# CS 594 – Advanced Crypto (Spring 2026)

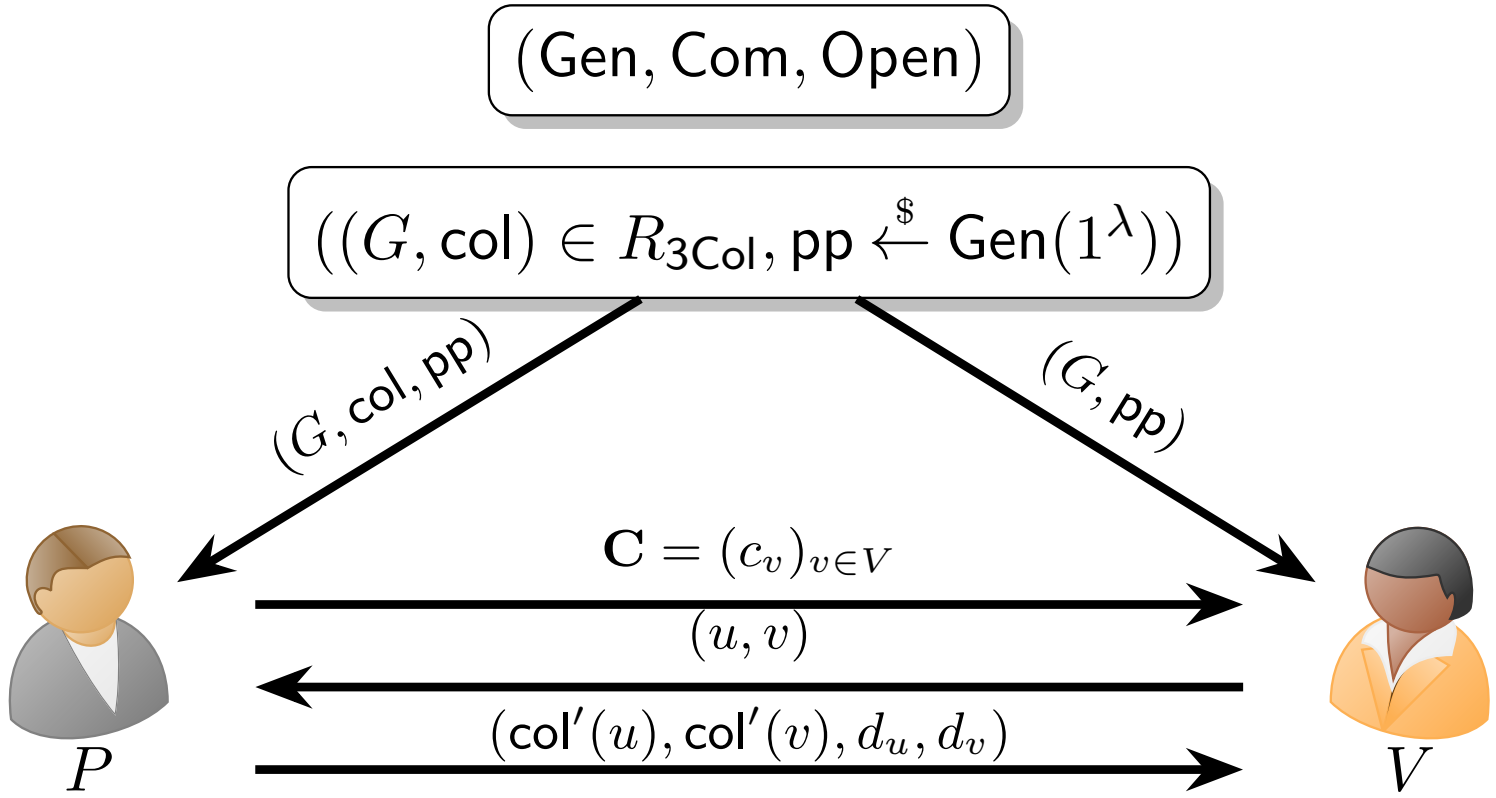
Alex Block

Lecture 14

March 04, 2026

# Sequential Repetition and Zero-knowledge

# Recall: 3Col Proof




(P1) Sample random permutation $\pi$ and compute $(c_v, d_v) \leftarrow \mathsf{Com}(\mathsf{pp}, \mathsf{col}'(v))$, $\forall v \in V$

(V1) Sample $(u, v) \xleftarrow{\$} E$.

(V2) Check the following.

(1) $\mathsf{col}'(u), \mathsf{col}'(v) \in \{0, 1, 2\}$

(2) $\mathsf{col}'(u) \neq \mathsf{col}'(v)$

(3) $\mathsf{Open}(\mathsf{col}'(u), c_u, d_u) = 1$

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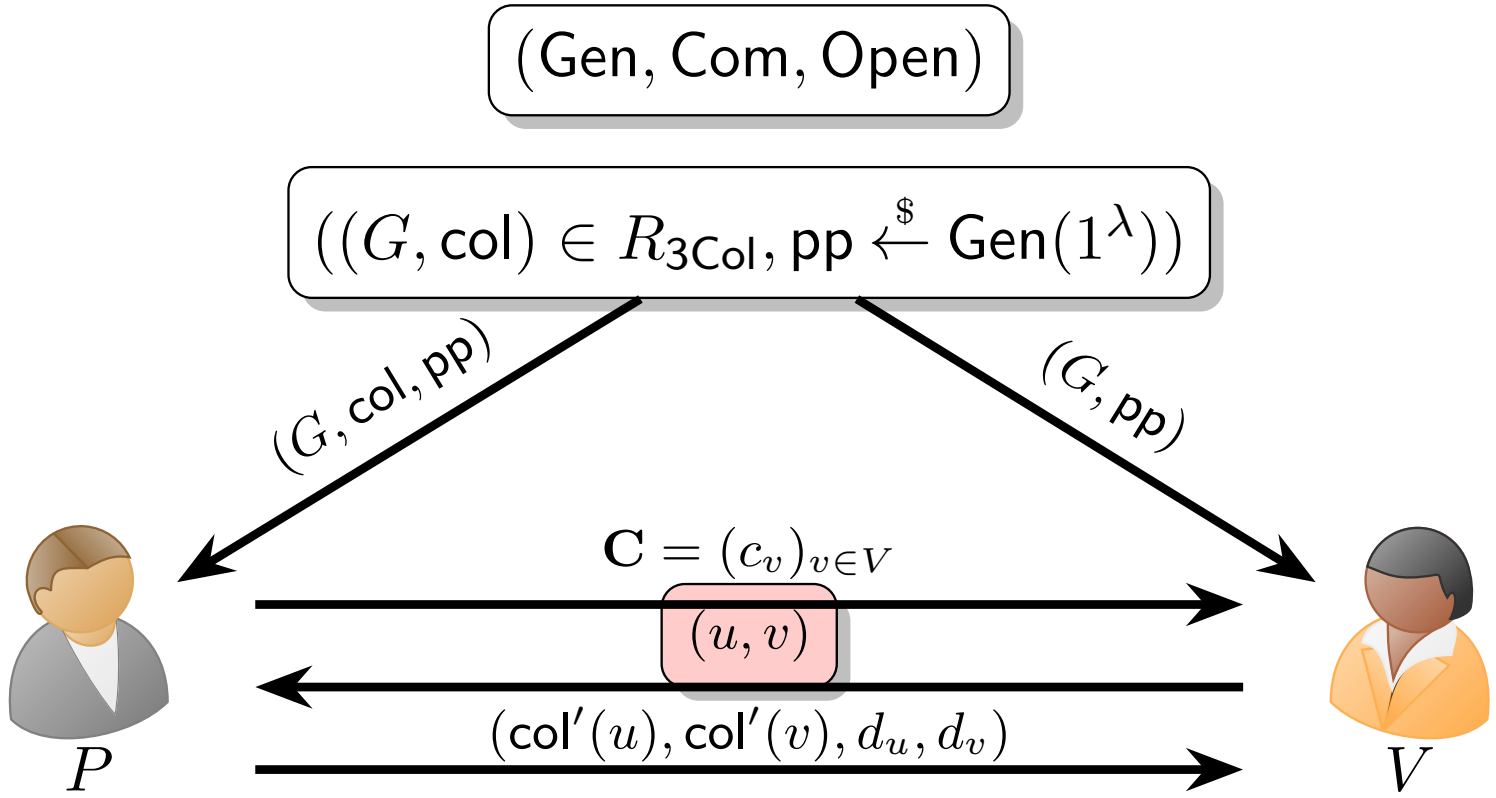


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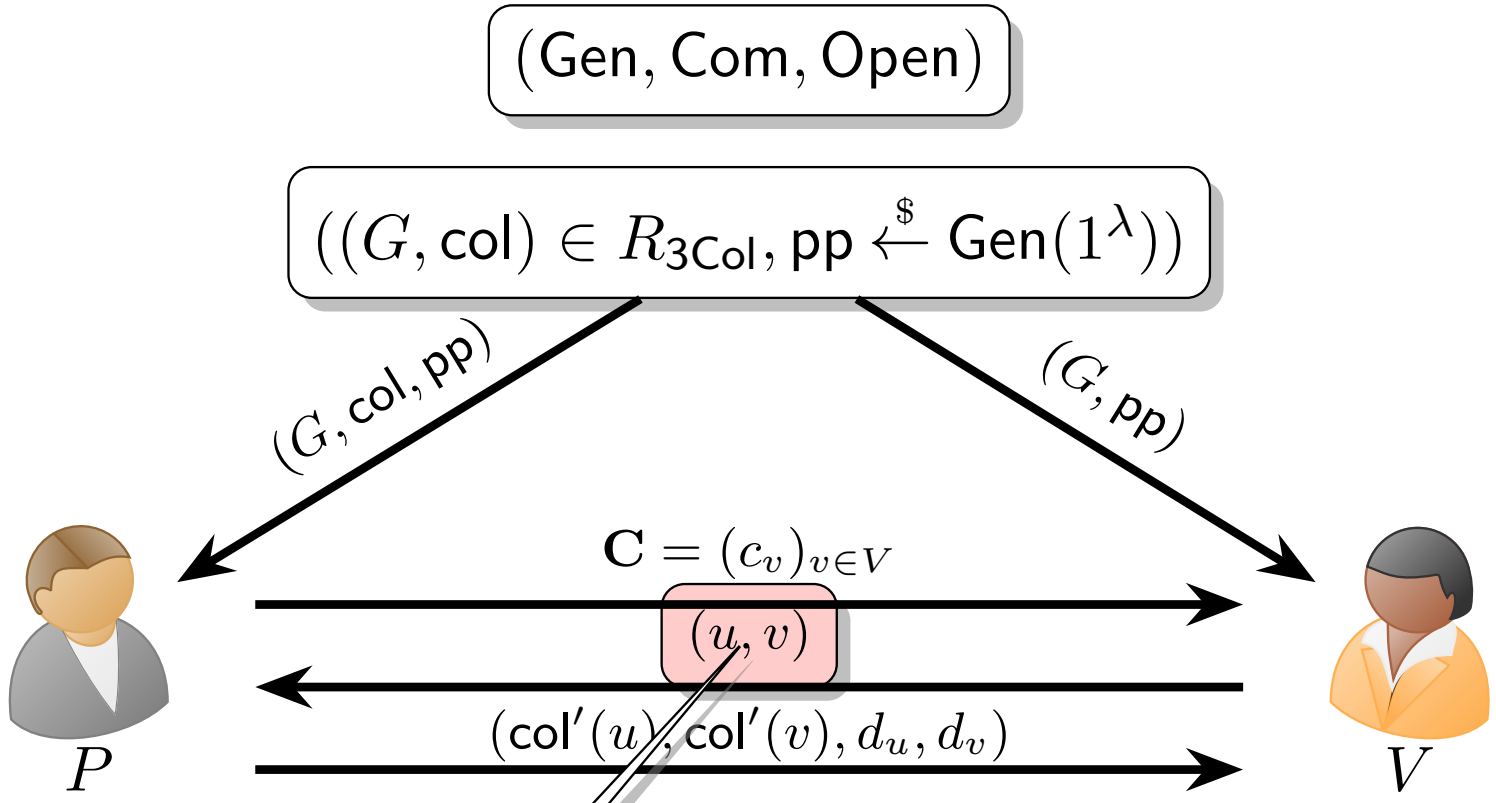


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Soundness error $1 - 1/|E|$

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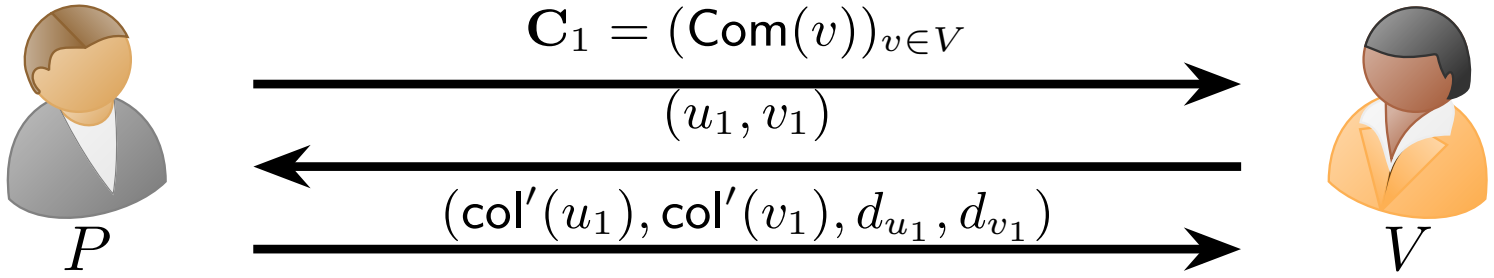

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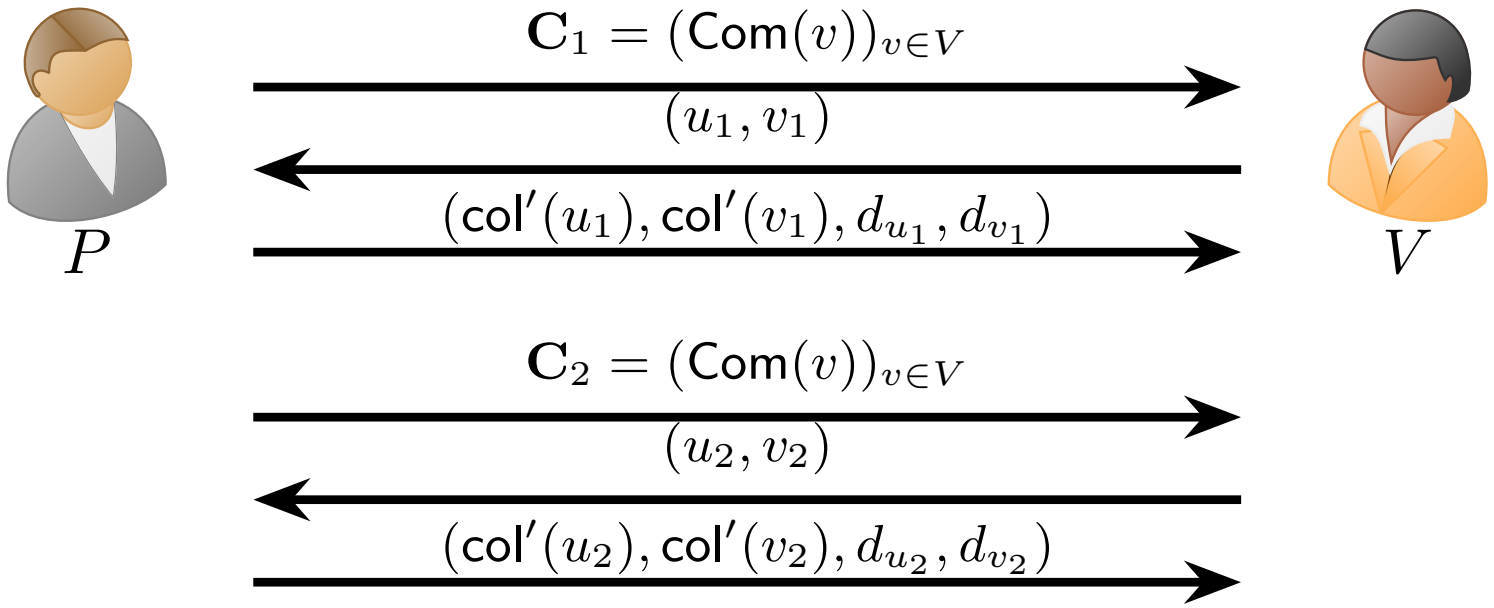

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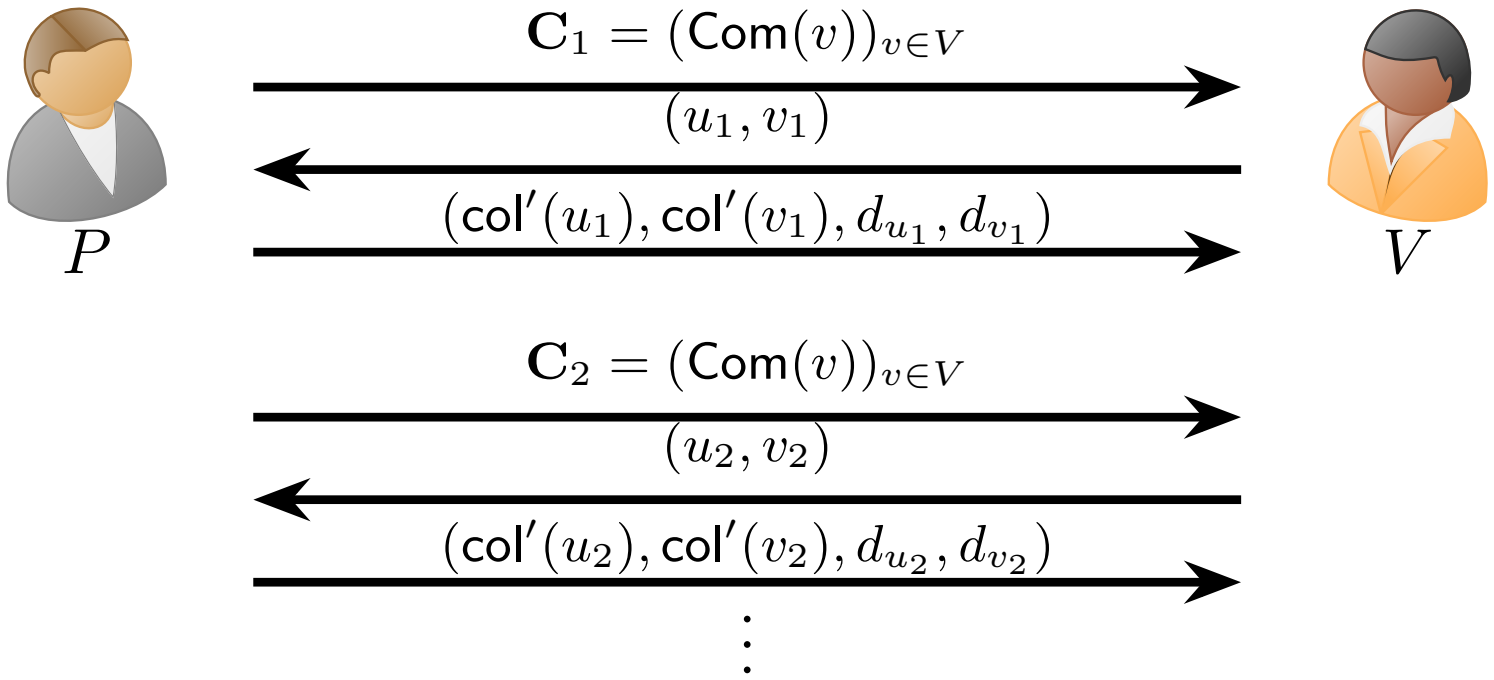

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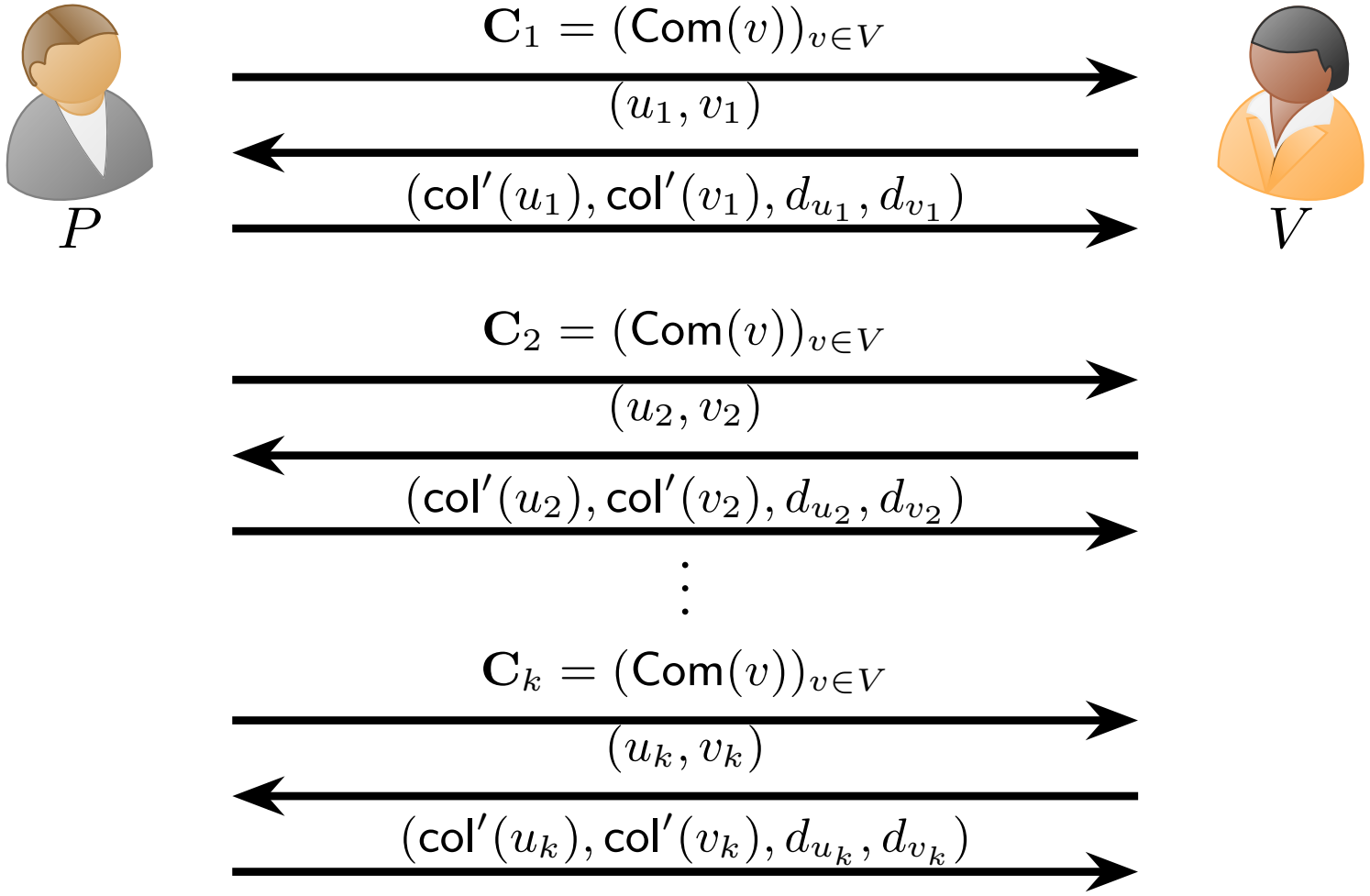

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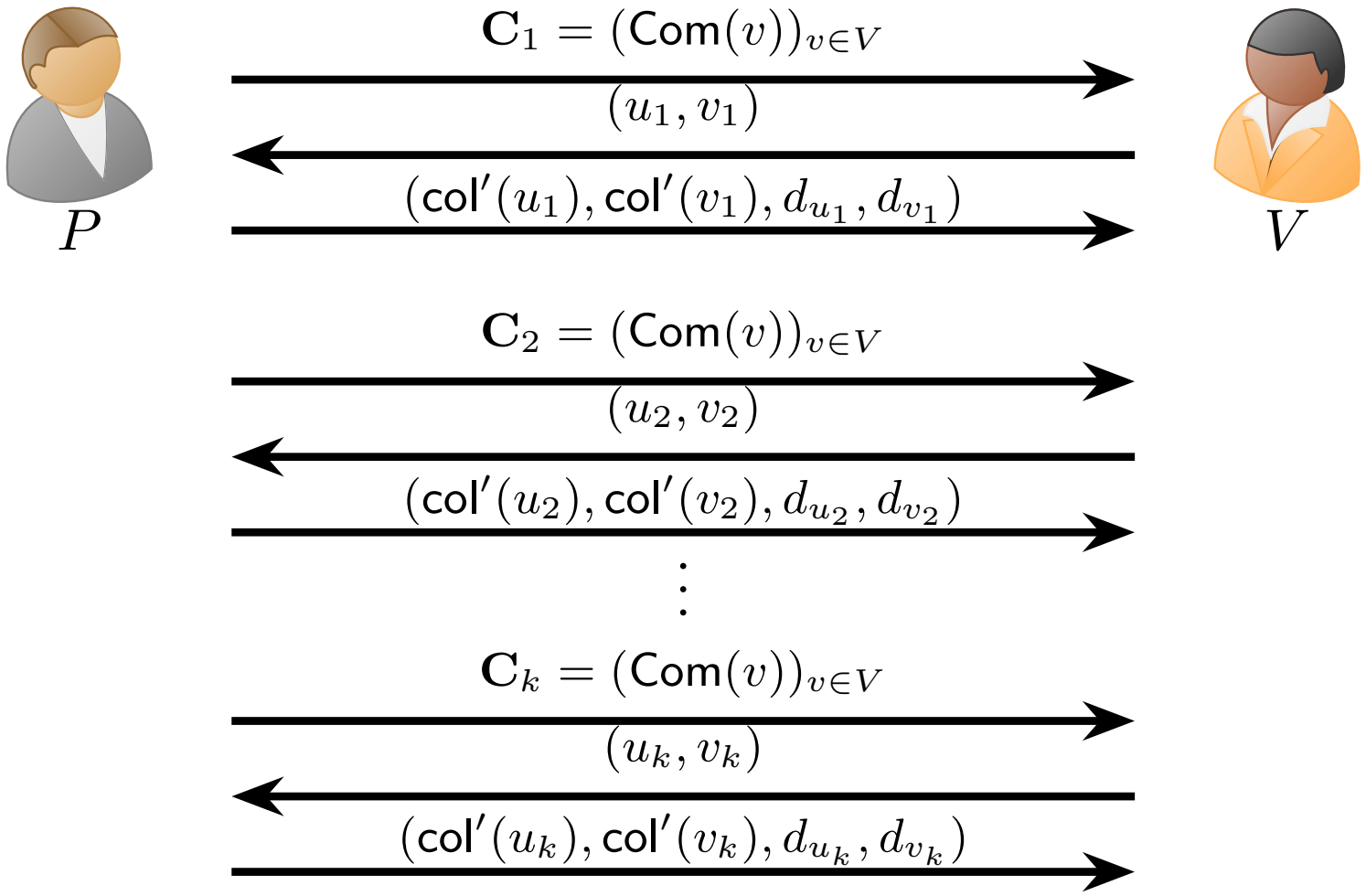


**Soundness Error**

$$(1 - 1/|E|)^k$$

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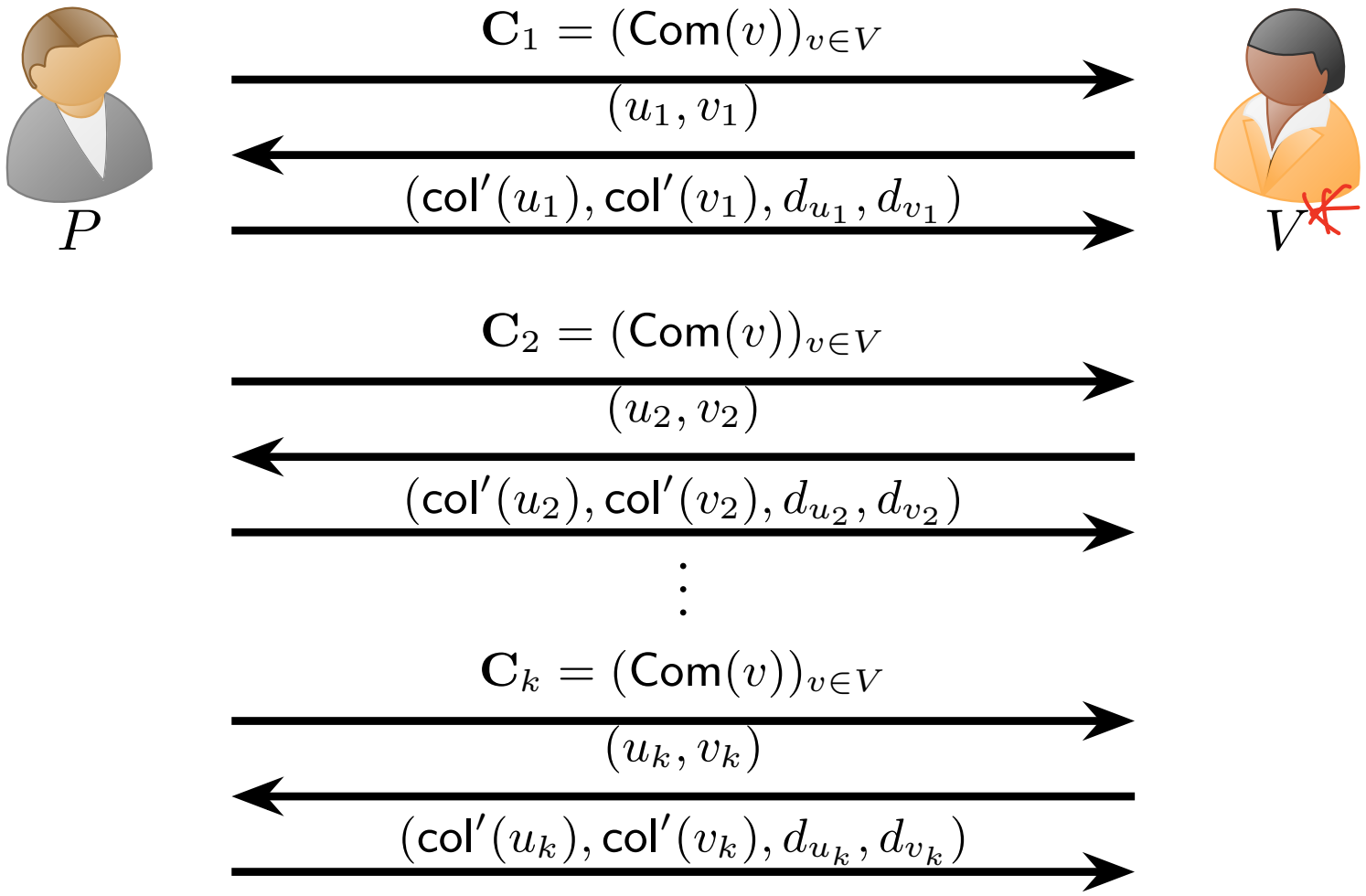


**Soundness Error**

$(1 - 1/|E|)^k$

**Zero-knowledge?**

Some issues!

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- In some sense, this is the "right" definition of zero-knowledge.

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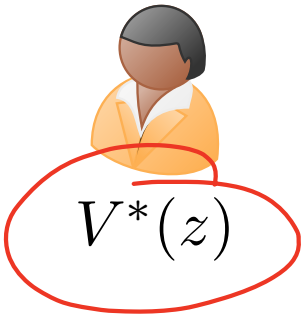
$P$



$V^*(z)$

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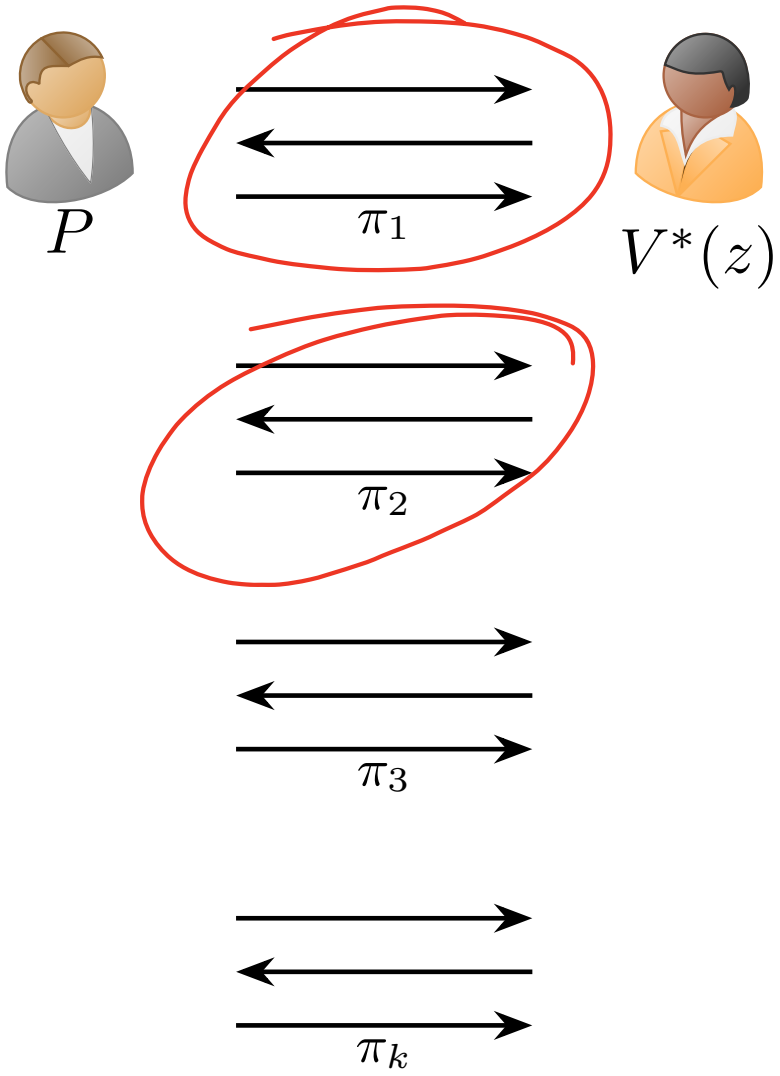

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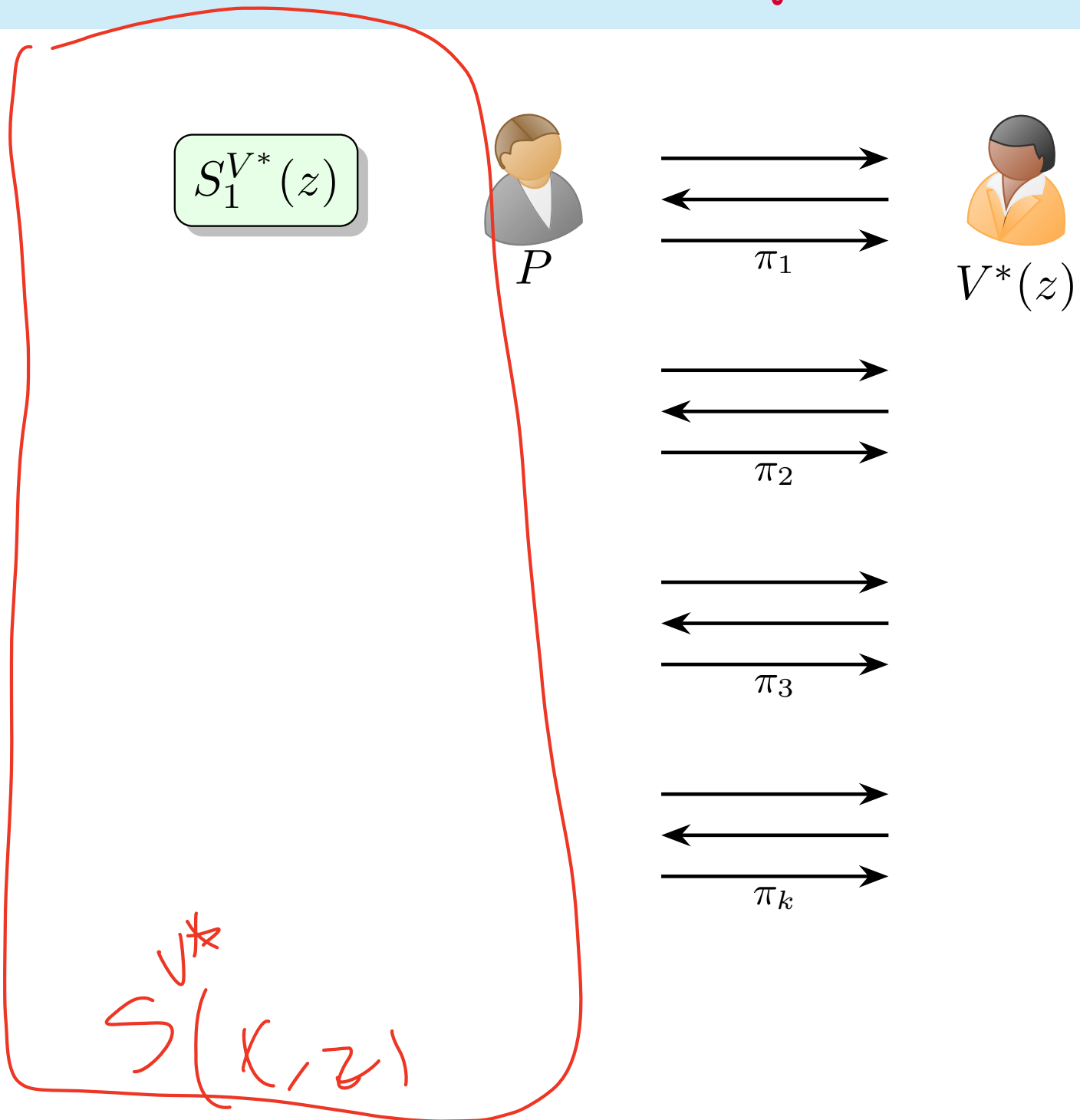

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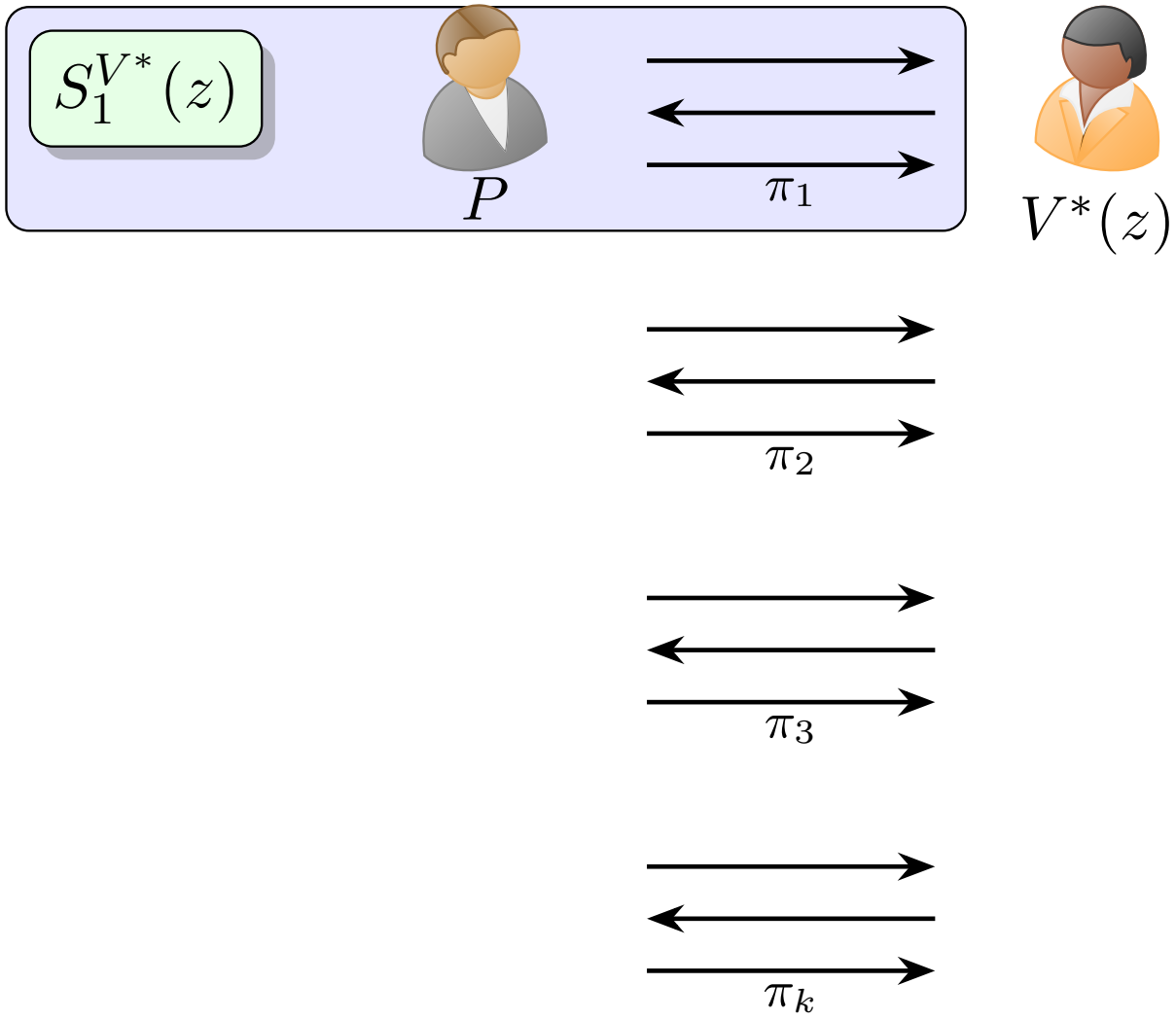

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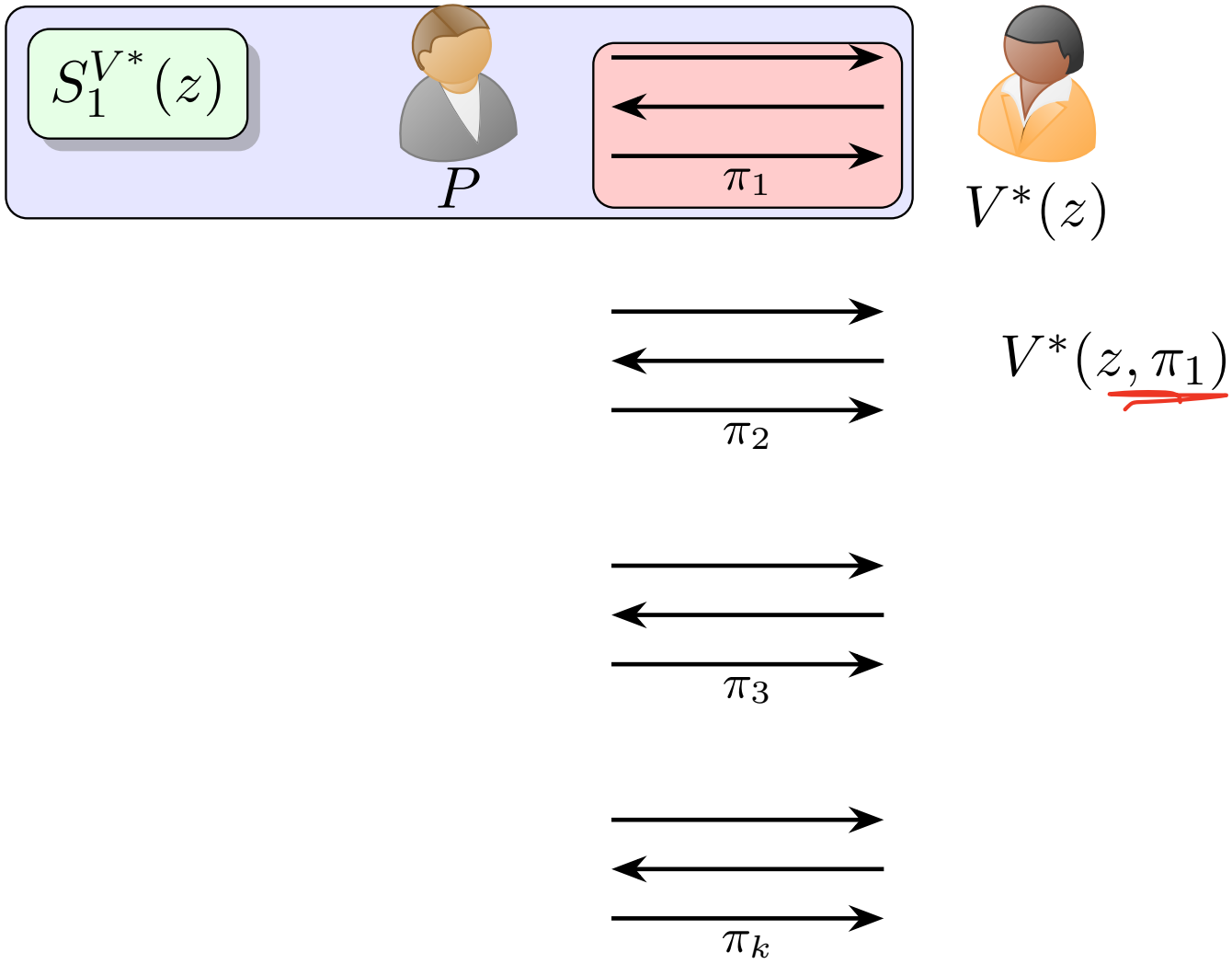

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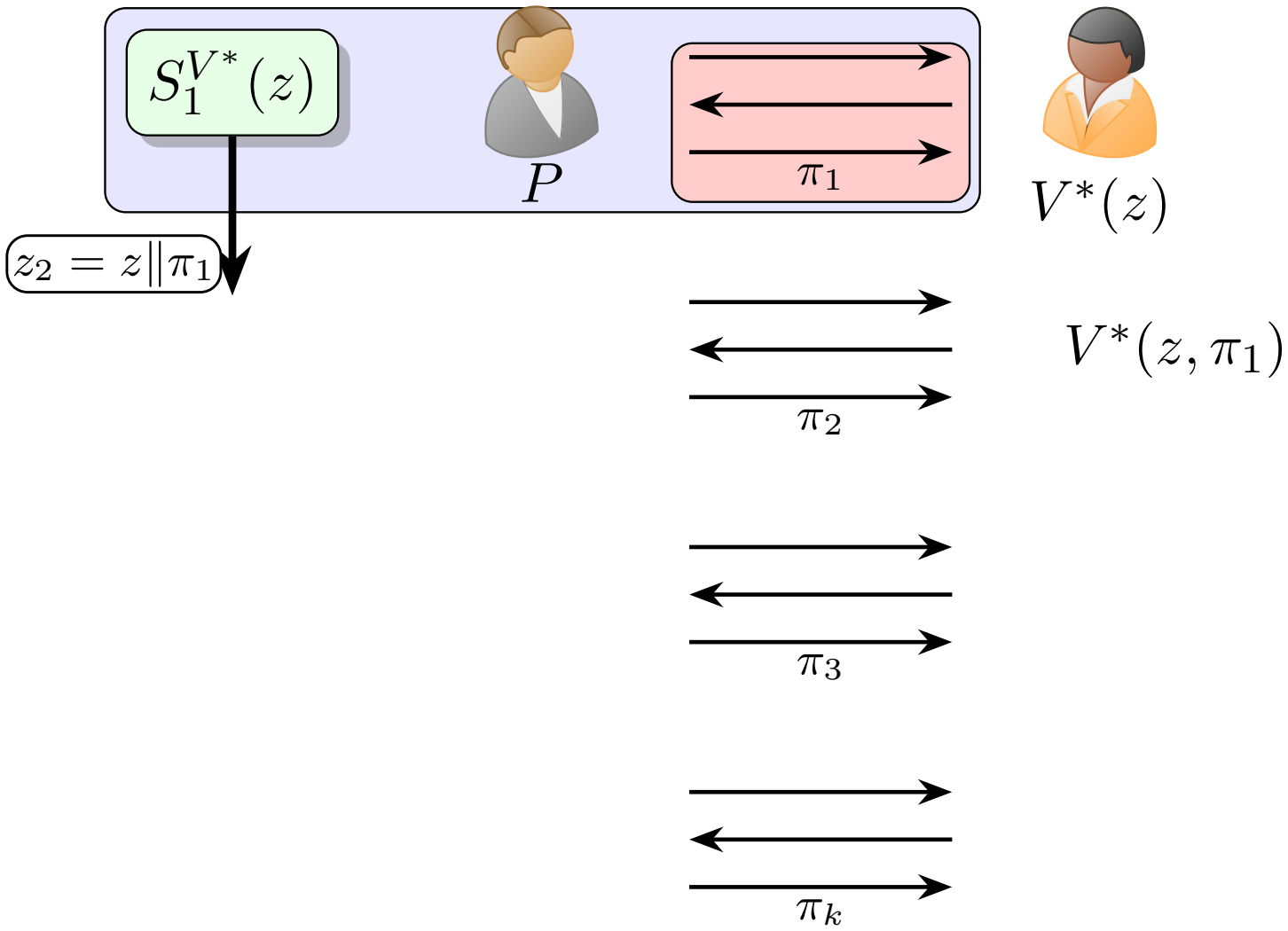

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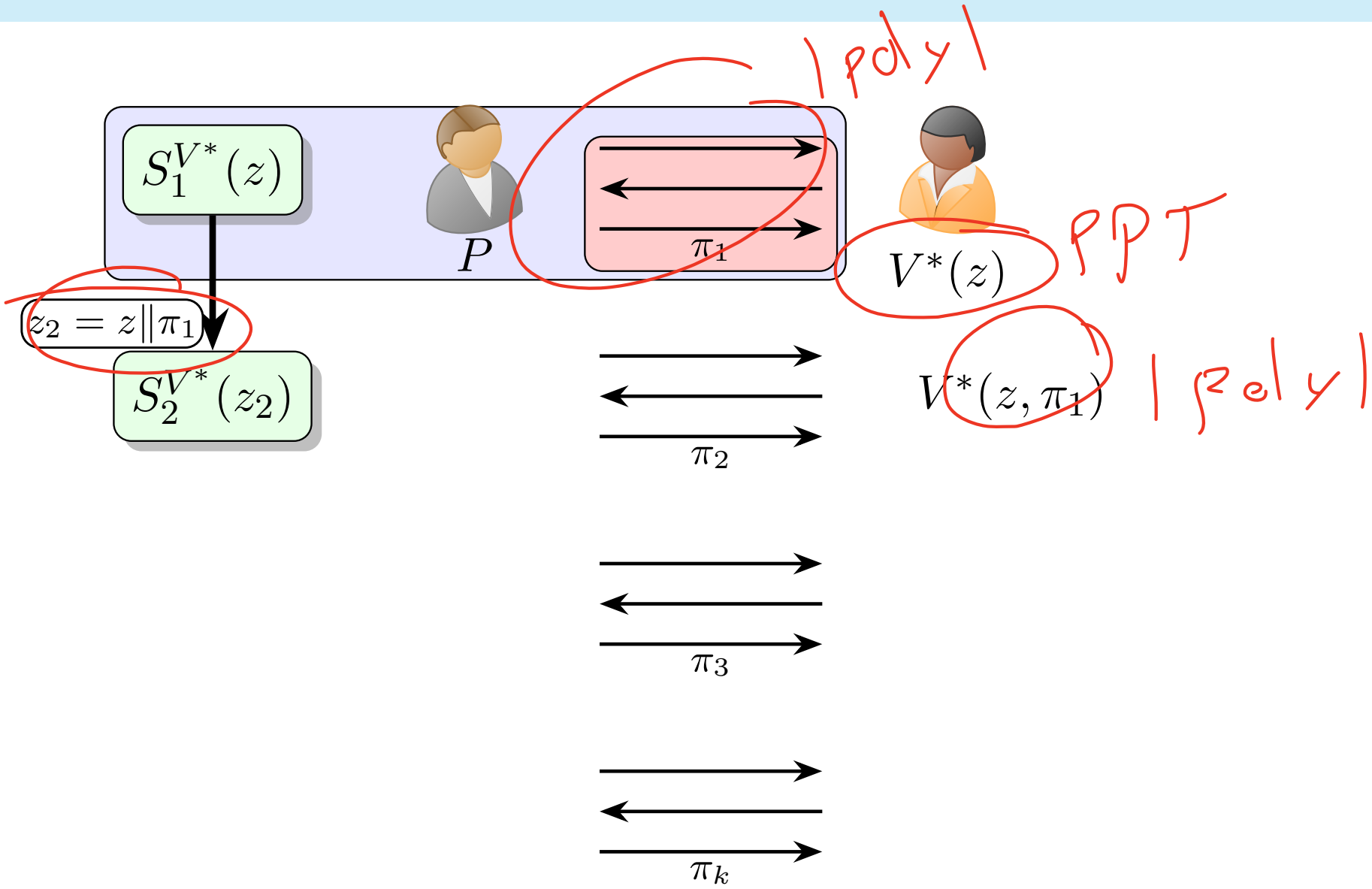

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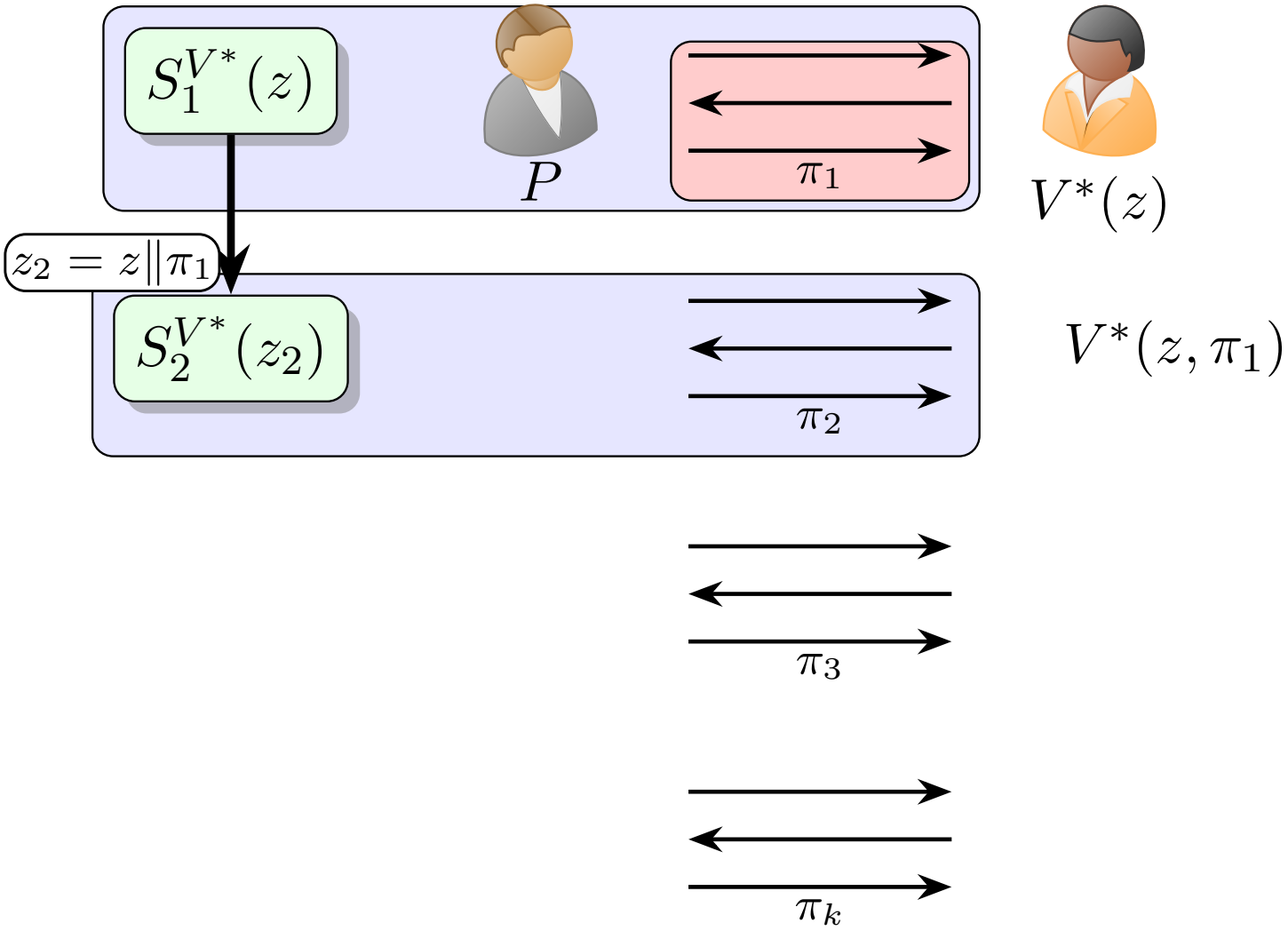

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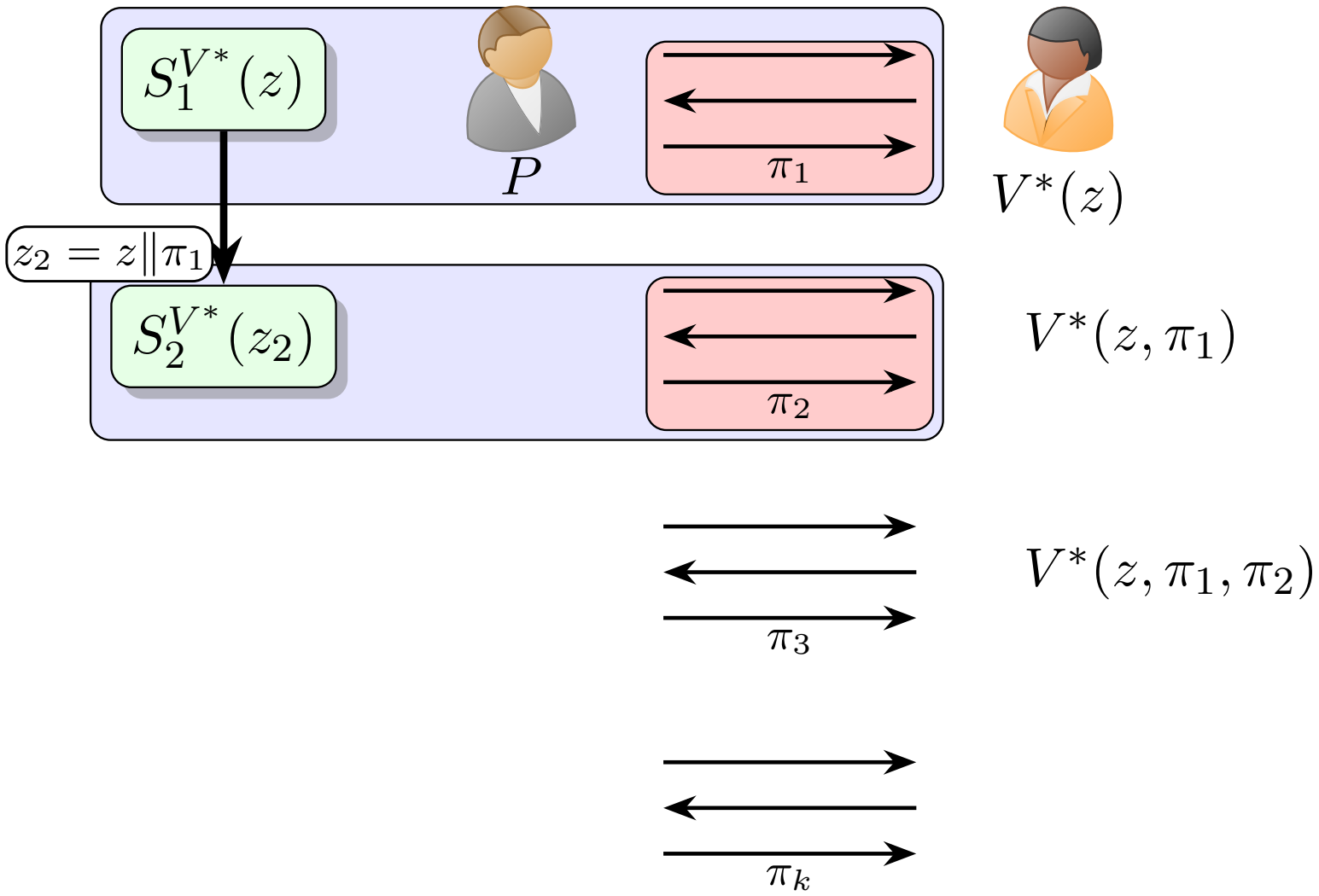

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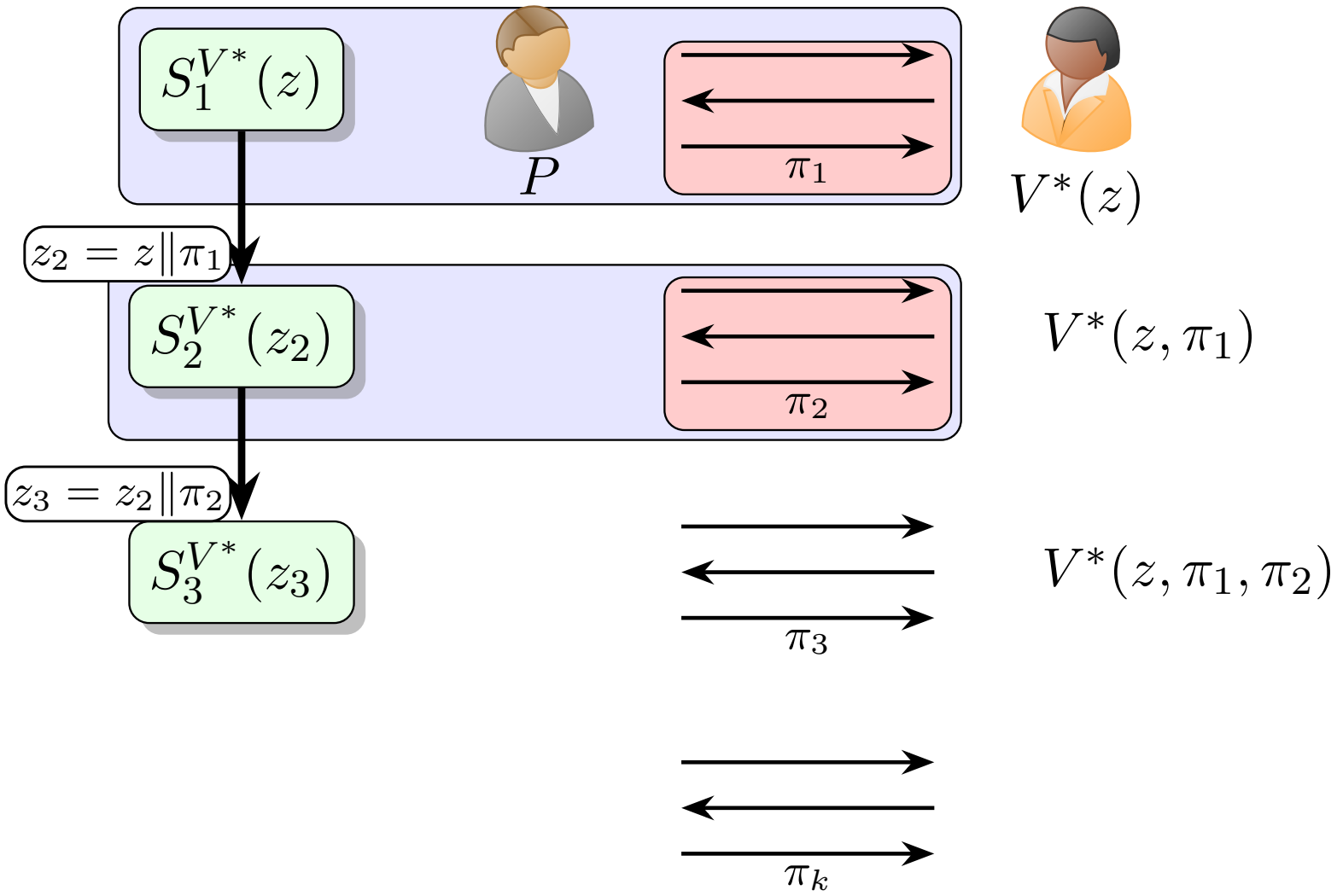

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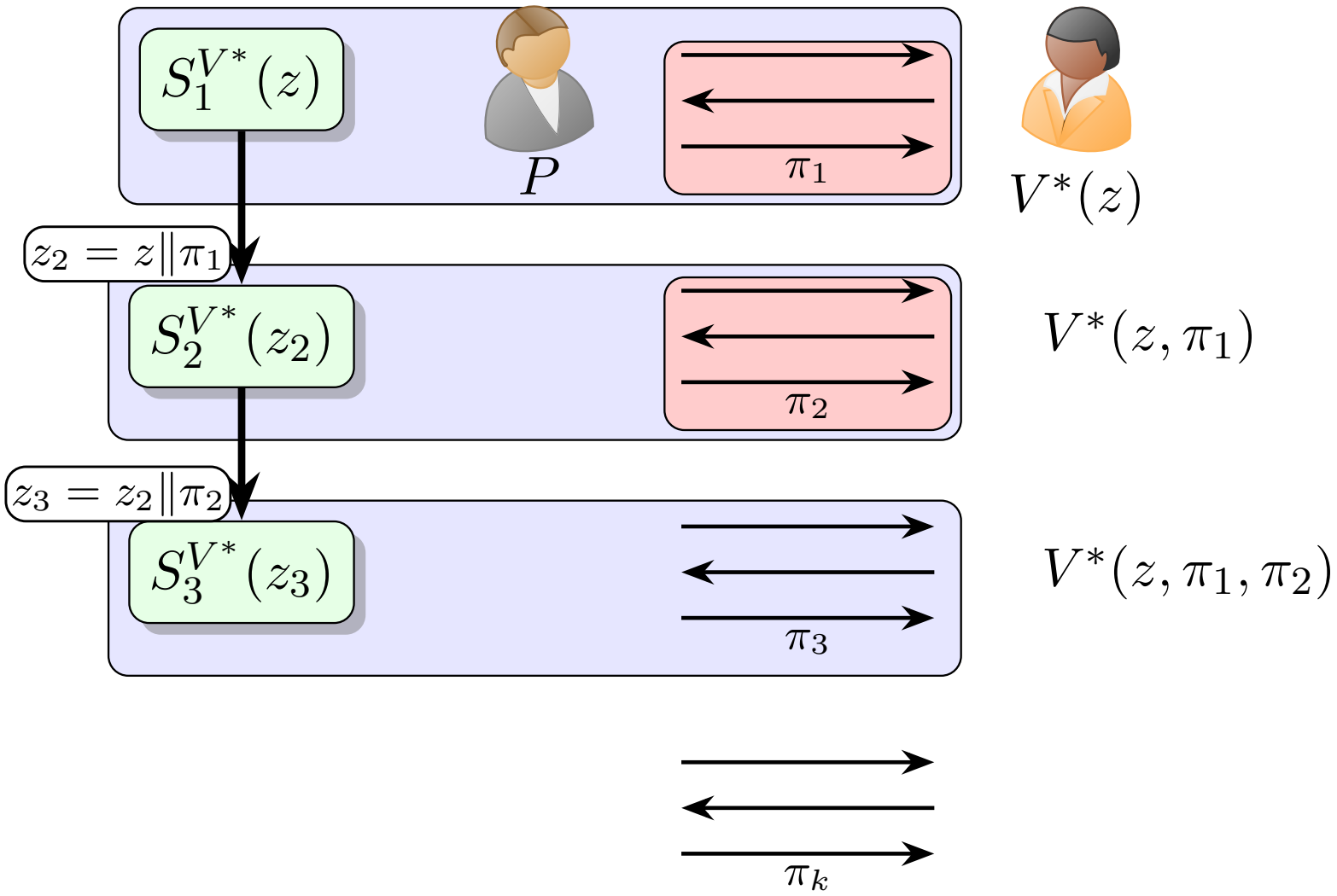

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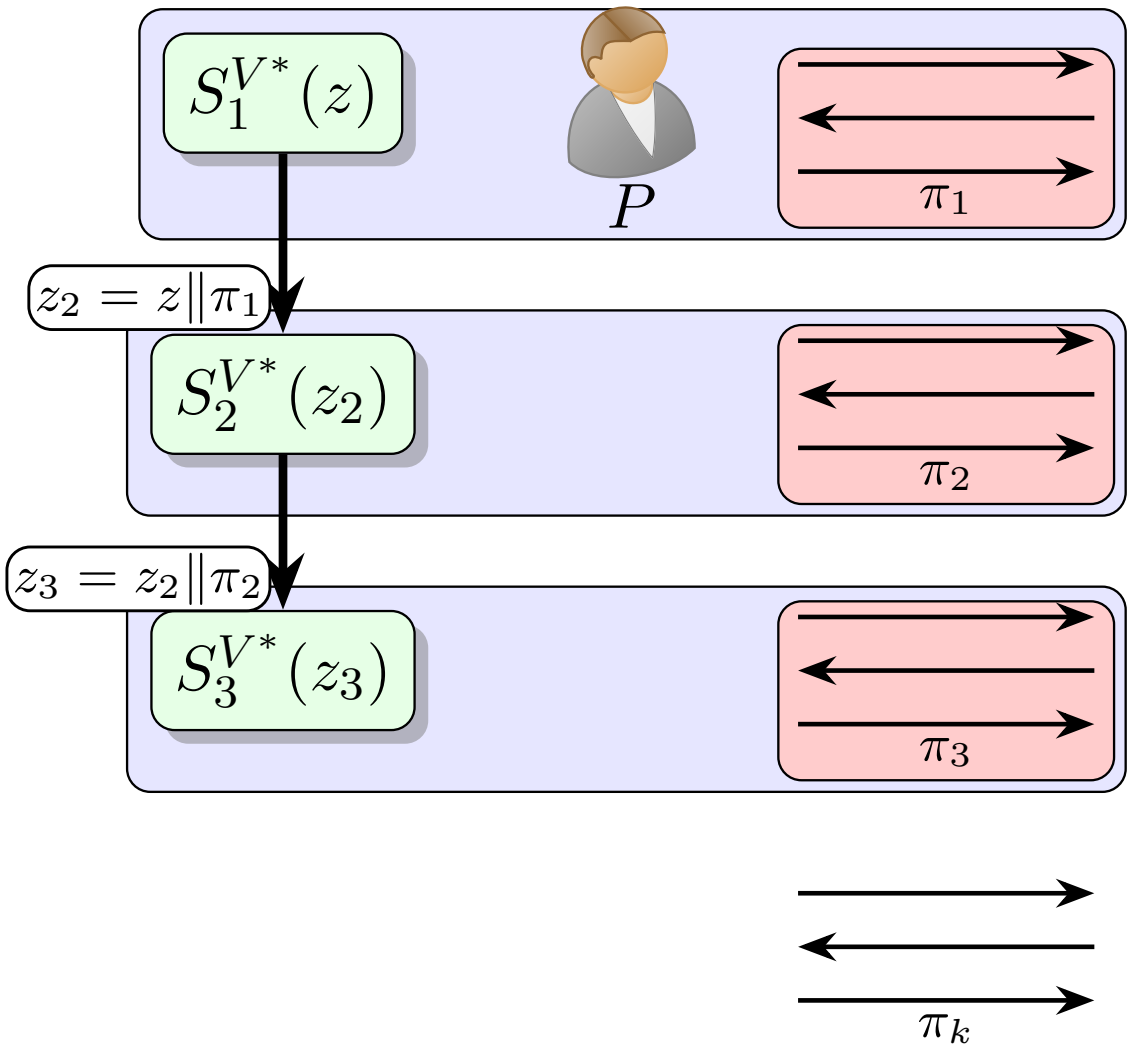


$V^*(z)$

$V^*(z, \pi_1)$

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$V^*(z, \pi_1, \pi_2, \ldots, \pi_{k-1})$

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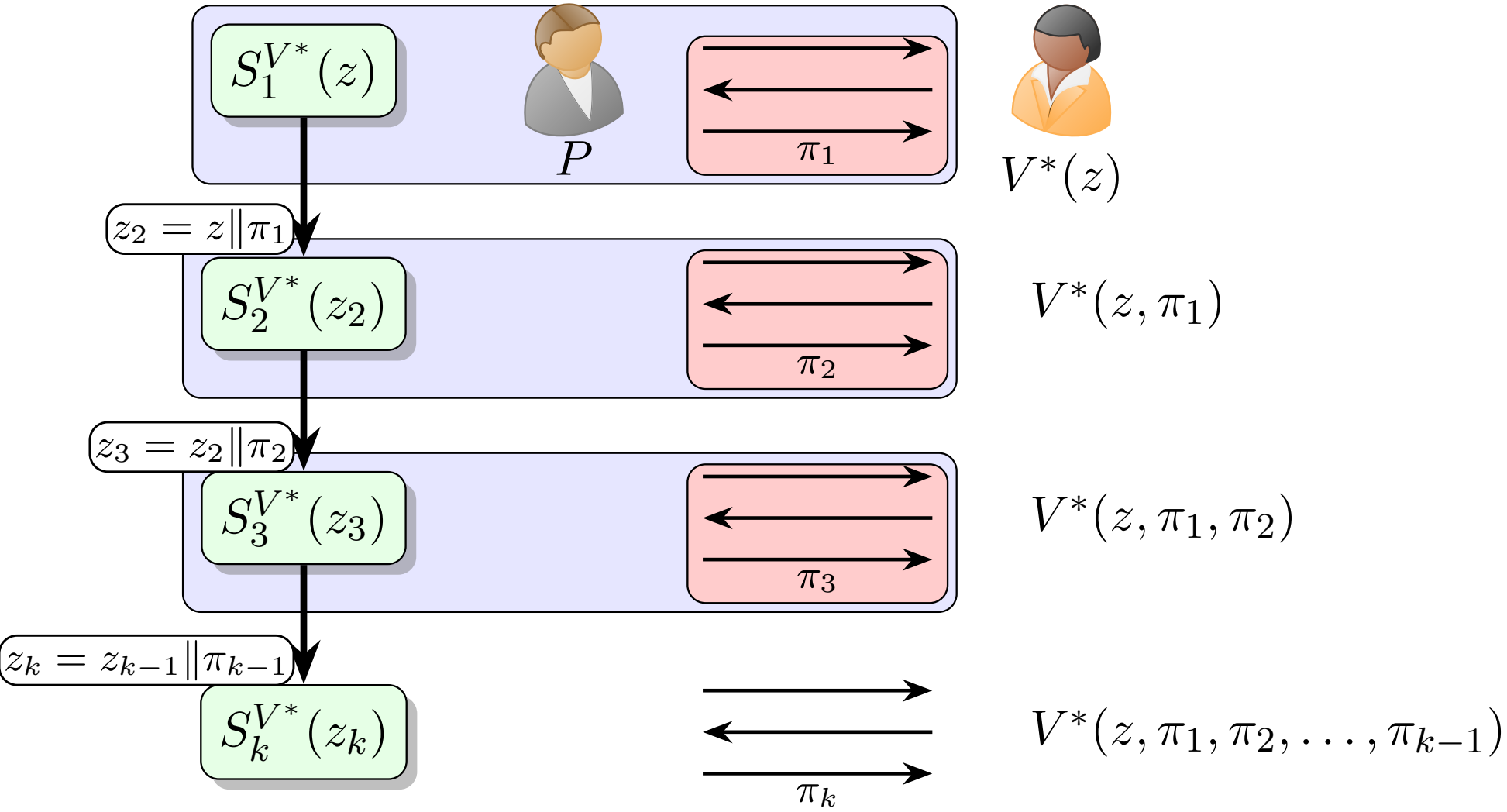

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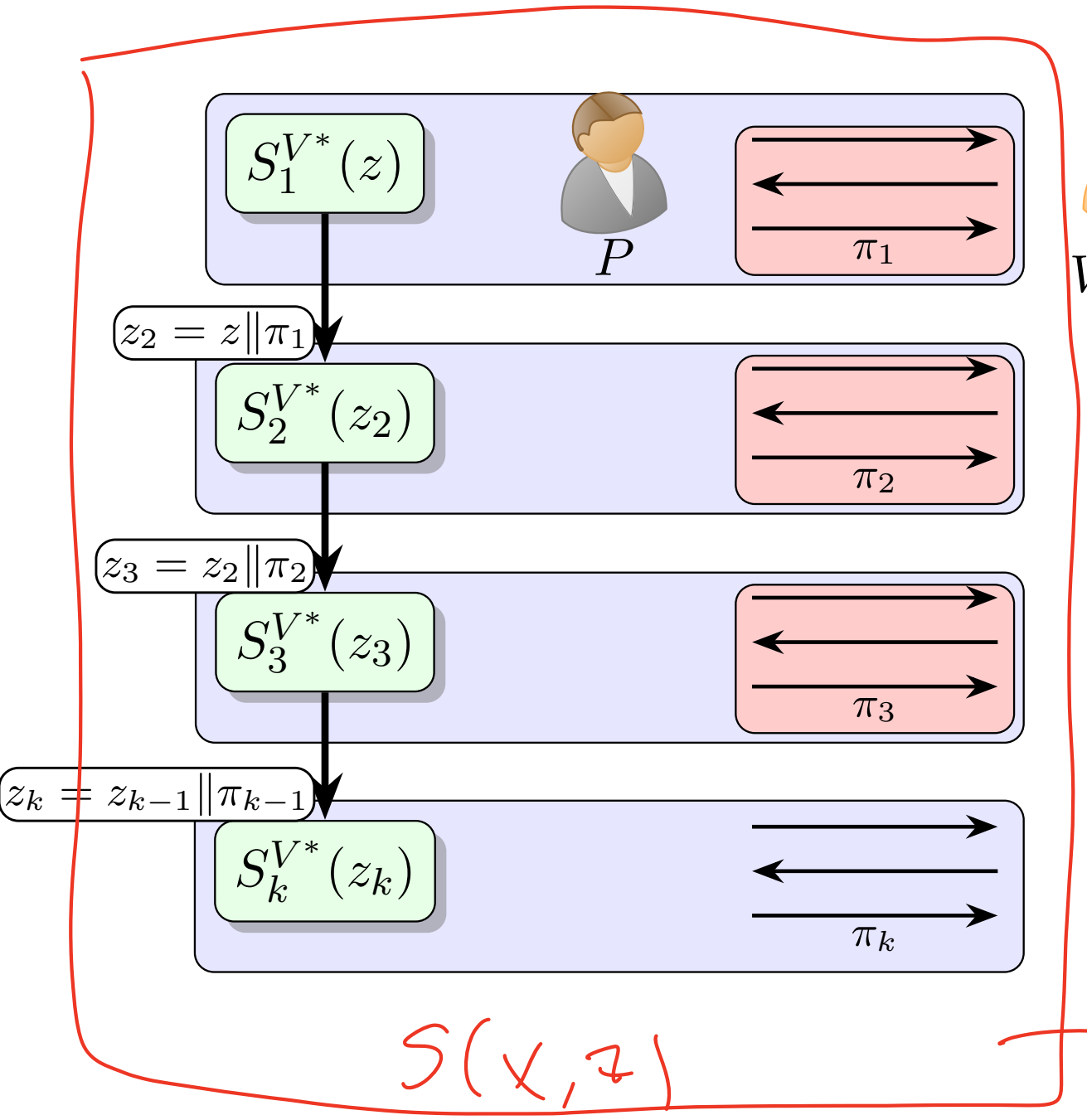


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$(\pi_1, \pi_2, \ldots, \pi_k)$

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$\|$

$(\pi_1^*, \pi_2^*, \ldots, \pi_k^*)$

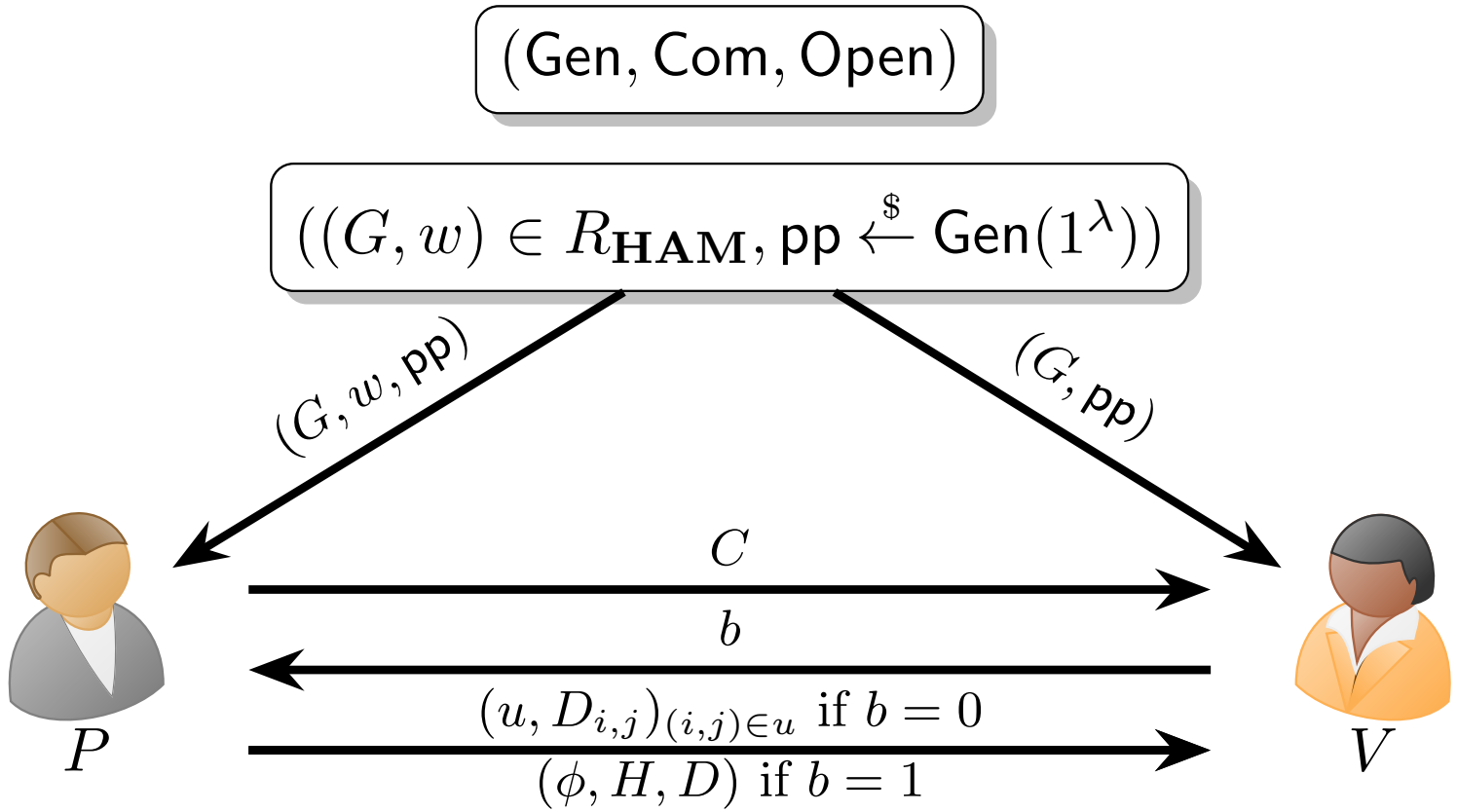
# Constant-round Zero-knowledge for NP

# More ZK for NP?

Goal: proofs for every $L \in \mathbf{NP}$

- in computational ZK;
- with *negligible* soundness error; and
- a constant number of rounds

# Recall: CZK Proof for HAM




(P1) Sample random permutation $\phi$ and compute $(C, D) \leftarrow \mathsf{Com}(\phi(G))$

(P2) Set $u = \phi(w)$.

(V1) Sample $b \xleftarrow{\$} \{0, 1\}$.

(V2-0) $b = 0$: check that $u$ is a cycle and $\forall (i, j) \in u$, check $1 = \mathsf{Open}(1, C_{i,j}, D_{i,j})$

(V2-1) $b = 1$: check that $H = \phi(G)$ and $1 = \mathsf{Open}(H, C, D)$

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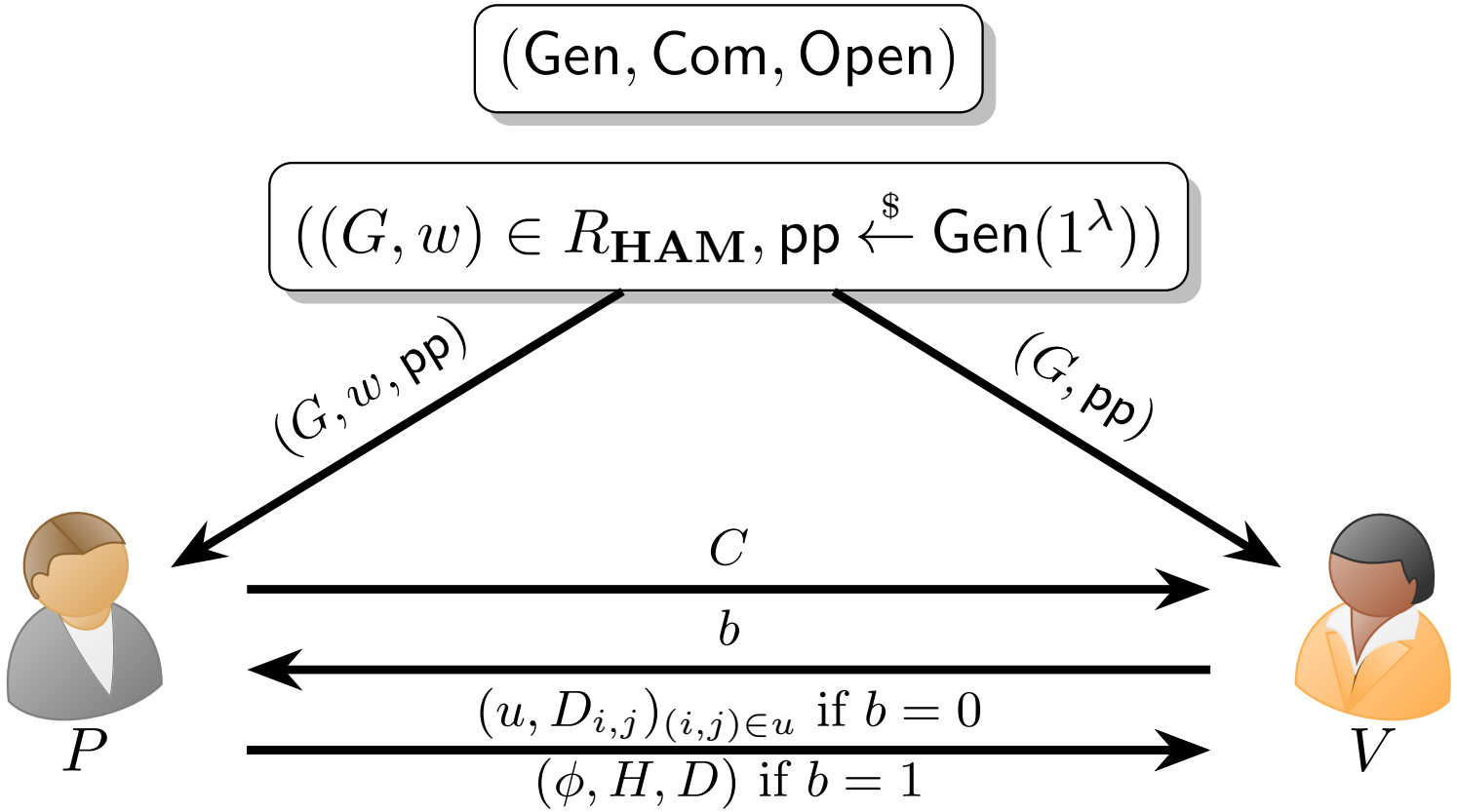


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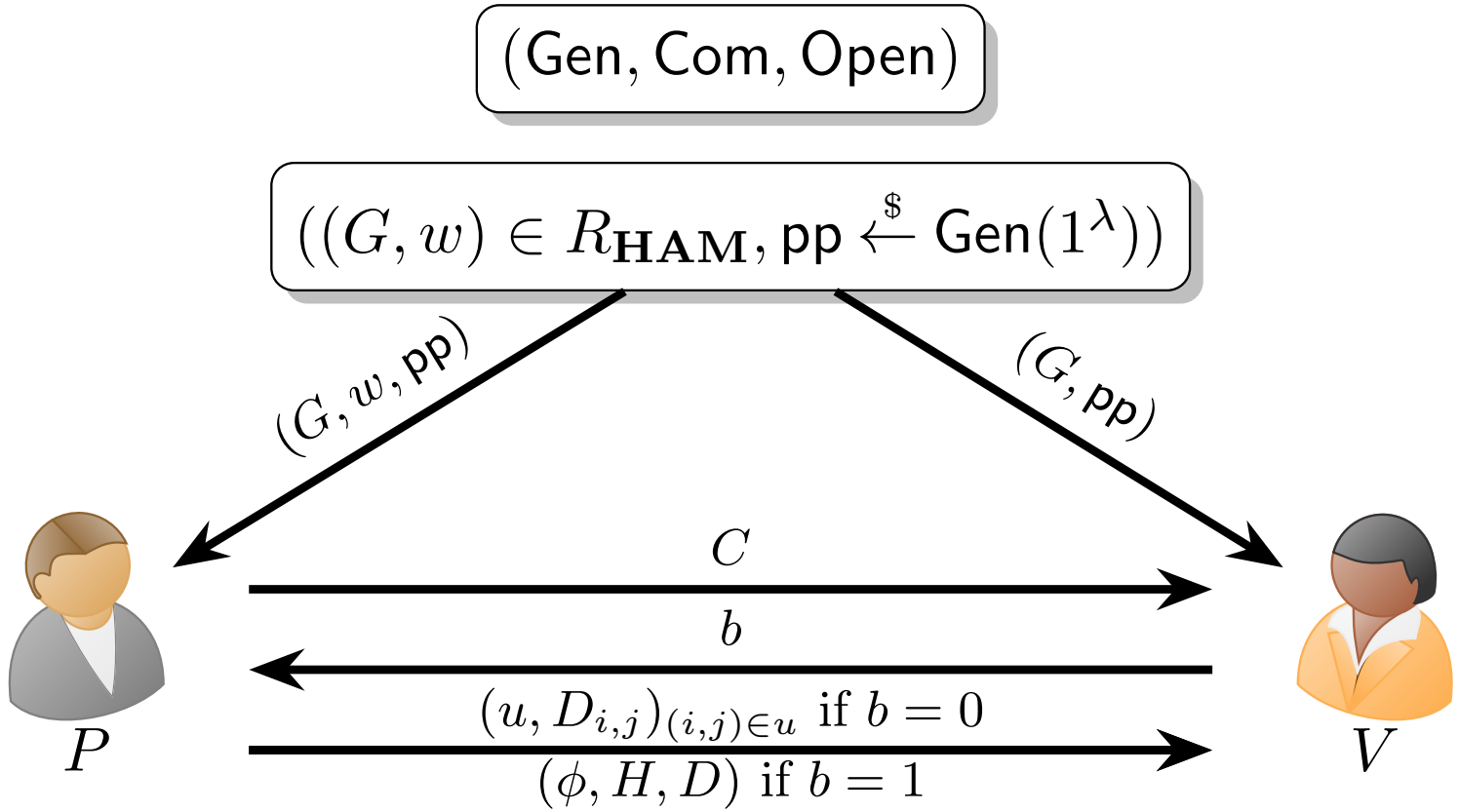


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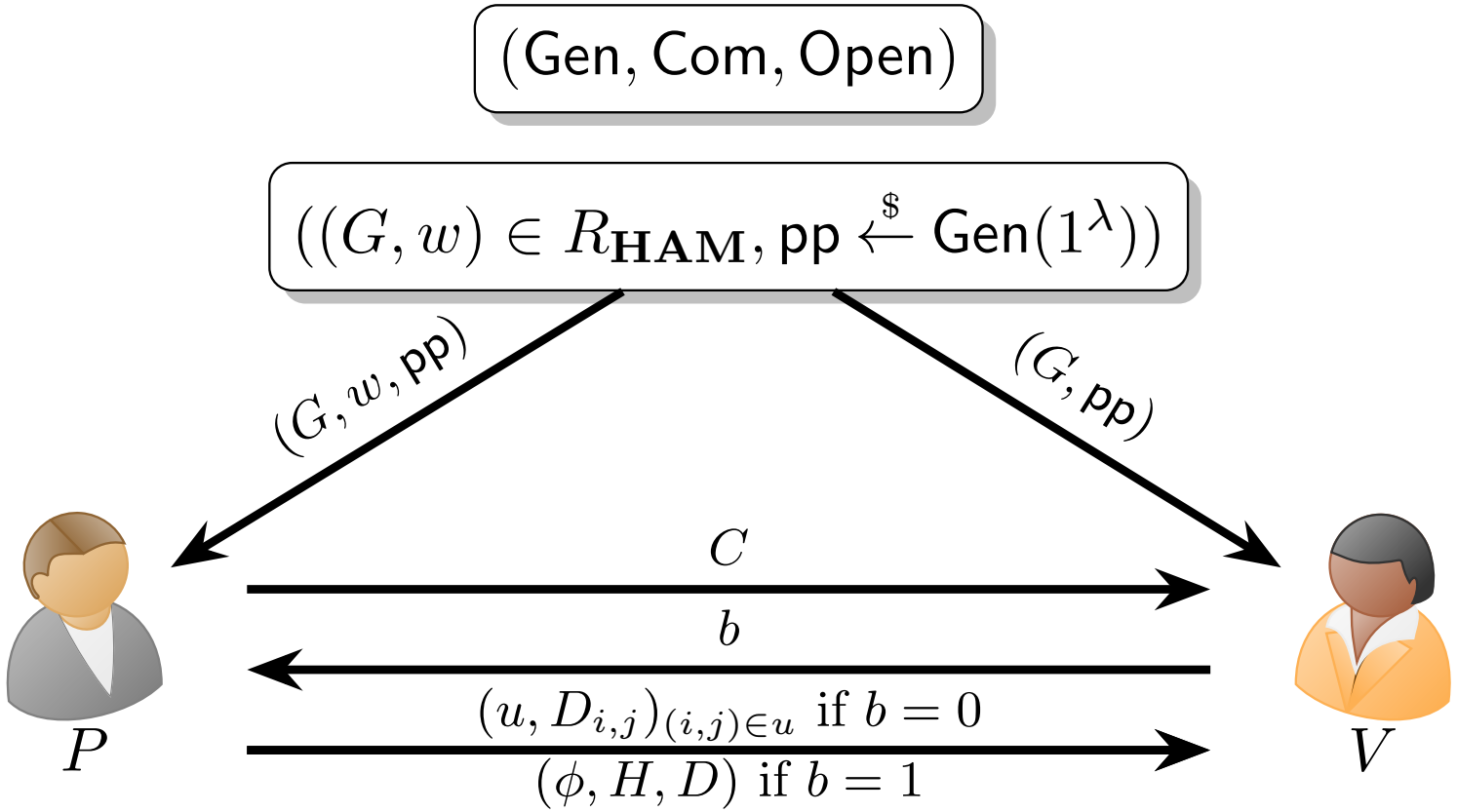
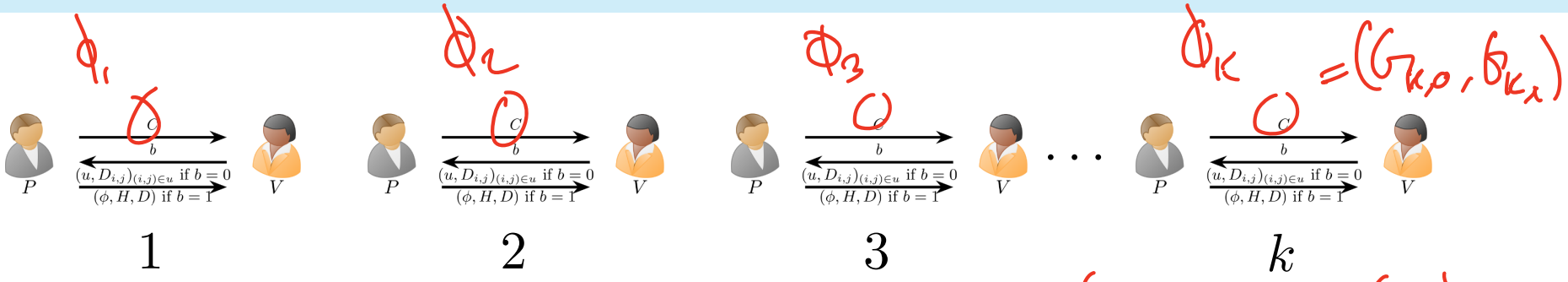



- To get negligible soundness error, sequential repetition for $k = \text{poly}(\lambda)$.
- Not constant round!
  - What about parallel repetition?

# Parallel Repetition: Exponential Simulation!

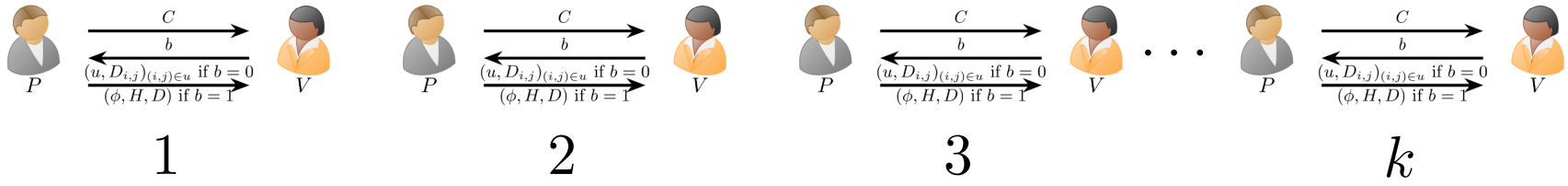



$\vec{b} \in \{0,1\}^k$

$\sigma_{i,0} = \phi_i(\omega)$

$\sigma_{i,1} = \phi_i(\sigma)$

# Parallel Repetition: Exponential Simulation!




✓ Soundness error $2^{-k}$!

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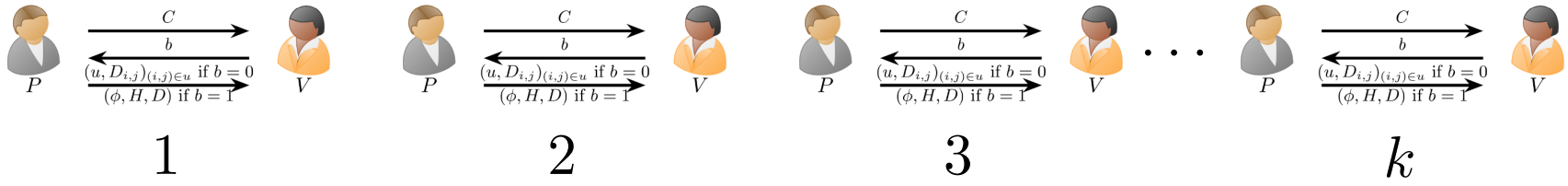


- ✓ Soundness error $2^{-k}$!
- ✓ Constant number of rounds! (3 rounds).

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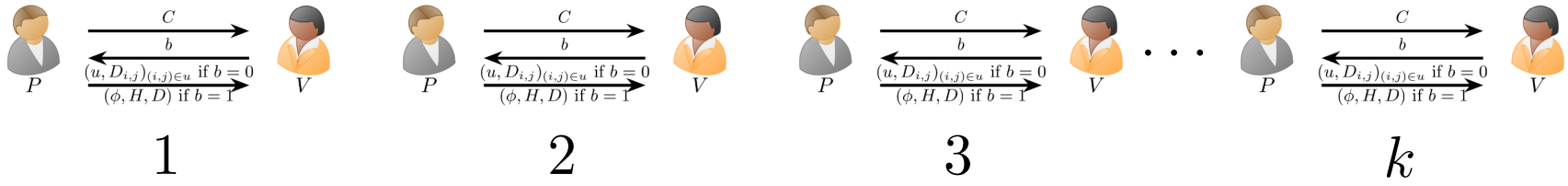


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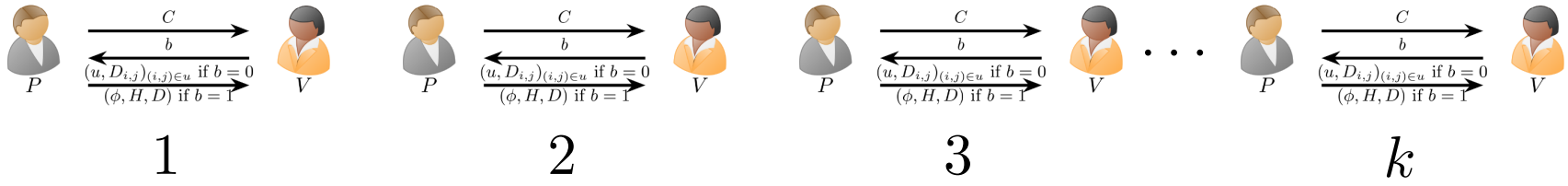


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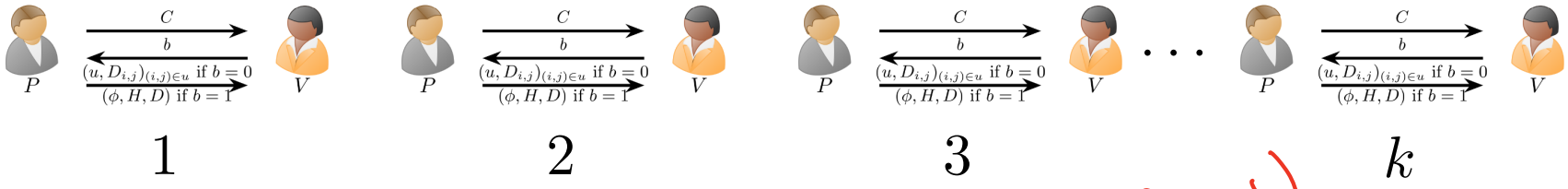


poly(|x|)
↑
$\in L/\bar{L}$

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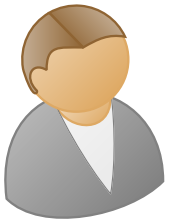
- ✓ Constant number of rounds! (3 rounds).
- × Can't simulate in expected polynomial time!
  - In original simulation, $S$ guessed the bit $b$ from $V^*$.
  - In parallel repetition, $S$ must guess $\mathbf{b} \in \{0,1\}^k$, which gives $2^k$ expected number of simulation attempts.

# Fix (with no proof)



$P$

$V$

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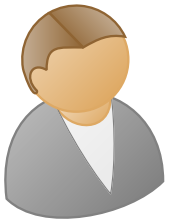
- Simple fix: add 2 more rounds!



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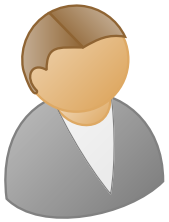

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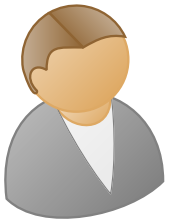

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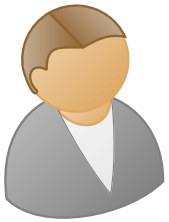

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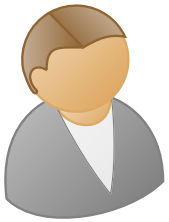

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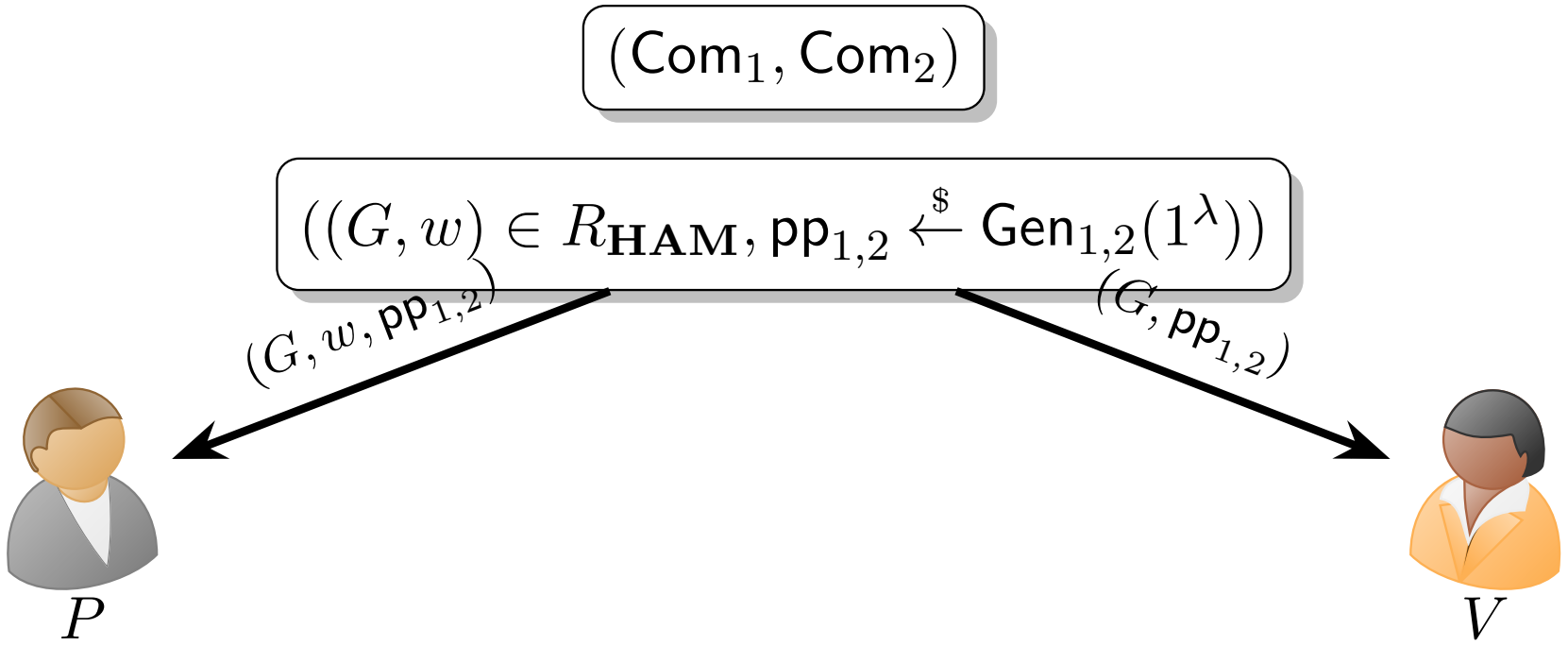

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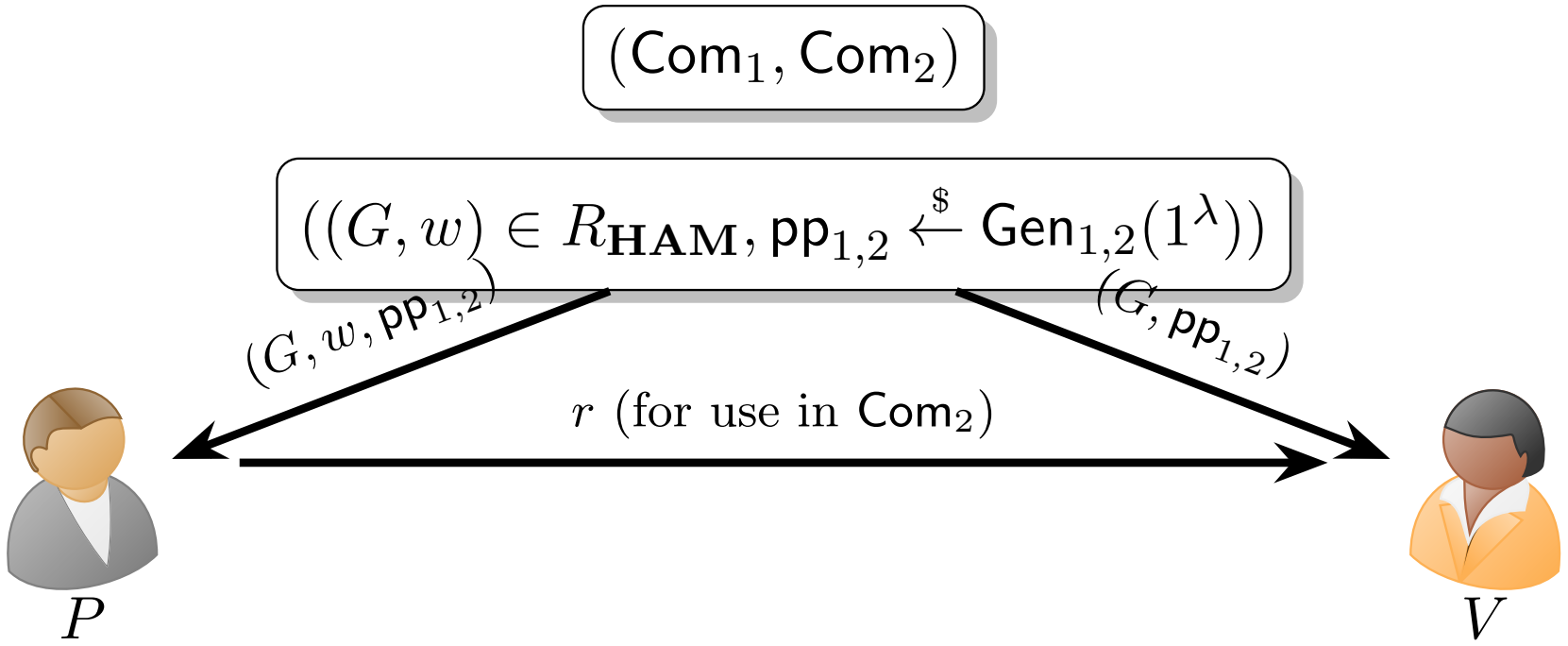

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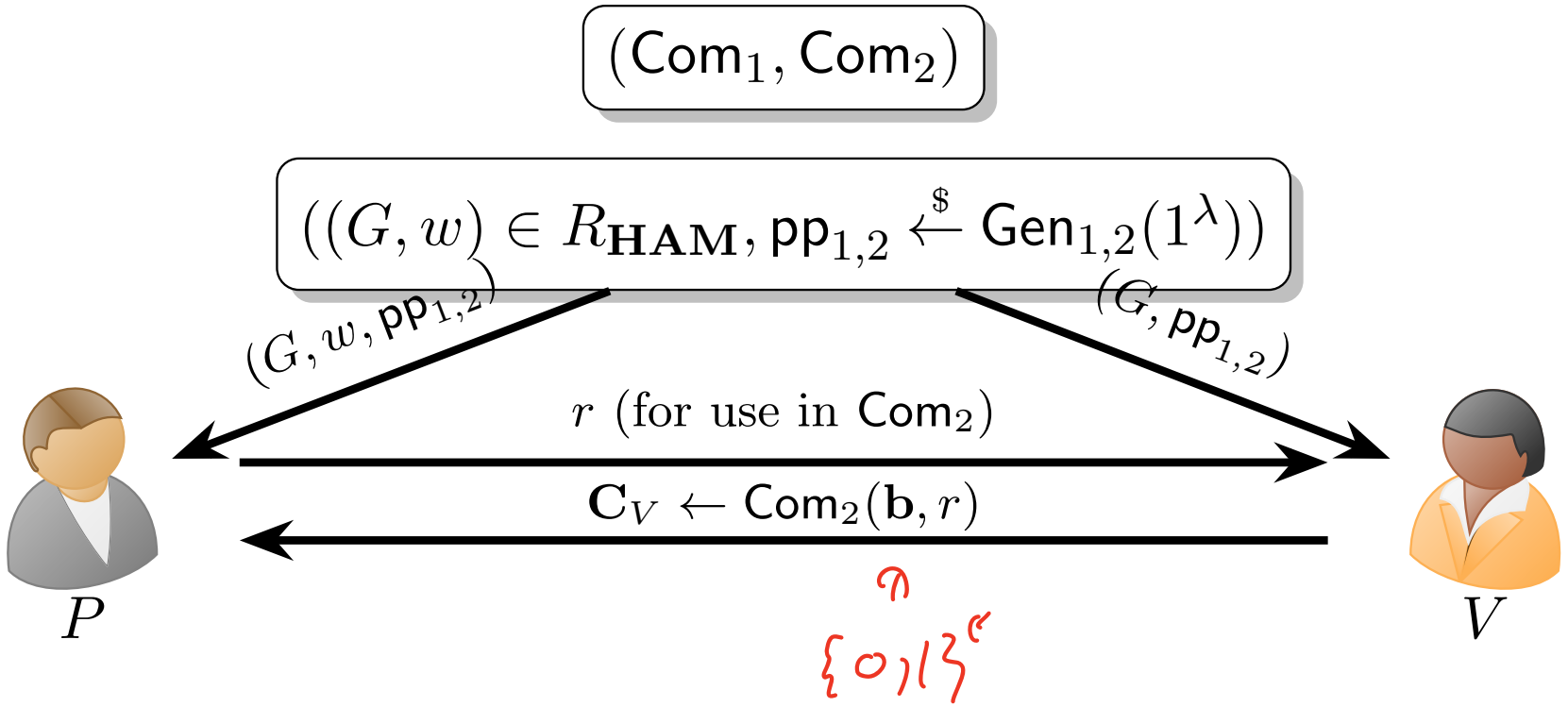

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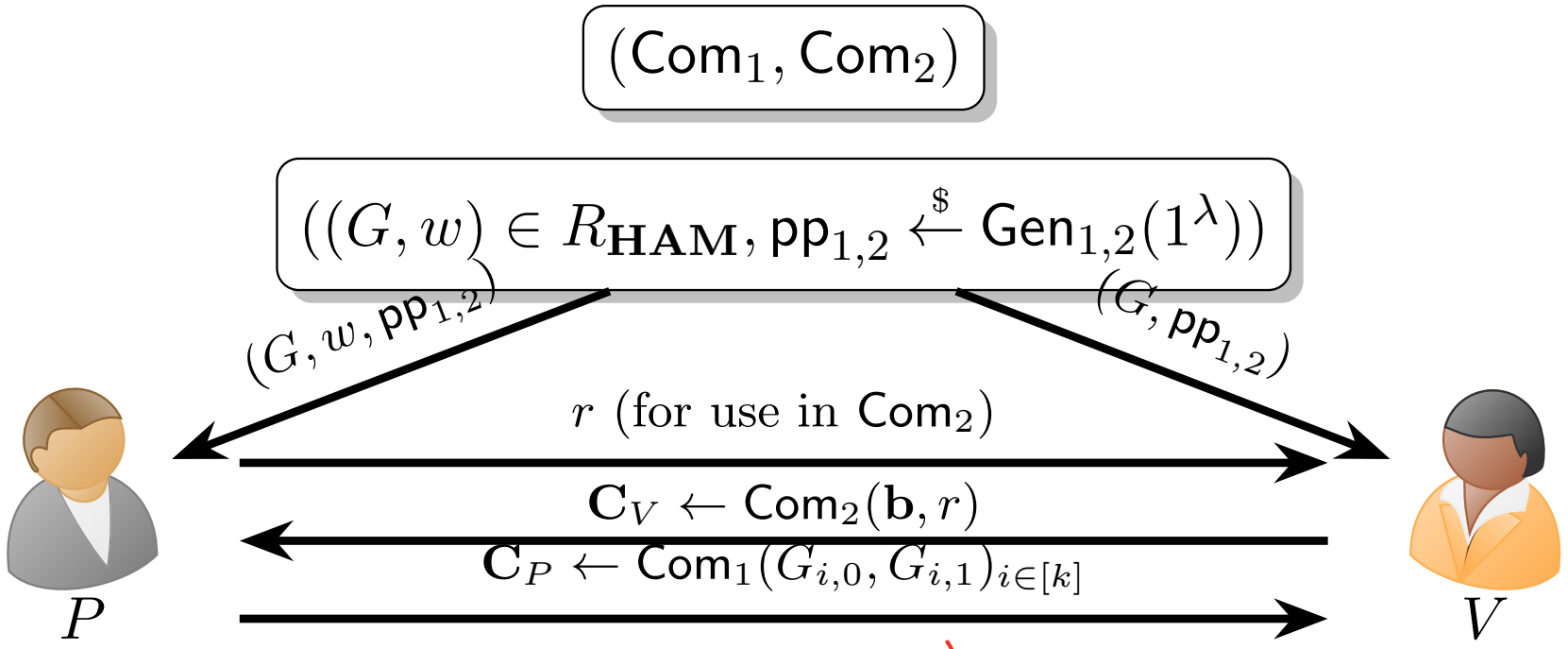


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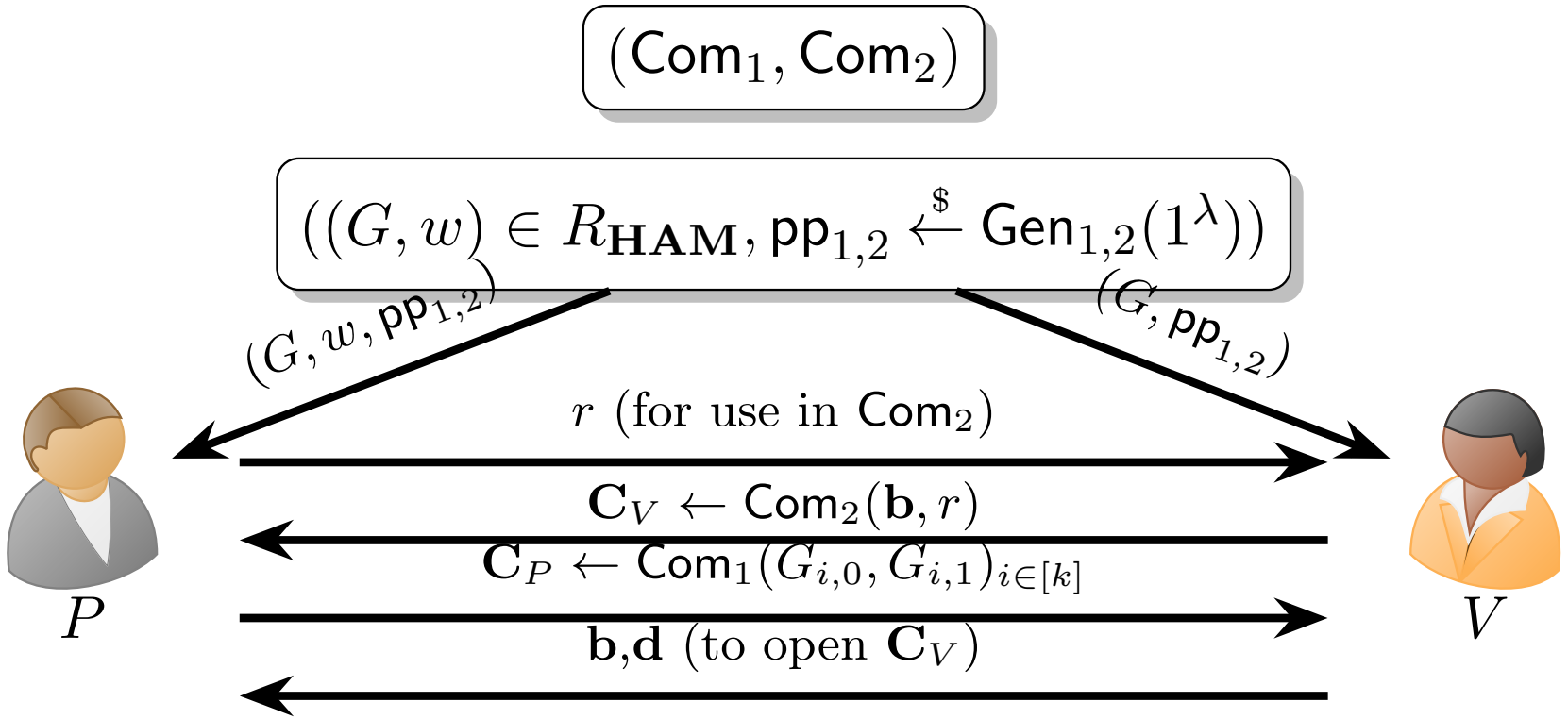

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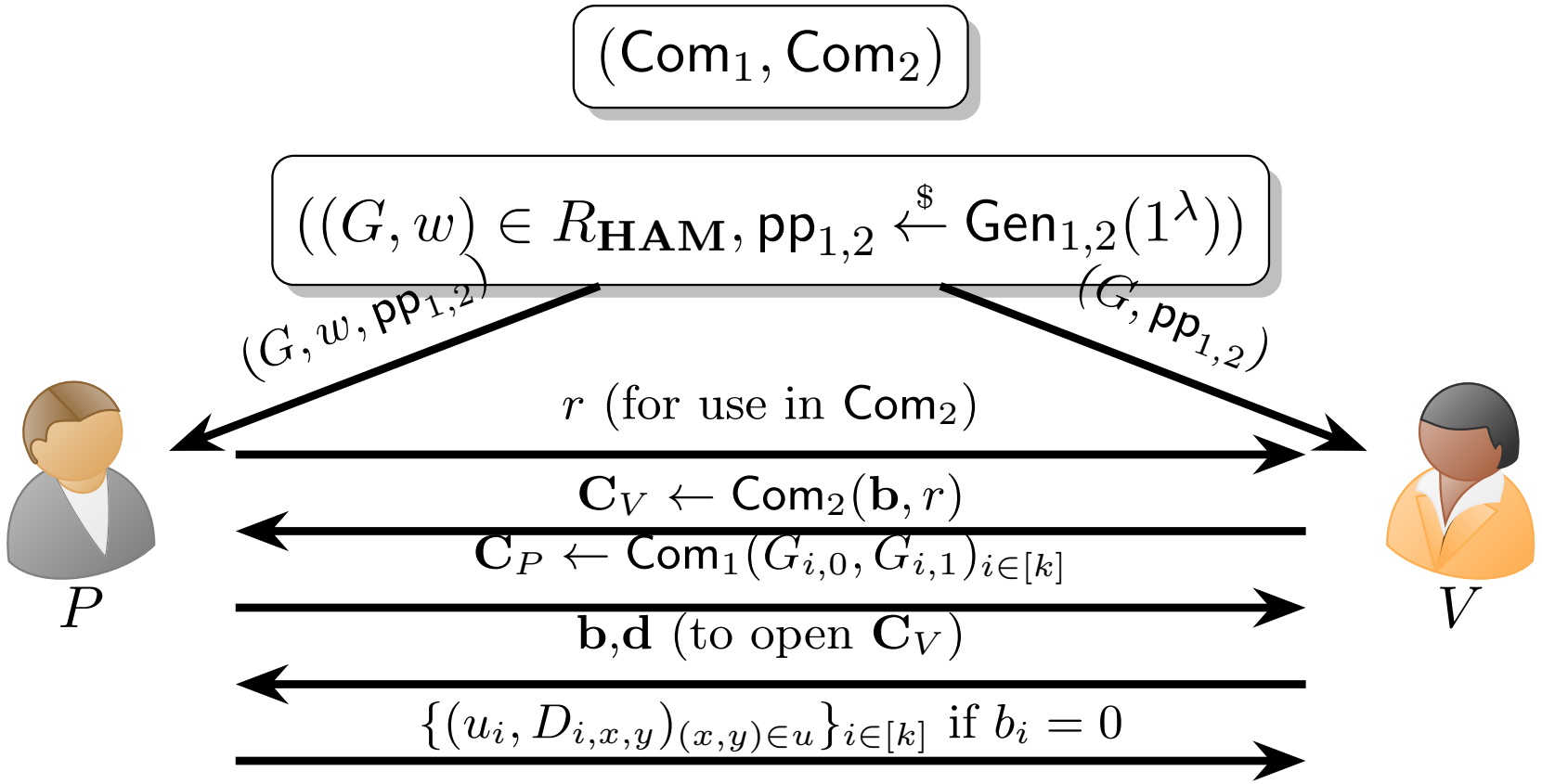

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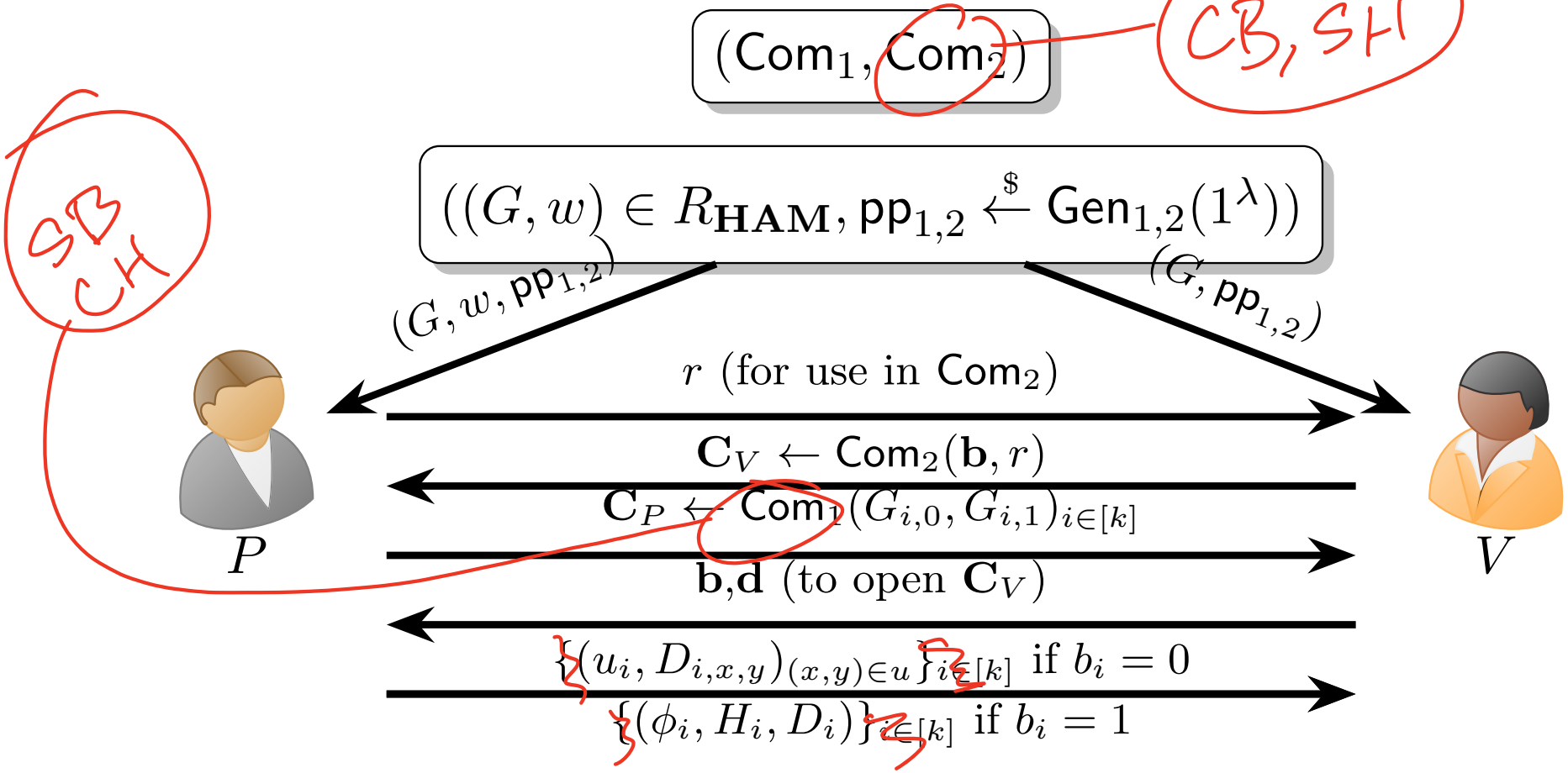

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- As of 2019, the GM**W** result showing **NP** $\subseteq$ **CZK** is **W**idgerson's favorite result by far.

# Next Time: Secure Multi-party Computation