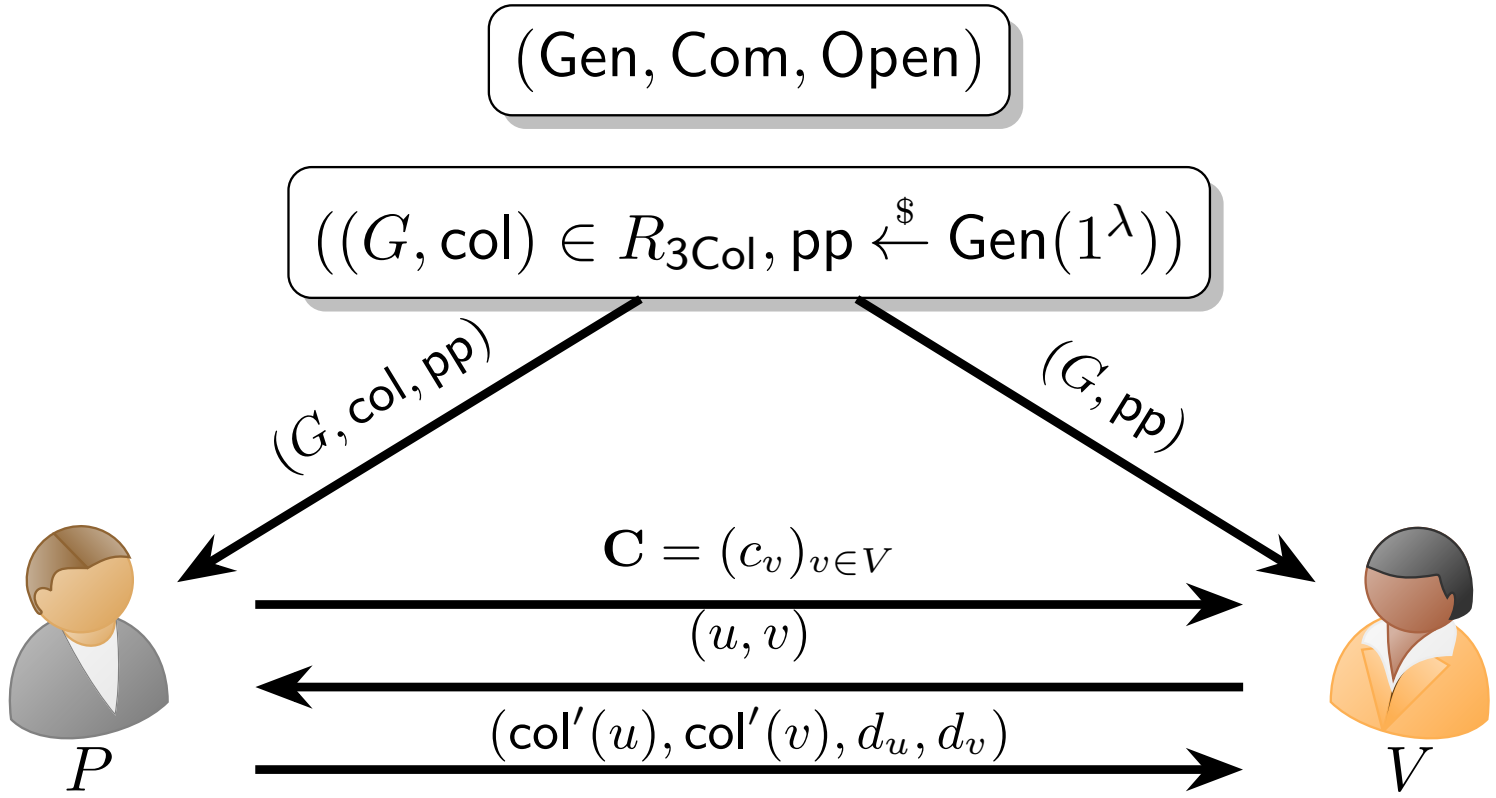

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(V1) Sample $(u, v) \xleftarrow{\$} E$.

(V2) Check the following.

(1) $\mathsf{col}'(u), \mathsf{col}'(v) \in \{0, 1, 2\}$

(2) $\mathsf{col}'(u) \neq \mathsf{col}'(v)$

(3) $\mathsf{Open}(\mathsf{col}'(u), c_u, d_u) = 1$

(4) $\mathsf{Open}(\mathsf{col}'(v), c_v, d_v) = 1$

$\mathsf{3Col} \in \mathrm{CZK}$: Completeness

- Let Π_{3Col} denote the protocol on the previous page.

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Lemma 4

Π_{3Col} *has perfect completeness.*

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Lemma 4

Π_{3Col} *has perfect completeness.*

Proof.

Follows immediately by completeness of Com and that col is a valid 3-coloring of G. Formal proof left as an exercise. □

$\mathsf{3Col} \in \mathrm{CZK}$: Soundness

Lemma 5

If Com *is statistically binding, then* Π_{3Col} *has soundness error at most* $1 - 1/|E|$.

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If G does not have a 3-coloring, then for all functions $\mathsf{col} \colon V \to \{0, 1, 2\}$, there exists an edge $(u, v) \in E$ such that $\mathsf{col}(u) = \mathsf{col}(v)$.

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- Note that soundness error $1 - 1/|E|$ is quite bad for large E, so one should amplify the soundness of the protocol by sequential repetition.

3Col $\in$ CZK: HVZK

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- We only show HVZK and leave the full malicious verifier ZK proof as an exercise.

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  - Hint: it is similar to the proof for **HAM**.

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Lemma 6

If Com *is computationally hiding, then* Π_{3Col} *is HVCZK.*

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- We only show HVZK and leave the full malicious verifier ZK proof as an exercise.
  - Hint: it is similar to the proof for **HAM**.

Lemma 6

If Com *is computationally hiding, then* Π_{3Col} *is HVCZK.*

- The proof formalizes the intuition of the protocol that the verifier only sees a randomly colored edge.

3Col $\in$ CZK: HVZK

- First, define our simulator.

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$S(G, \mathsf{pp})$

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1. Sample $(u, v) \xleftarrow{\$} E$.
2. Define col' as $\mathsf{col}'(x) = 0$ for all $x \neq u, v$, and set $\mathsf{col}'(u) \xleftarrow{\$} \{0, 1, 2\}$ and $\mathsf{col}'(v) \xleftarrow{\$} \{0, 1, 2\}$, resampling if $\mathsf{col}'(u) = \mathsf{col}'(v)$.

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3. Compute $(c_w, d_w) = \mathsf{Com}(\mathsf{pp}, \mathsf{col}'(w))$ for all $w \in V$, and set $\mathbf{C} = (c_w)_{w \in V}$.

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4. Output $(\mathbf{C}, (u, v), \mathsf{col}'(u), \mathsf{col}'(v), d_u, d_v)$.

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  - Define a hybrid simulator $\widetilde{S}$ which takes $G, \mathsf{col}, \mathsf{pp}$ as input.

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  - Define a hybrid simulator $\widetilde{S}$ which takes $G, \mathsf{col}, \mathsf{pp}$ as input.
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  - Computational hiding will show these two simulators are indistinguishable.

More on CZK

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- One-way functions are *necessary* for non-trivial zero-knowledge.

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Theorem 4 (Ostrovsky-Wigderson (90))

If there exists $L \in \mathbf{CZK} \setminus \mathbf{BPP}$, *then there exist functions with one-way instances.*

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If there exists $L \in \mathbf{CZK} \setminus \mathbf{BPP}$, *then there exist functions with one-way instances.*

Theorem 5 (Ostrovsky-Wigderson (90))

If there exists $L \in \mathbf{CZK}$ *that is hard-on-average, then one-way functions exist.*

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- Salil Vadhan (2006) gave a complete characterization of **CZK**.

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- **HVCZK** = **CZK**;
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- **CZK** *with imperfect completeness equals* **CZK** *with perfect completeness.*

Next Time: Zero-knowledge Wrap-up