CS 594 – Advanced Crypto (Spring 2026)

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Lecture 17

March 16, 2026

The GMW Protocol

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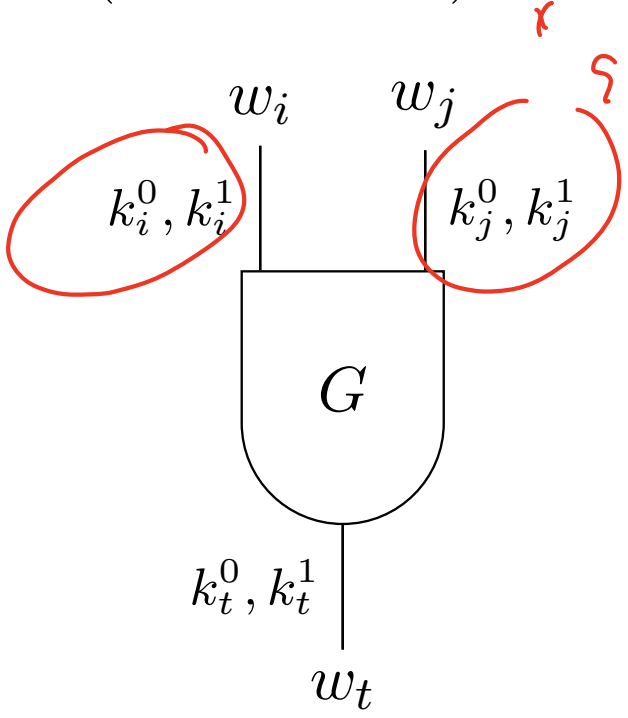

"secret shares"

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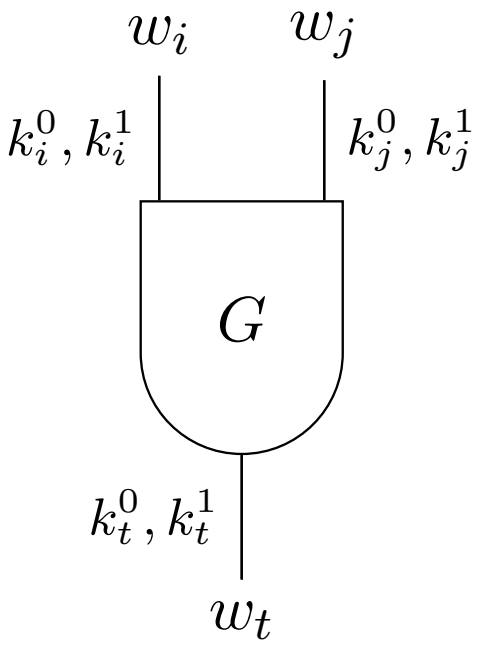

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- GMW Protocol: P_1 and P_2 both have *additive* secret shares of w_i for every wire.

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- For GMW over finite fields, we'll instead look at another protocol and then cast it as GMW (next time).

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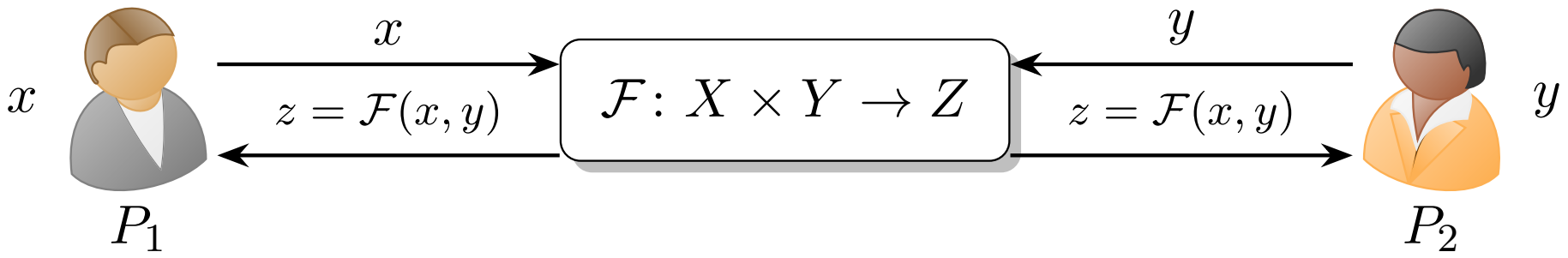

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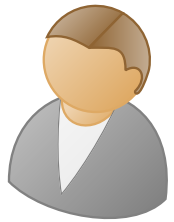

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$\mathbf{x} \in \{0,1\}^n$

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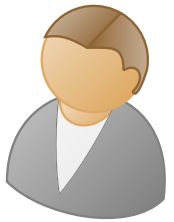

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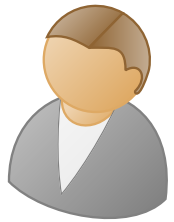

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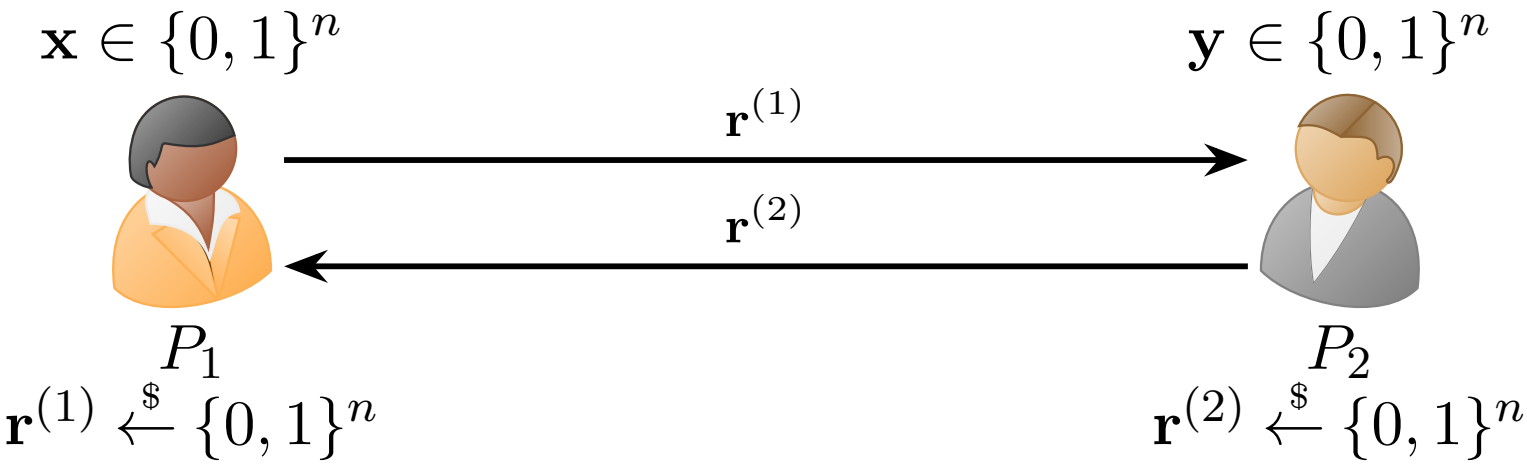

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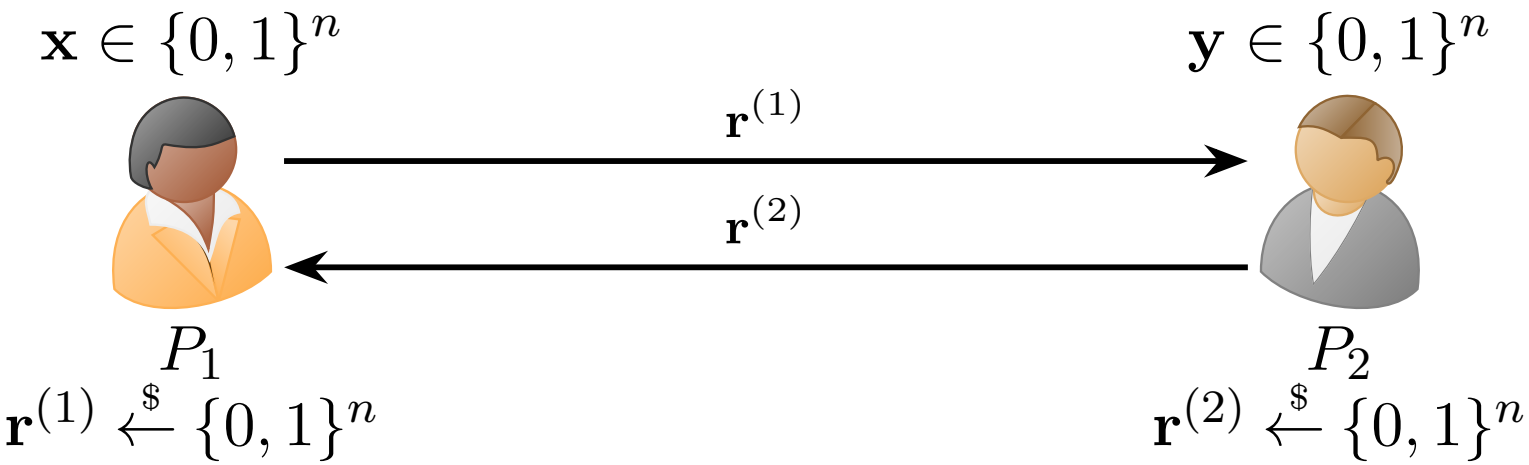

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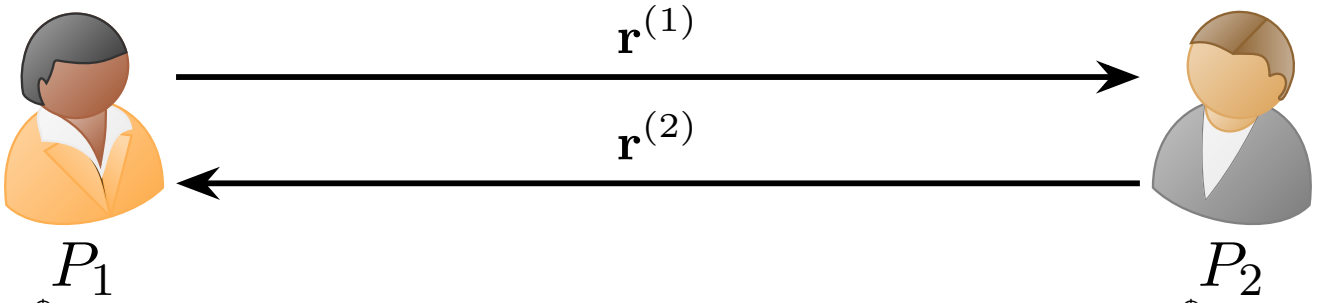

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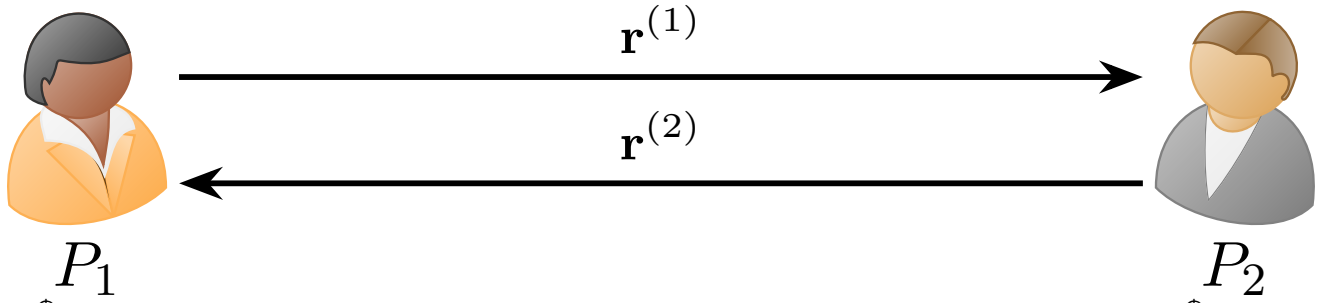

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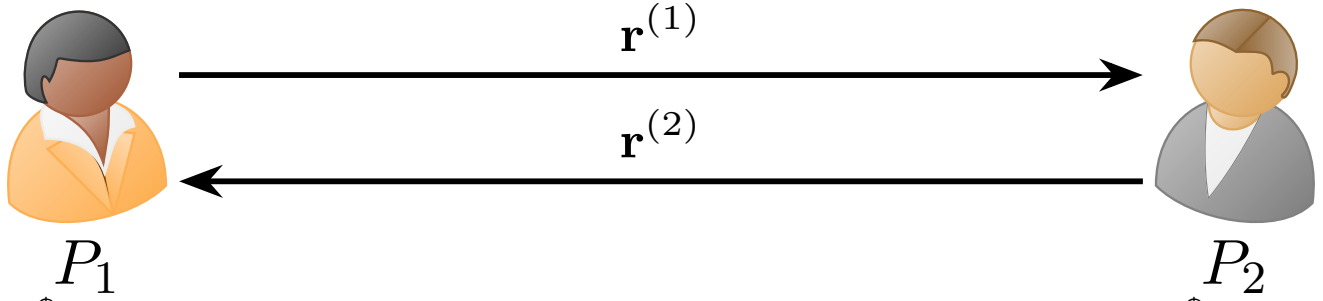

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- Next, *both* parties proceed gate-by-gate evaluating $\mathcal{C}$ for $\mathcal{F}$ using inputs $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$, respectively.

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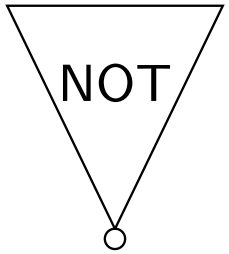

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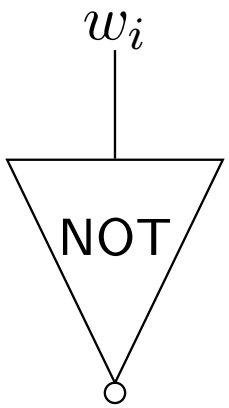

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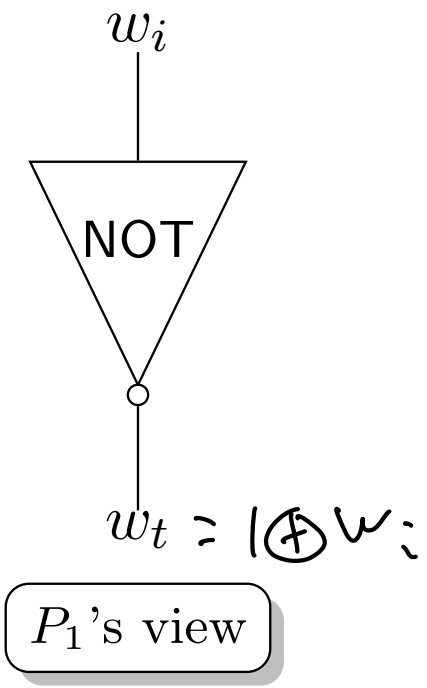

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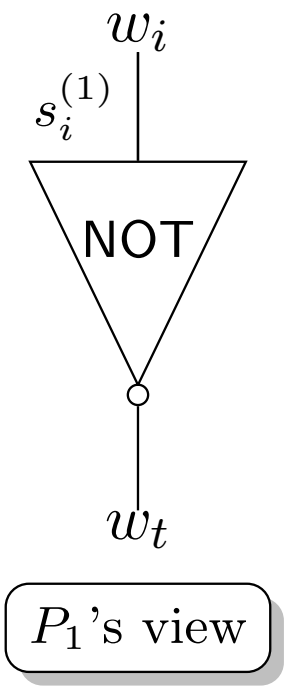

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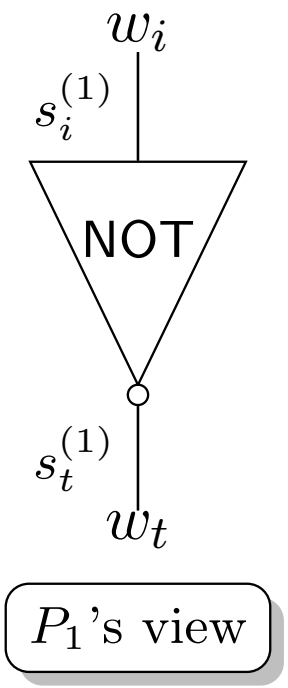

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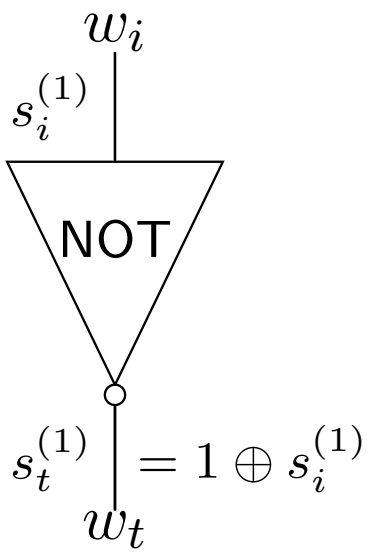

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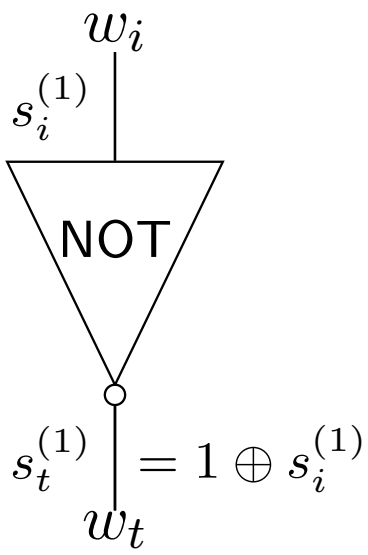

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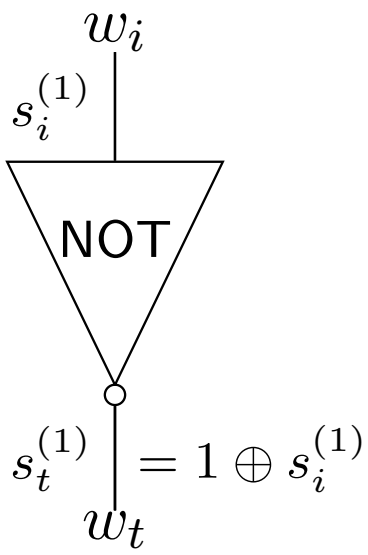

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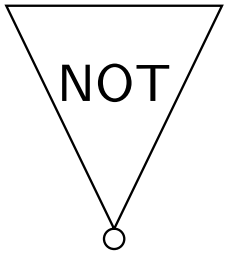

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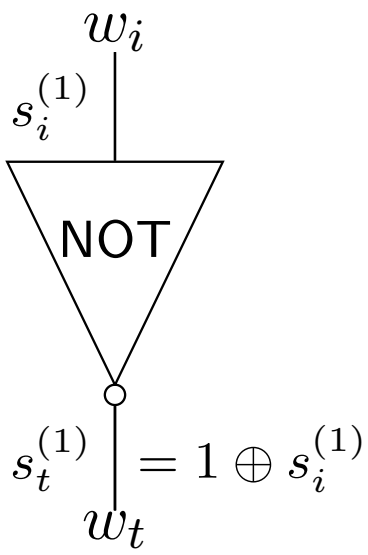

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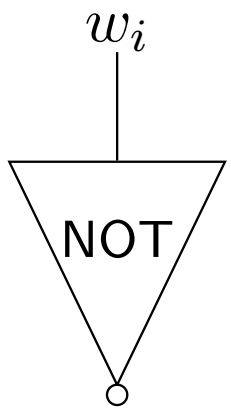

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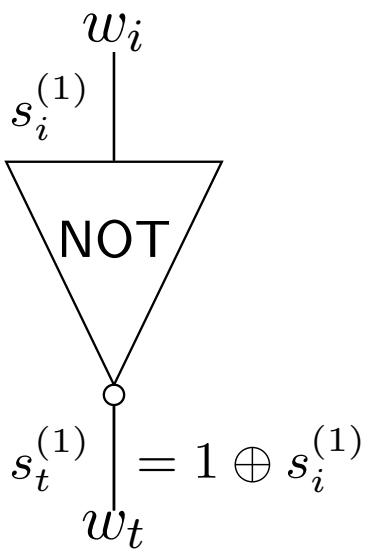

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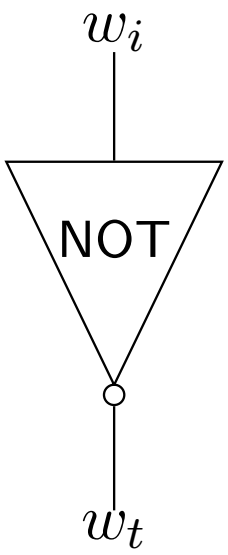

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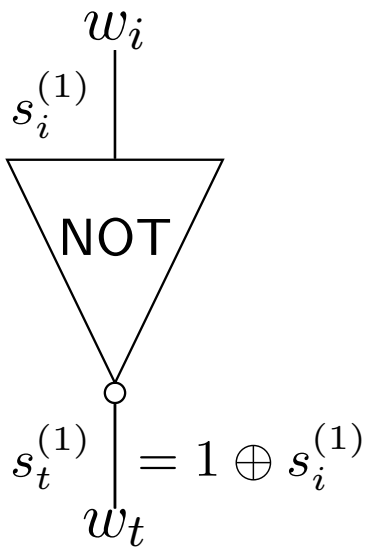

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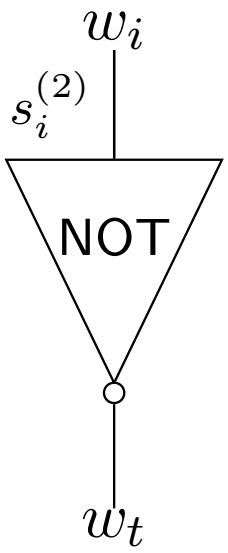

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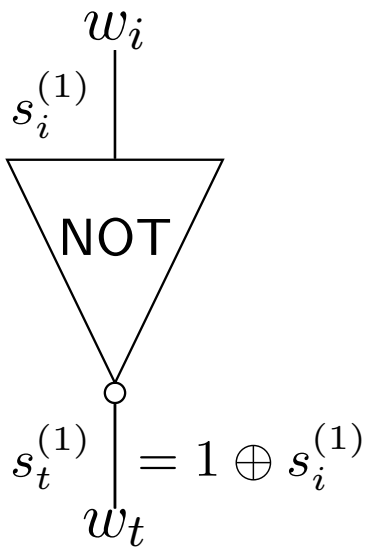

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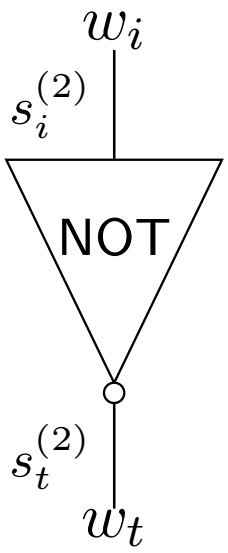

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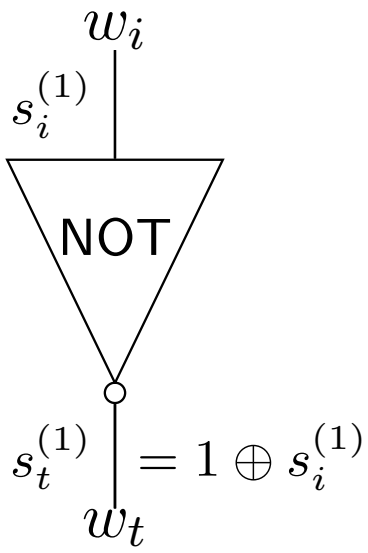

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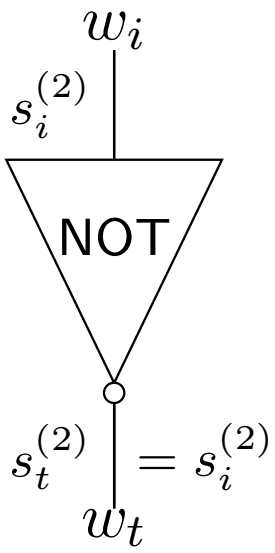

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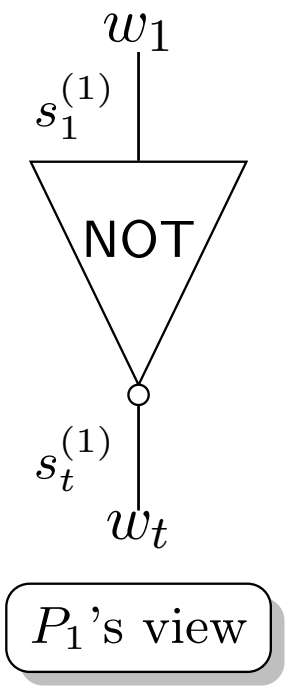

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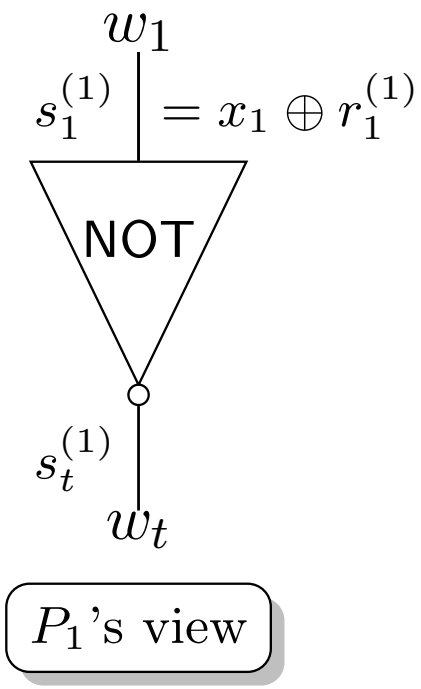

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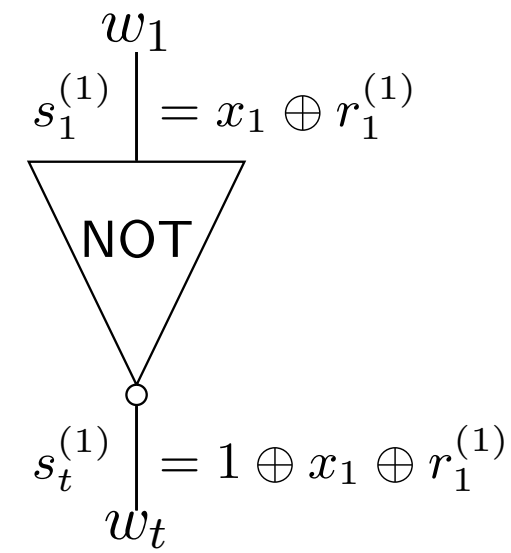

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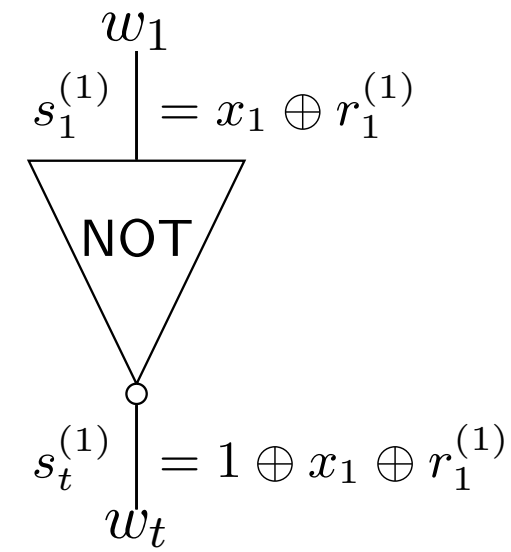

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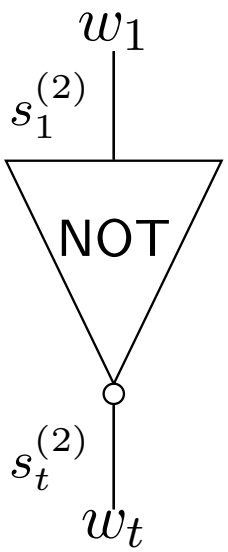

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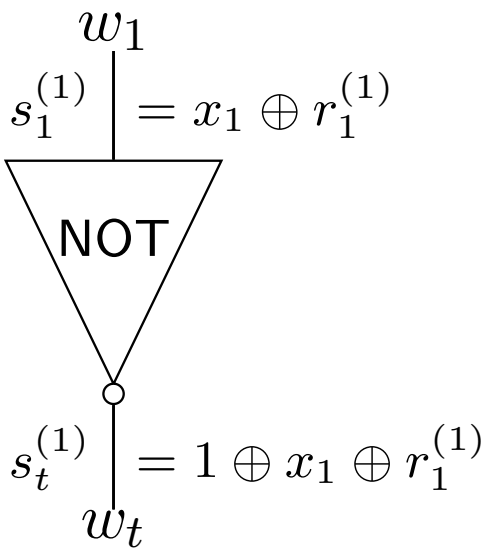

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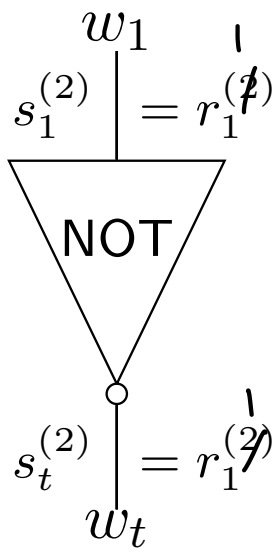

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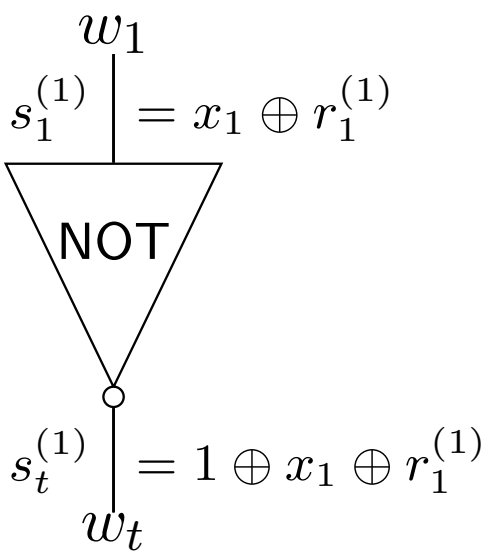

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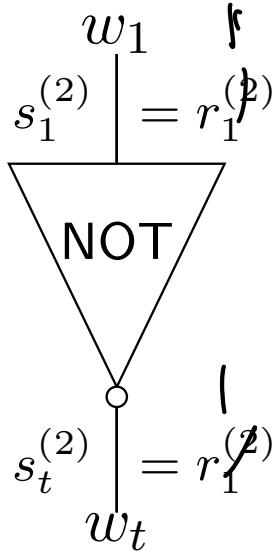

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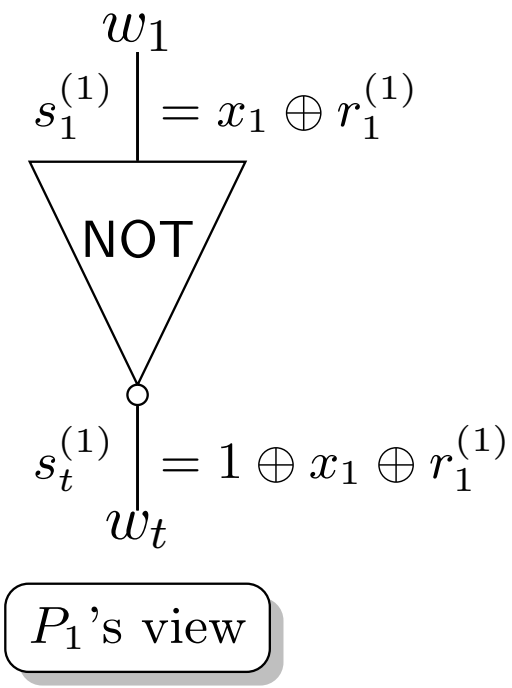

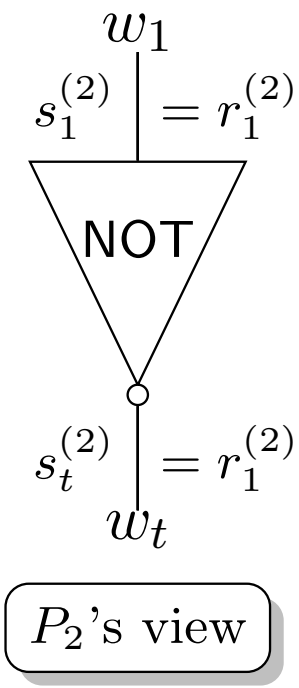

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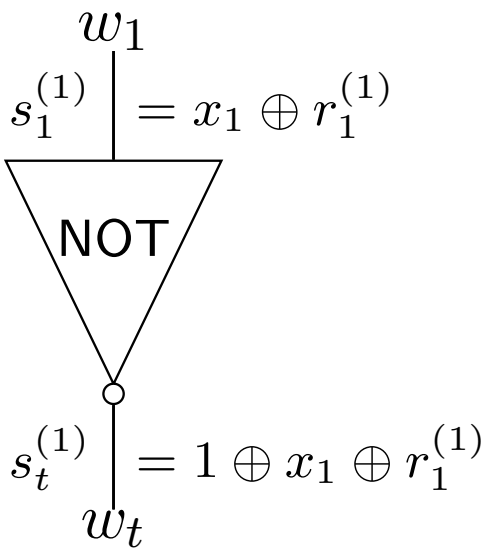

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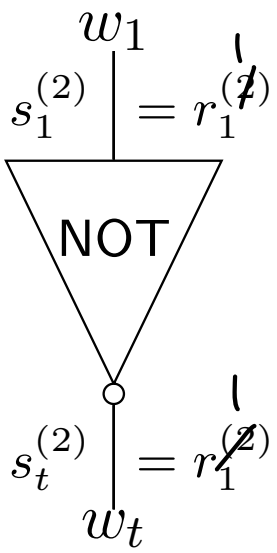

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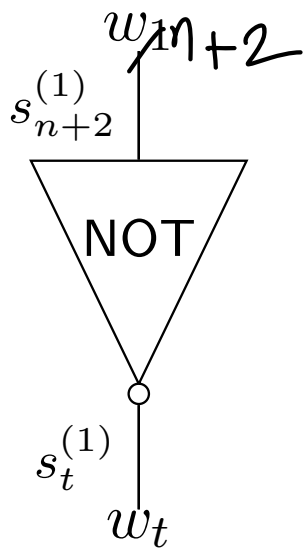

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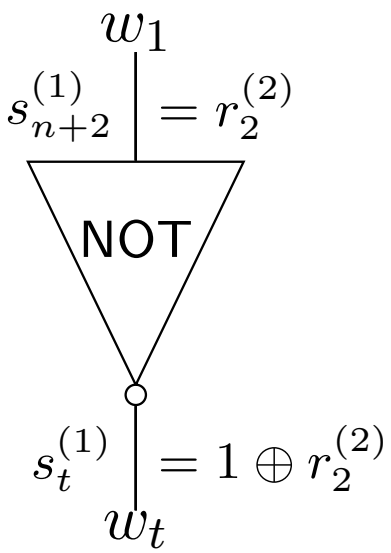

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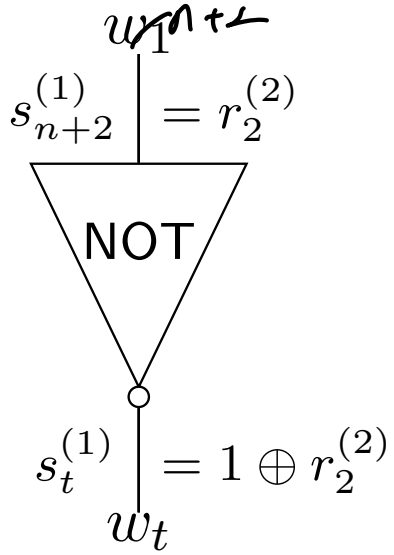

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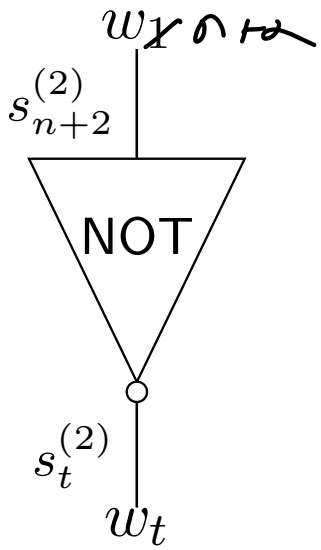

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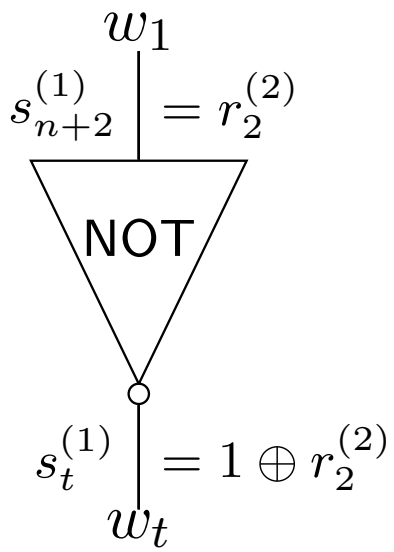

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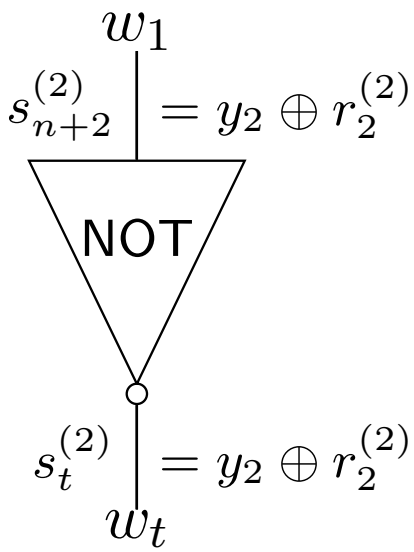

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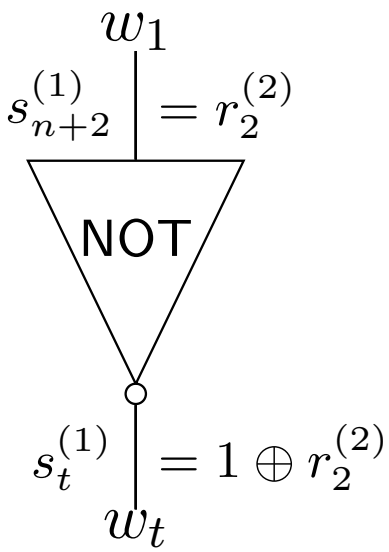

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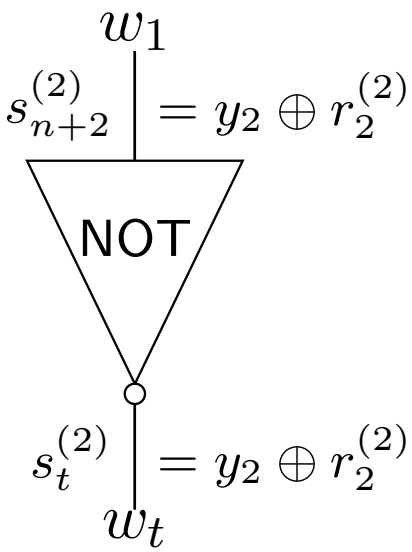

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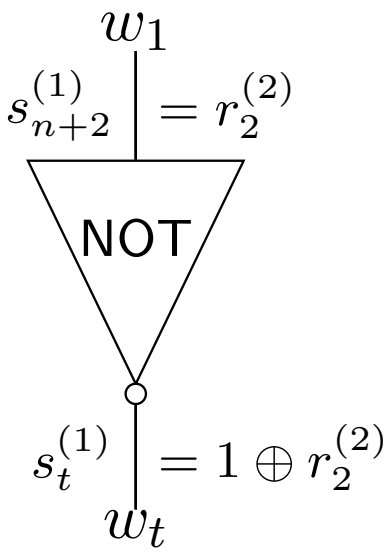

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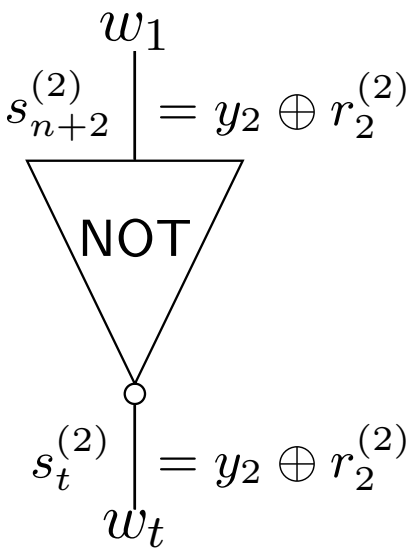

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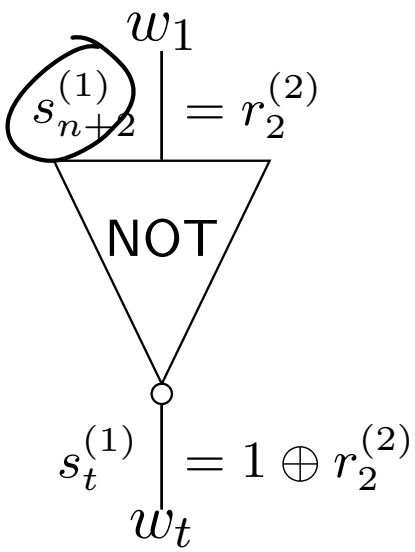

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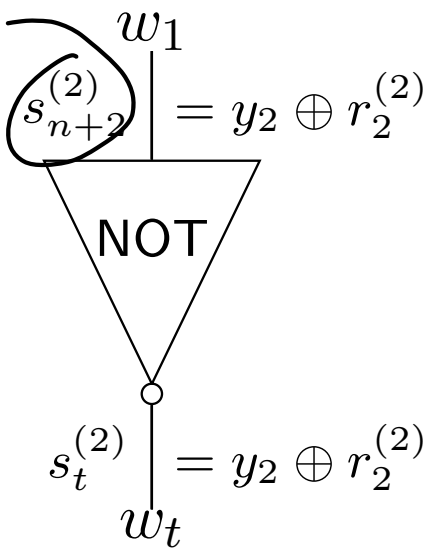

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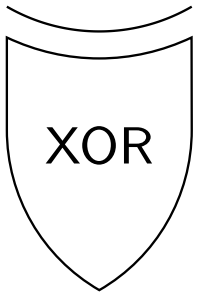

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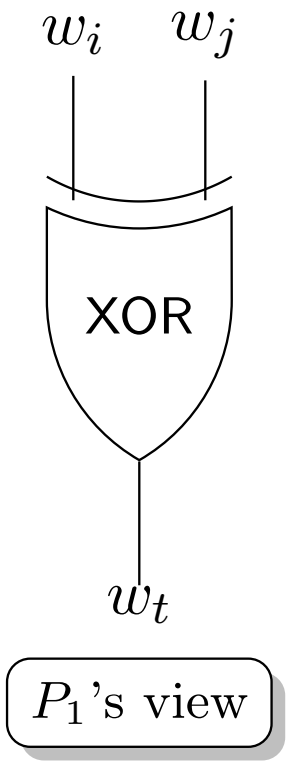

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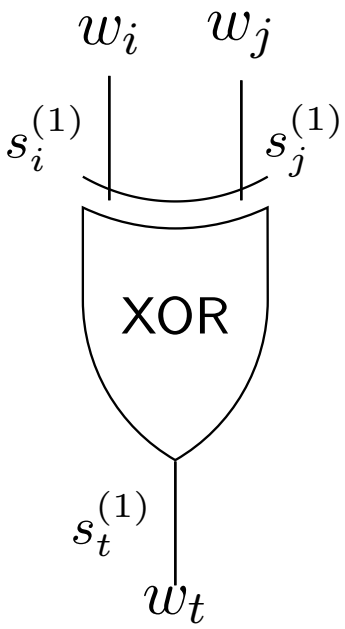

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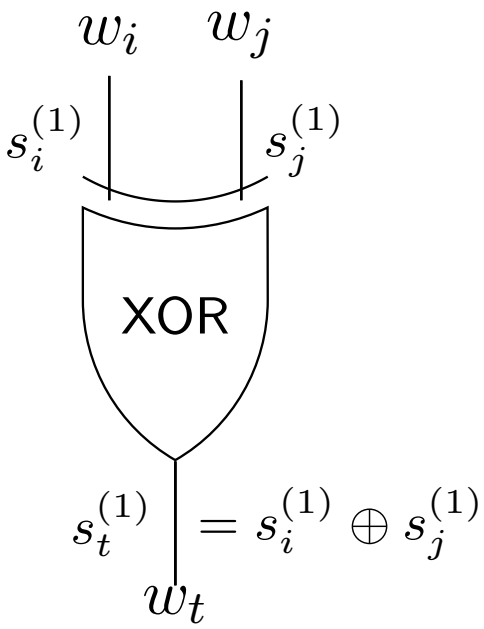

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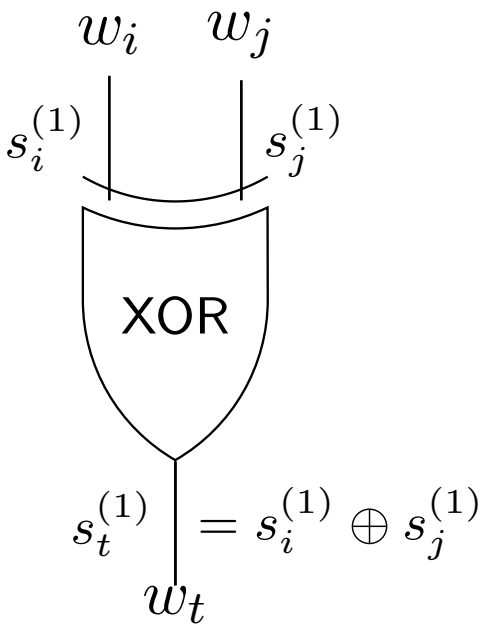

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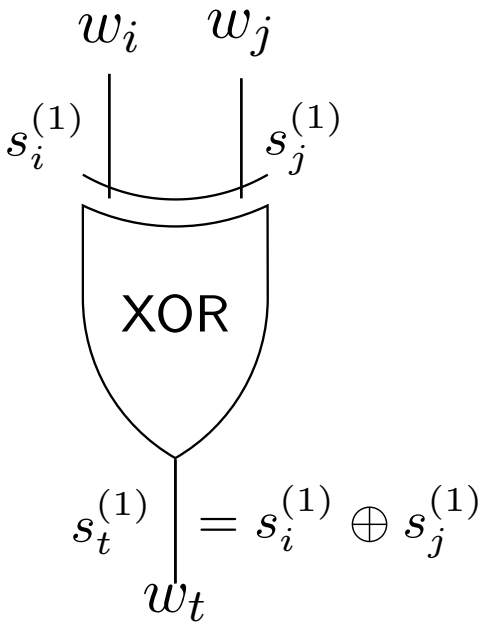

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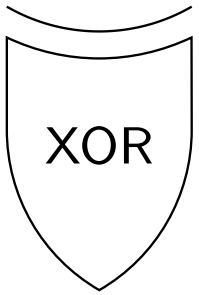

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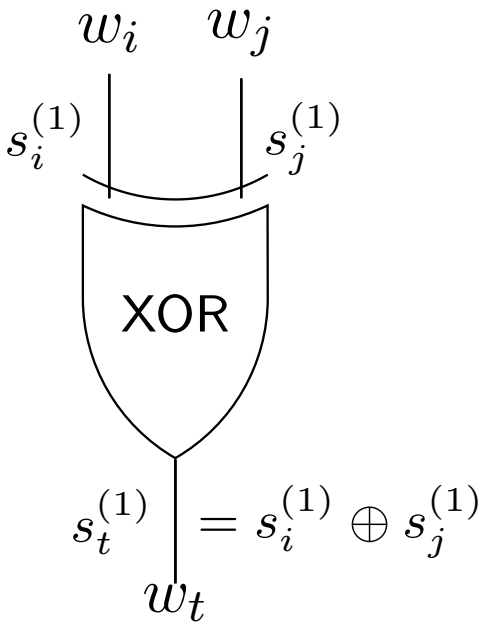

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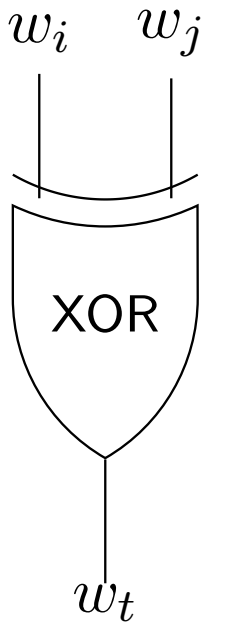

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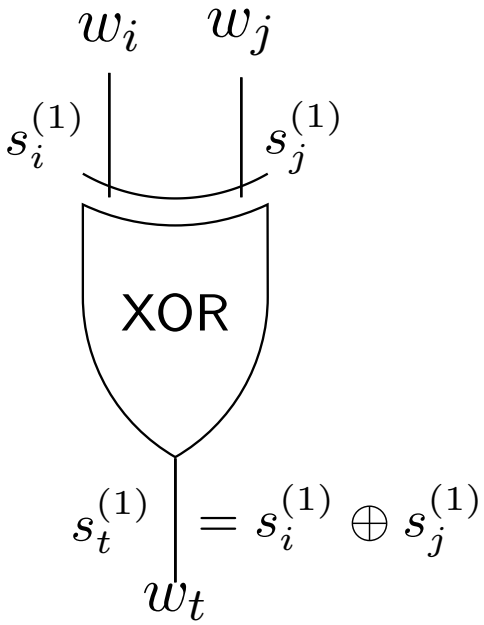

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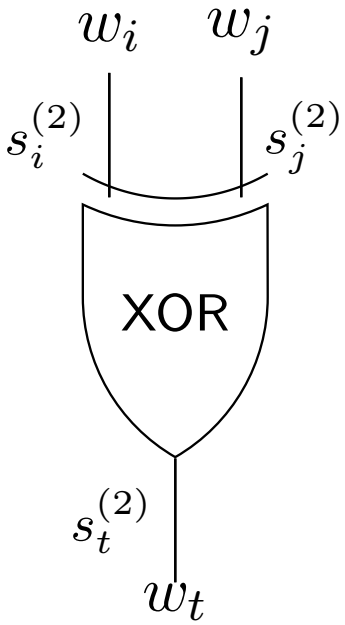

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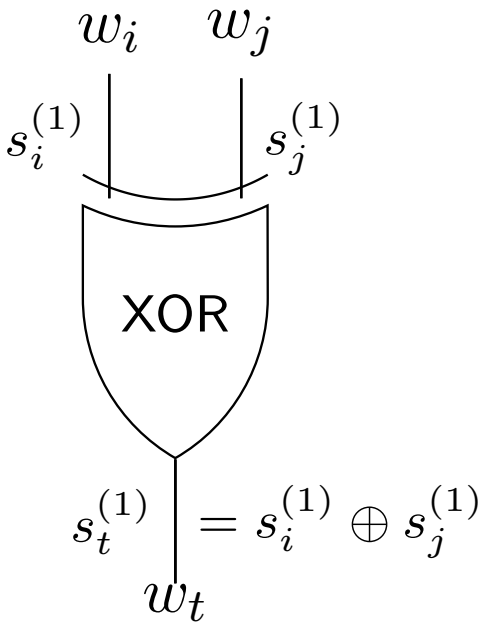

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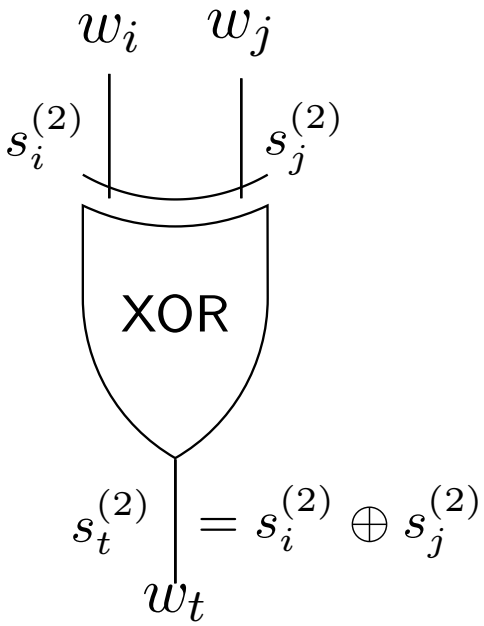

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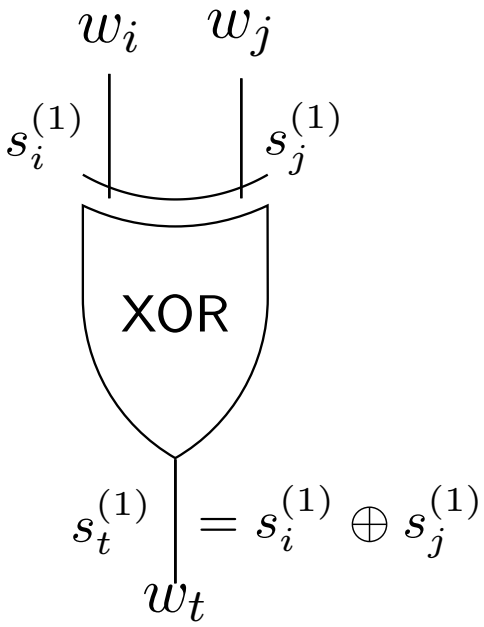

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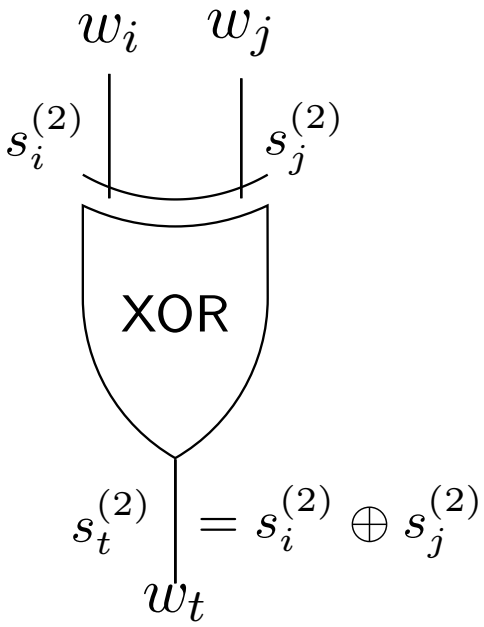

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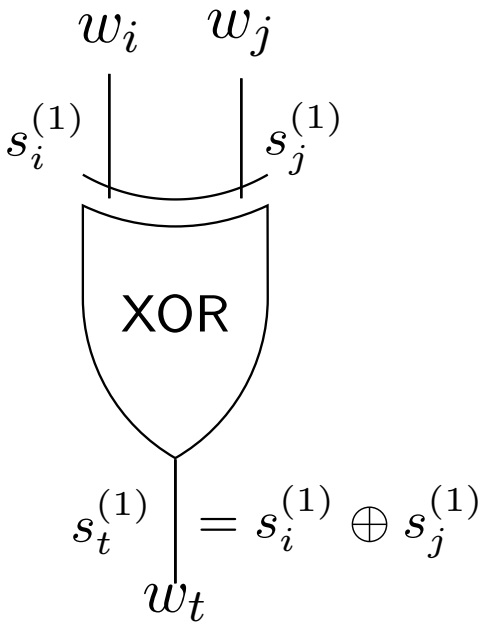

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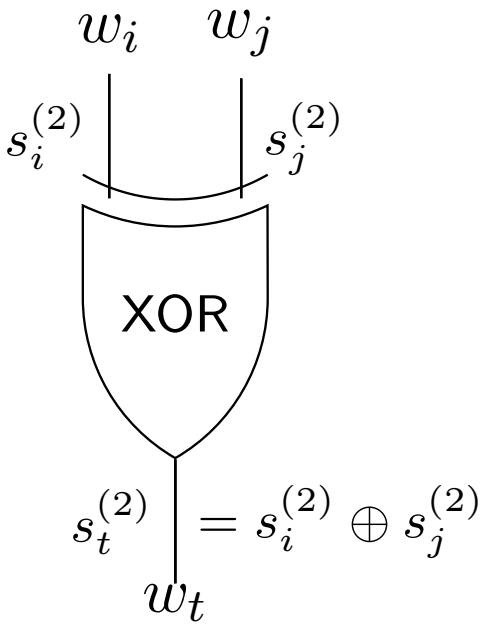

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- Here, $s_i^{(1)} \oplus s_i^{(2)} = w_i$ and $s_j^{(1)} \oplus s_j^{(2)} = w_j$.
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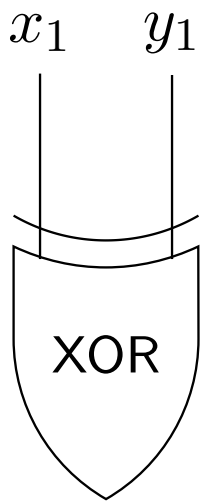

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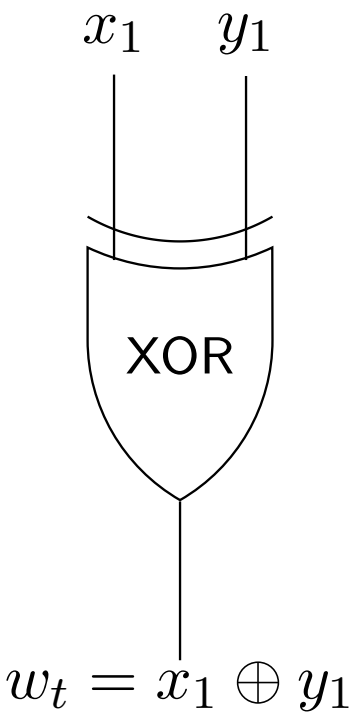

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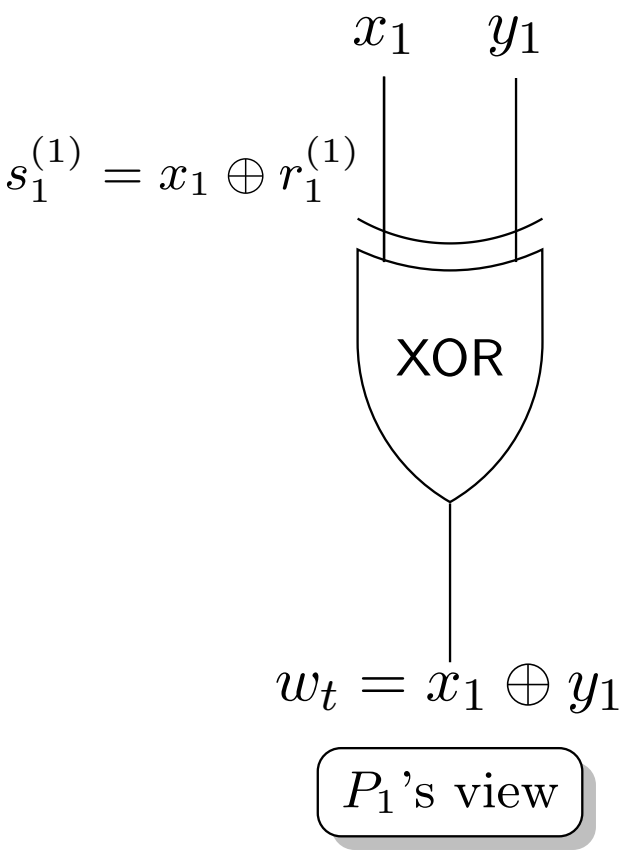

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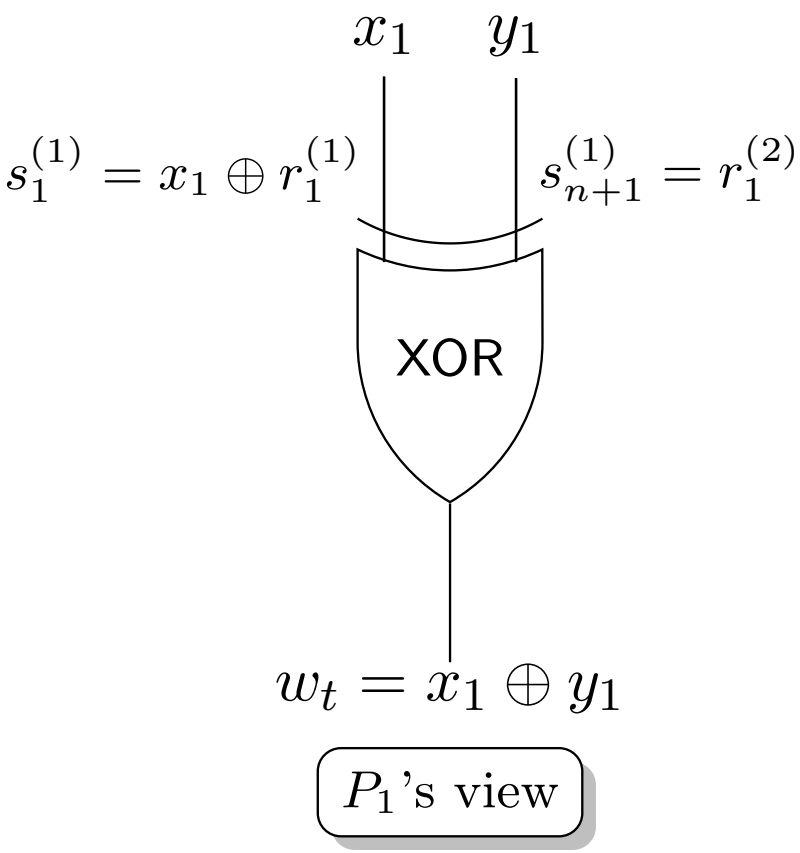

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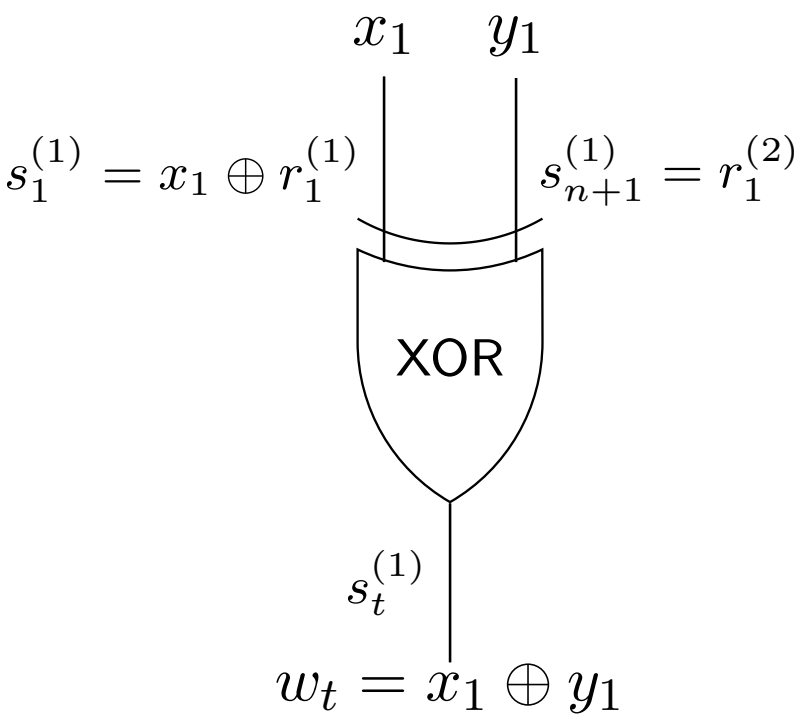

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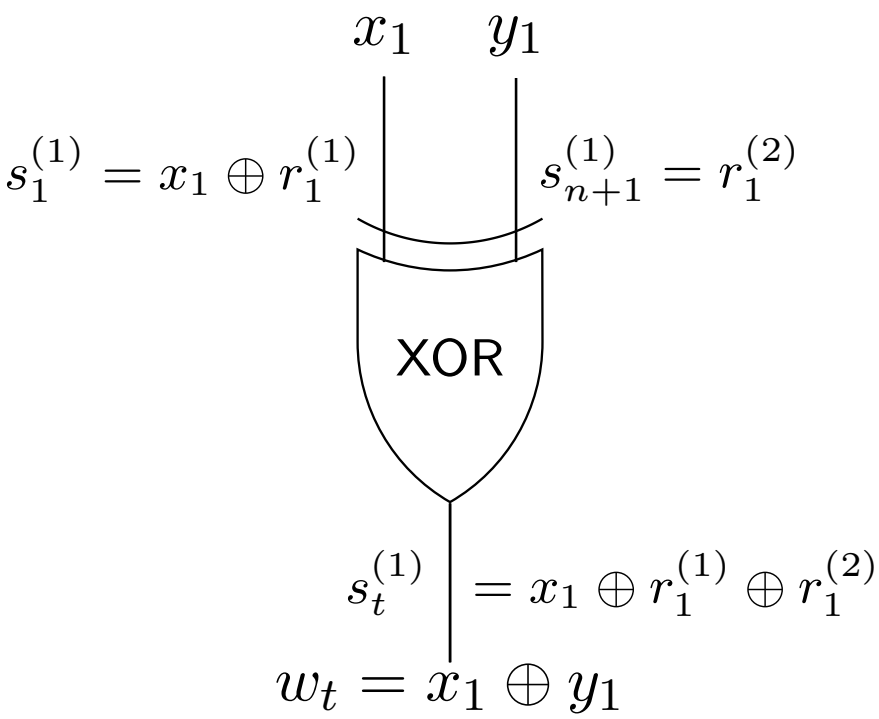

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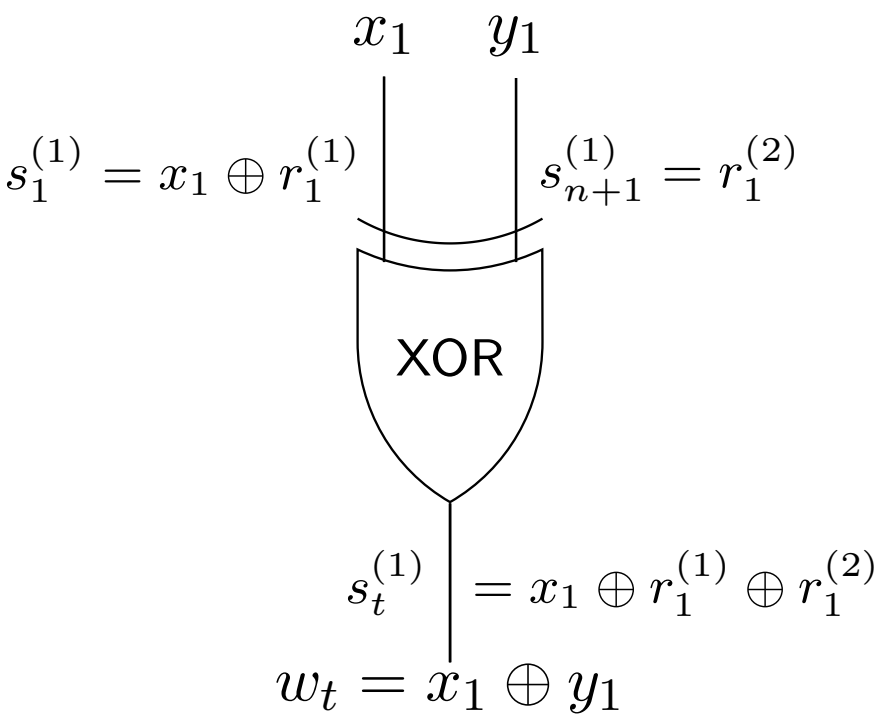

P_1's view

P_2's view

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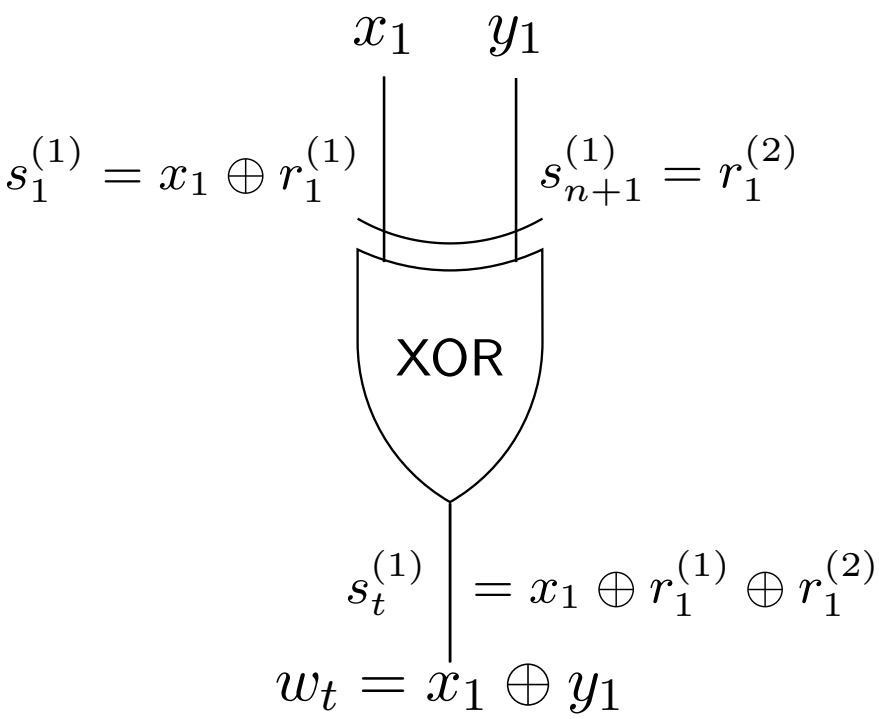

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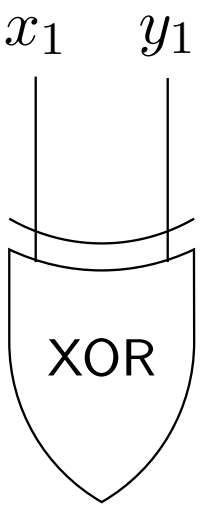

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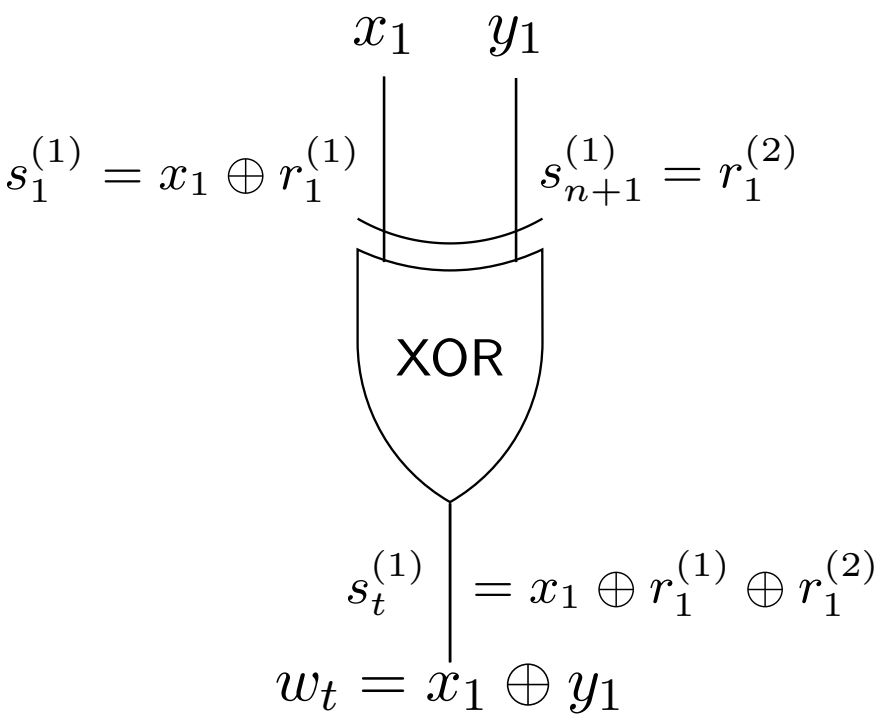

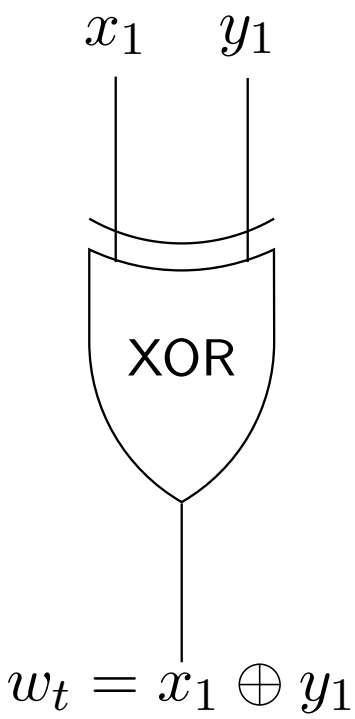

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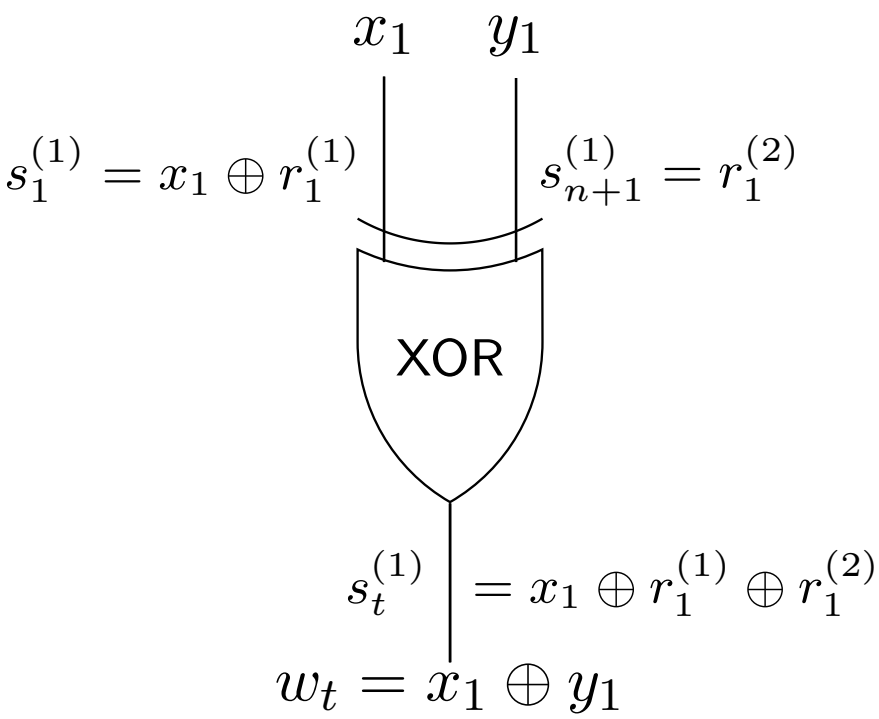

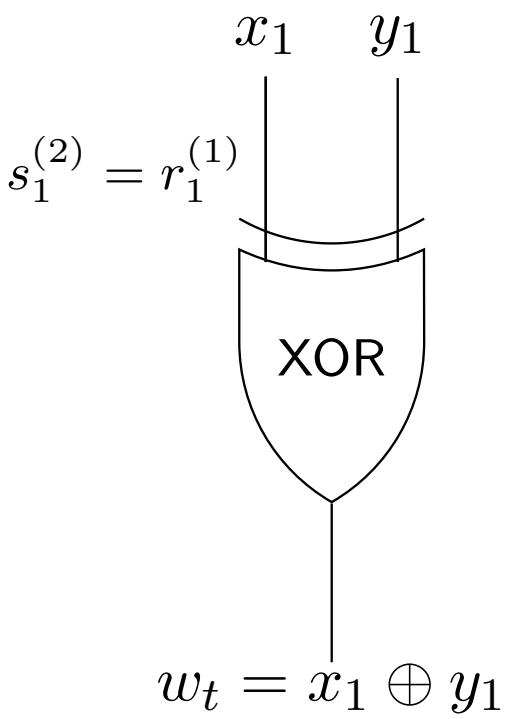

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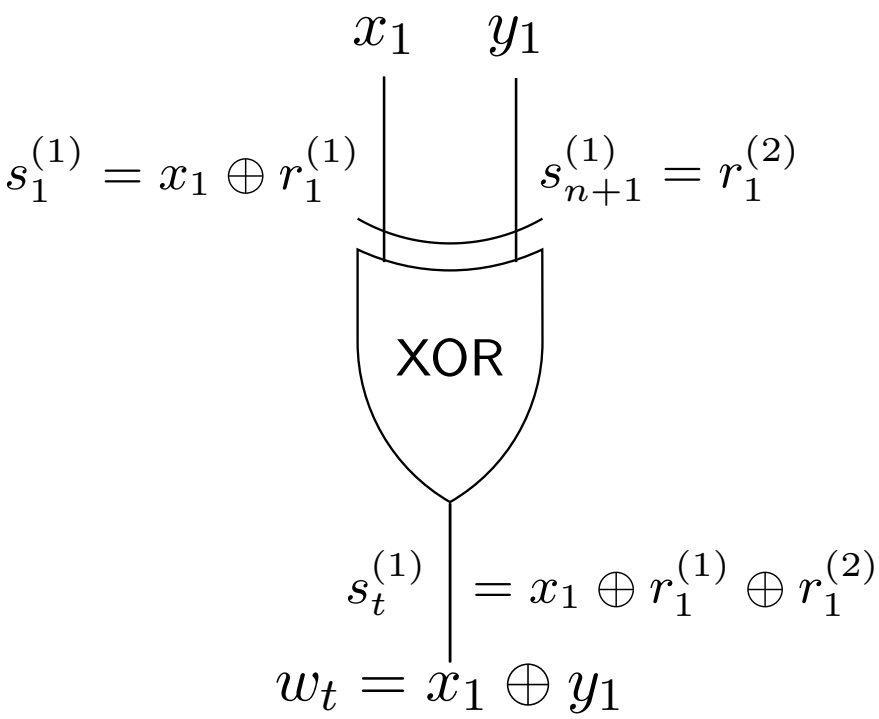

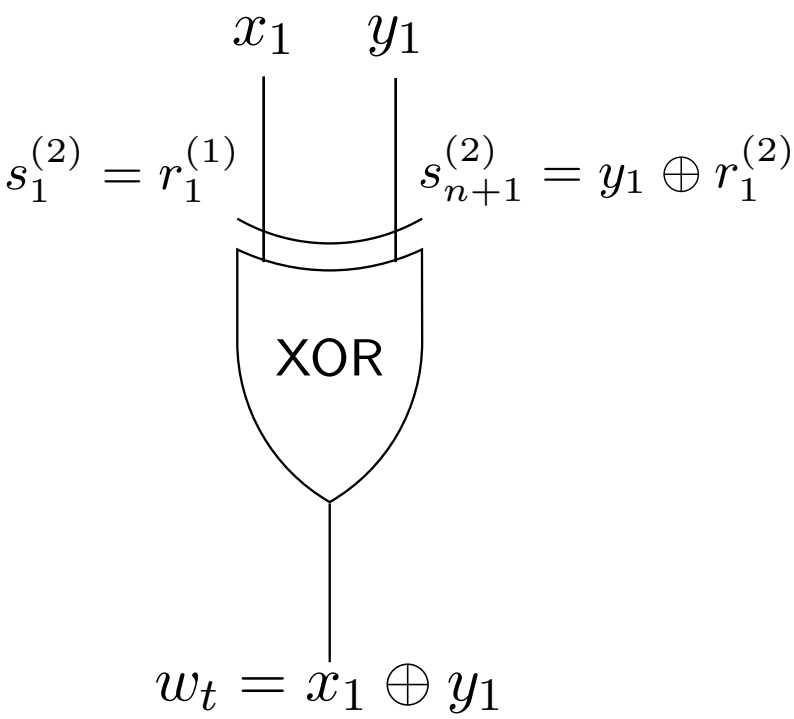

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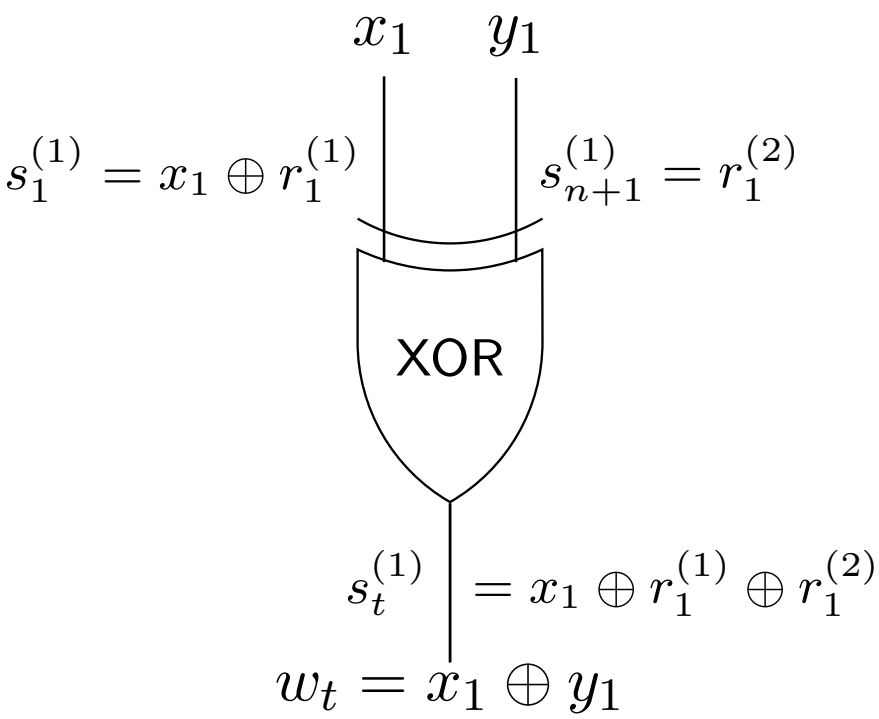

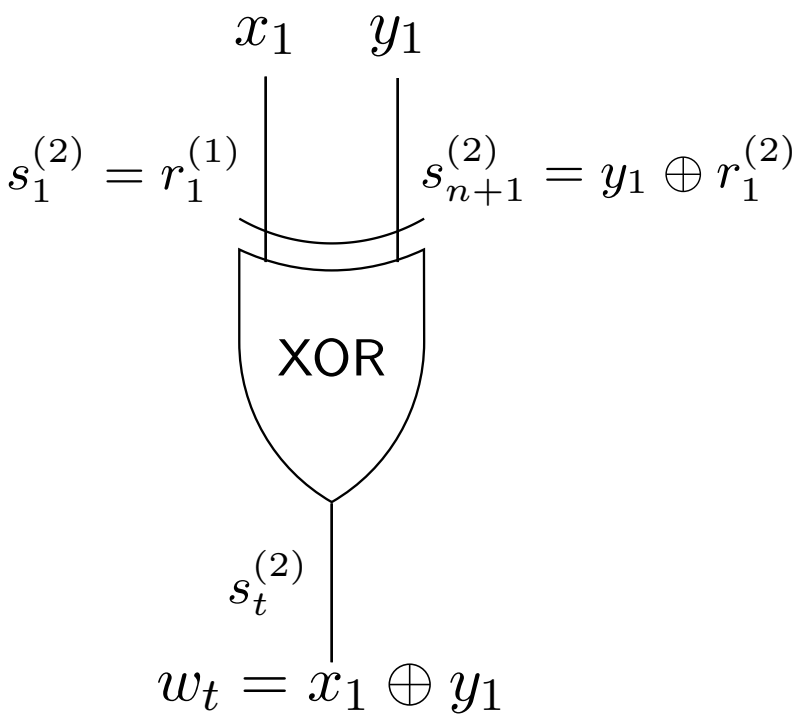

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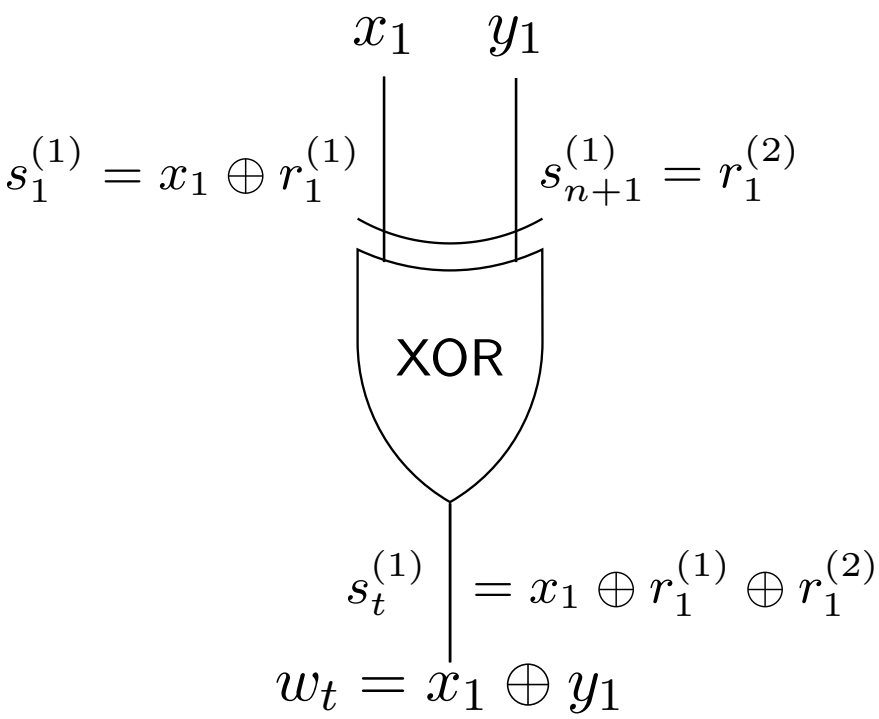

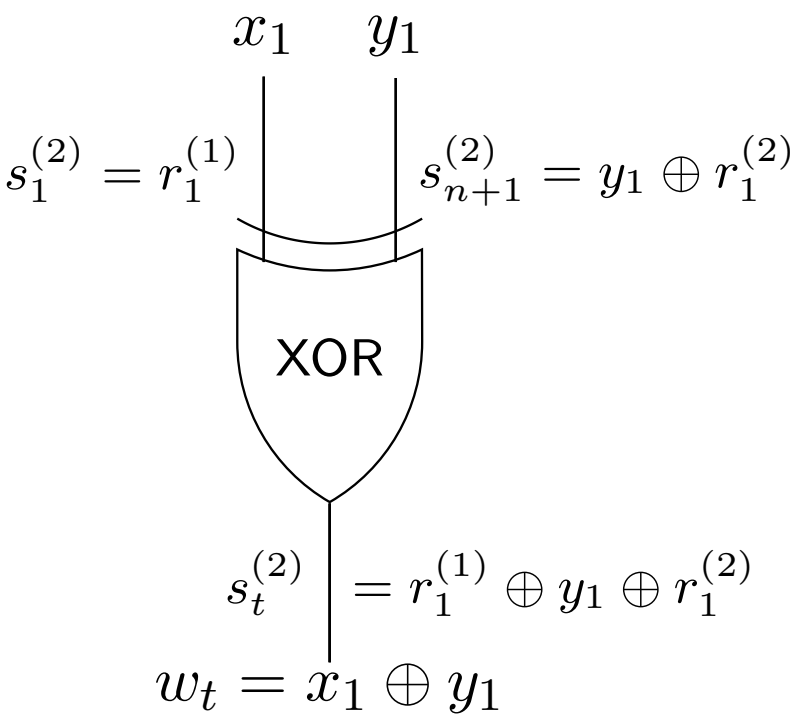

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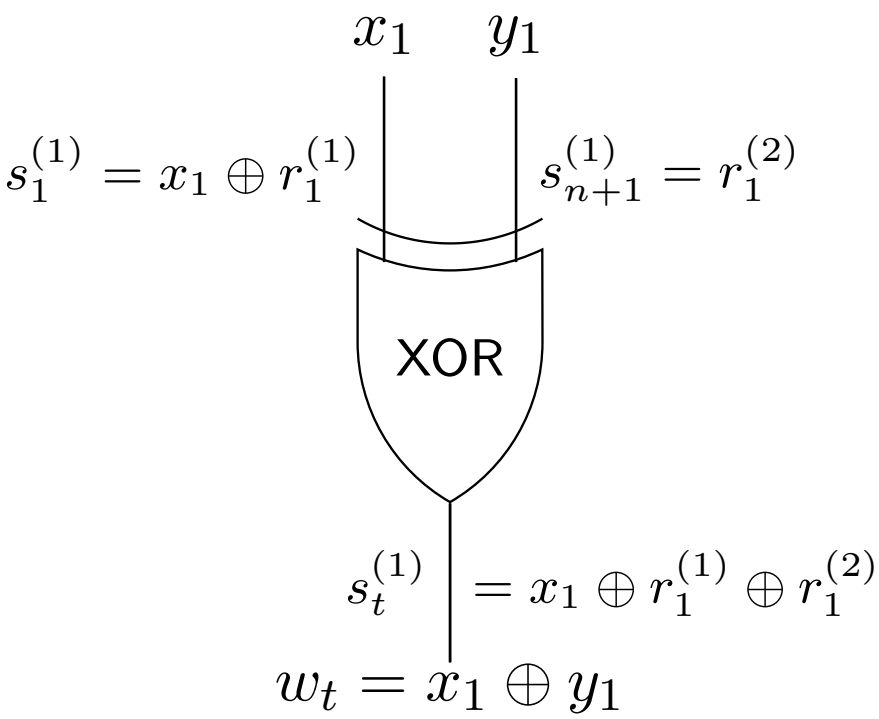

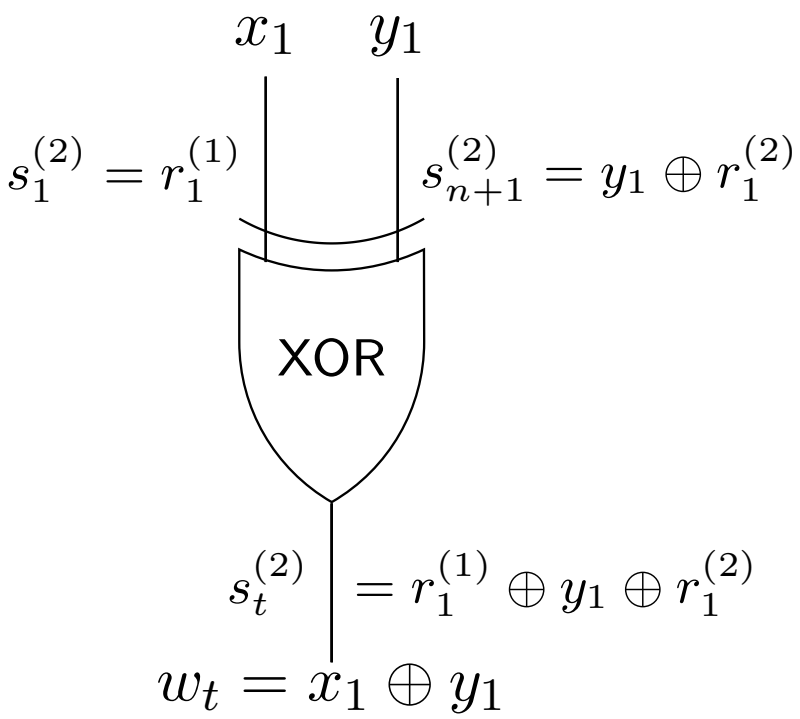

- Notice: $s_t^{(1)} \oplus s_t^{(2)} = (x_1 \oplus r_1^{(1)} \oplus r_1^{(2)}) \oplus (r_1^{(1)} \oplus y_1 \oplus r_1^{(2)}) = x_1 \oplus y_1.$

$\| \; w_t$

GMW Protocol: Evaluation of AND

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- AND gates require interaction between P_1 and P_2.

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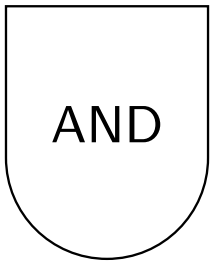

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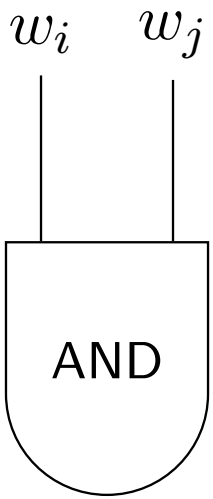

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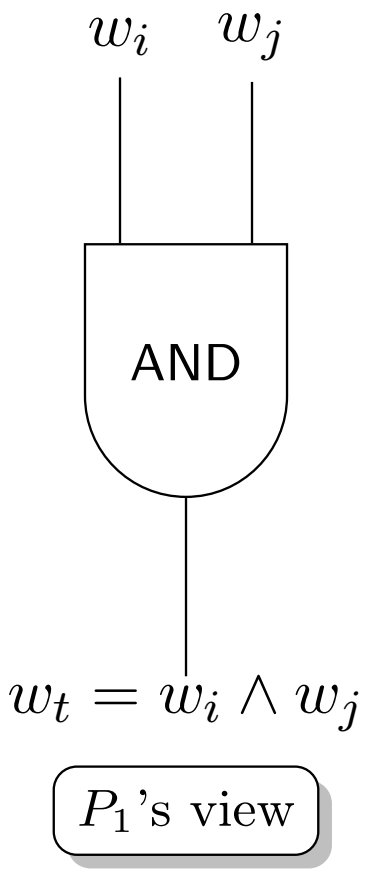

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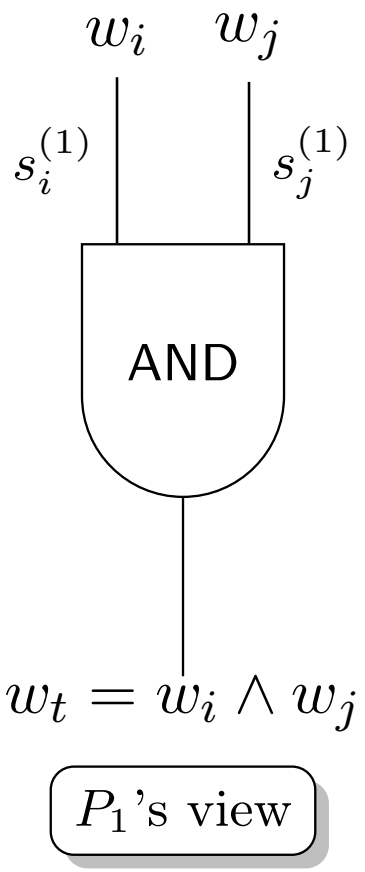

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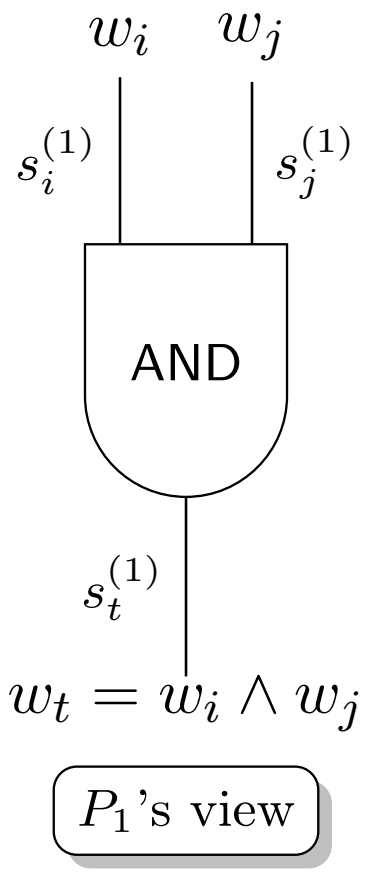

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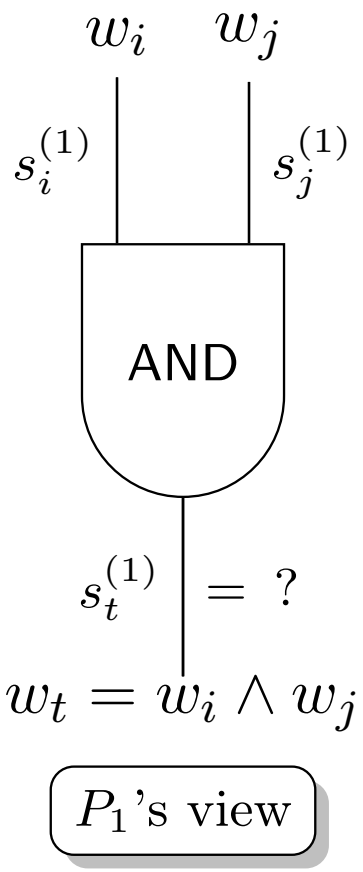

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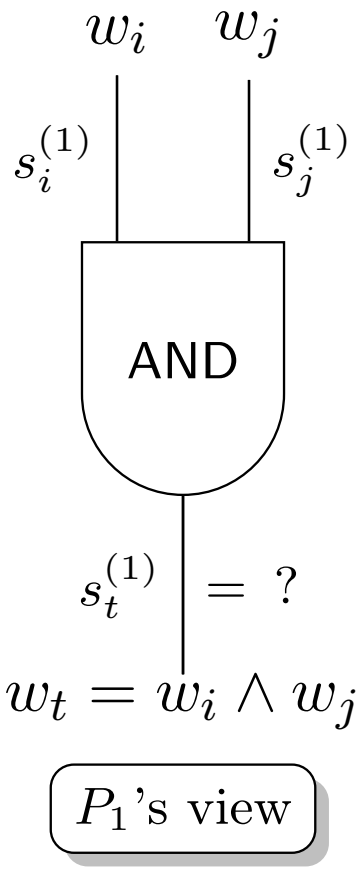

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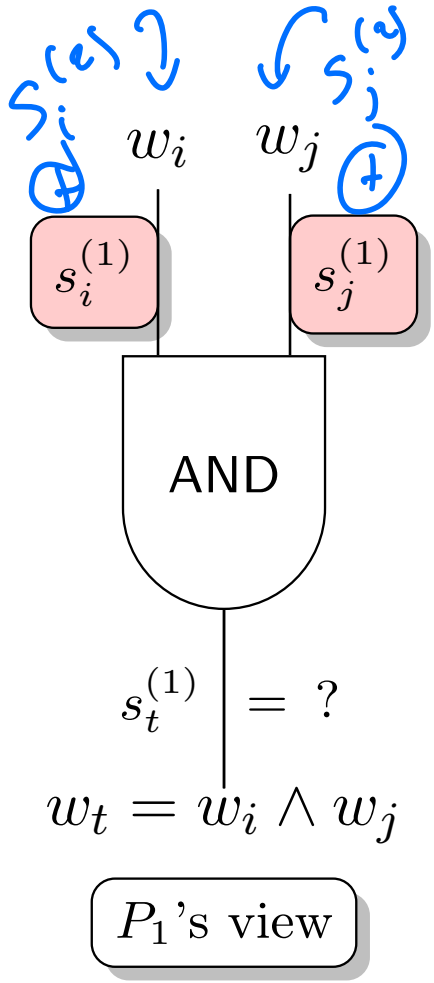

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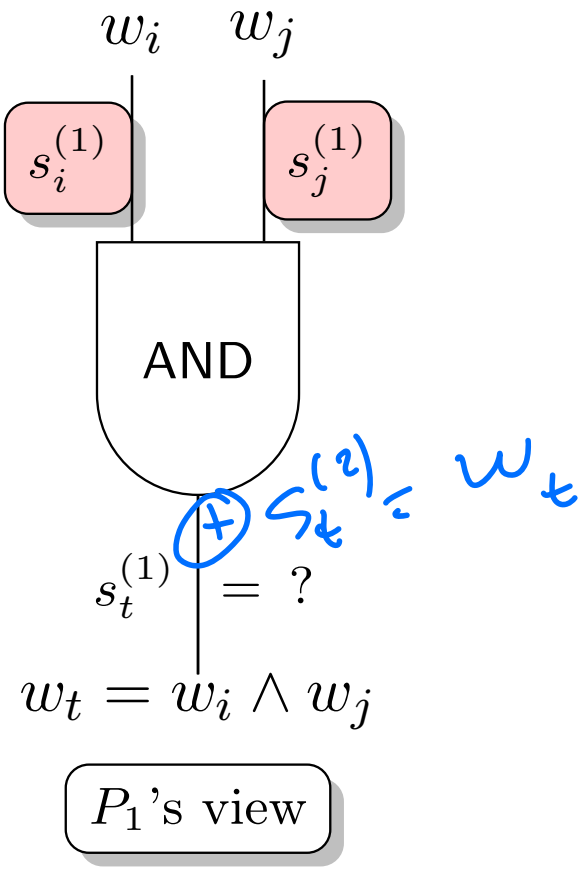

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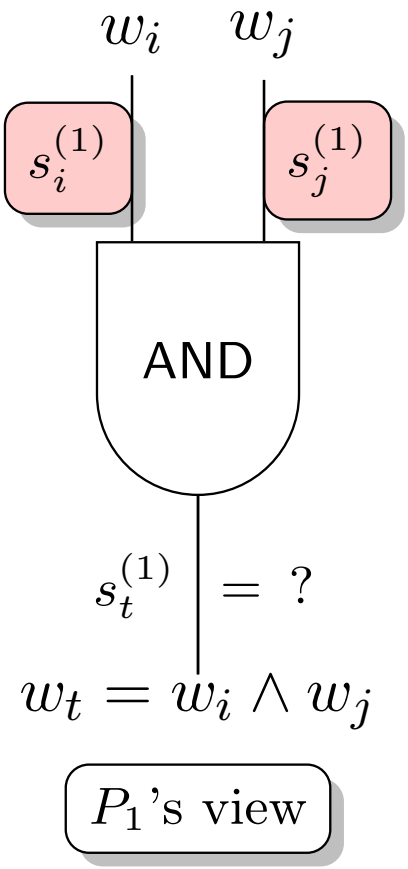

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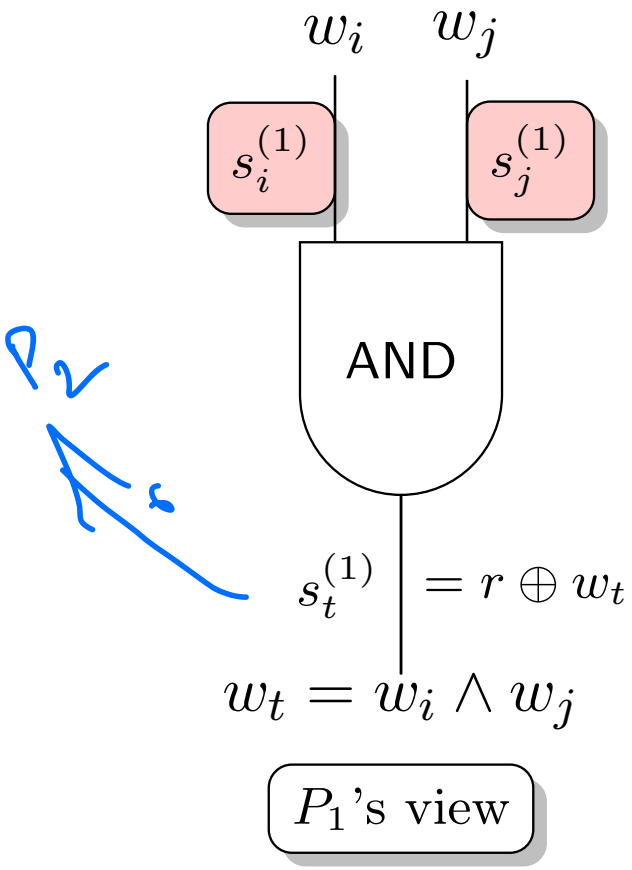

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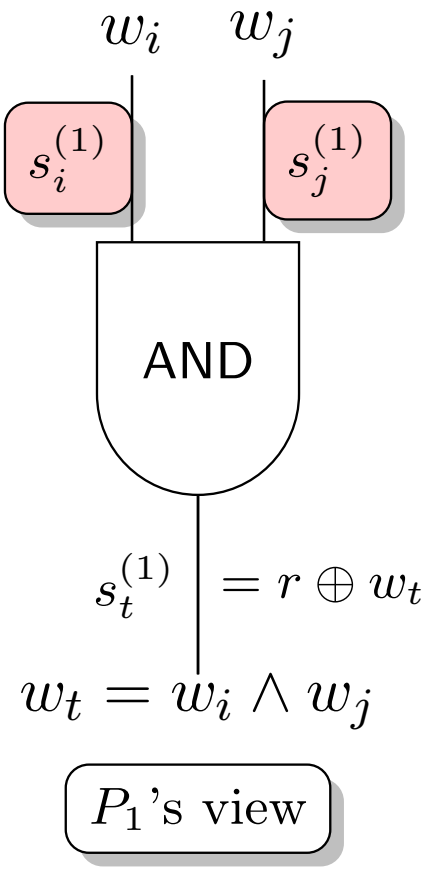

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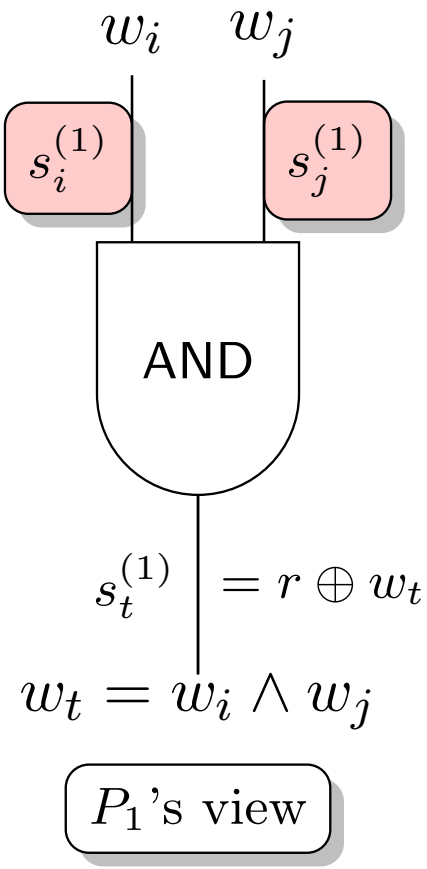

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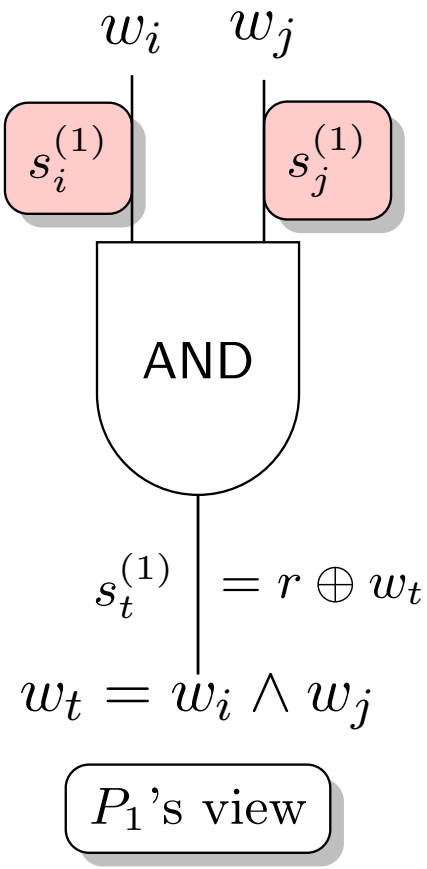

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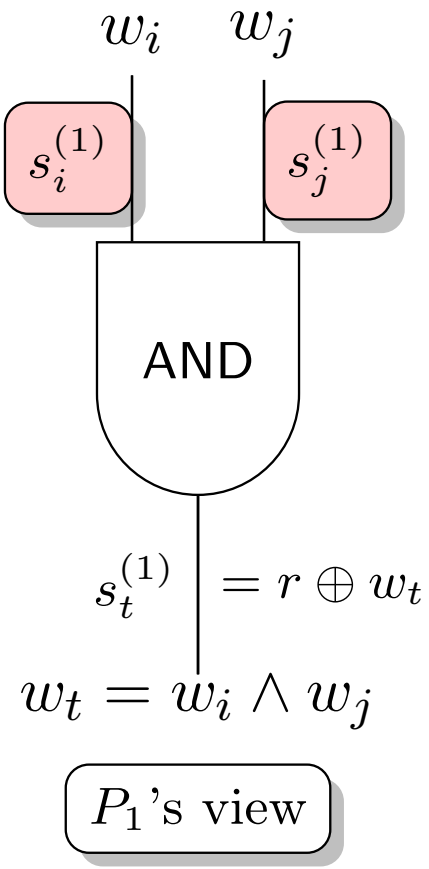

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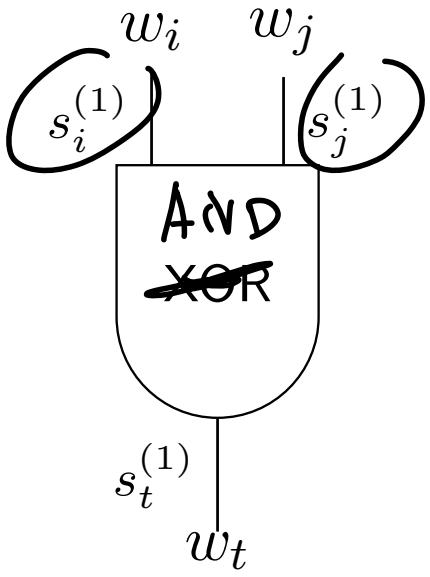
  - P_1 and P_2 then run a 1-out-of-4 OT to give P_2 the correct shares.

GMW Protocol: Evaluation of AND



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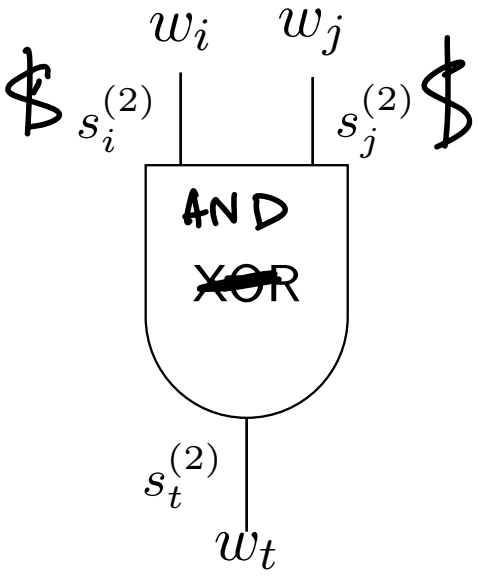

P_2's view

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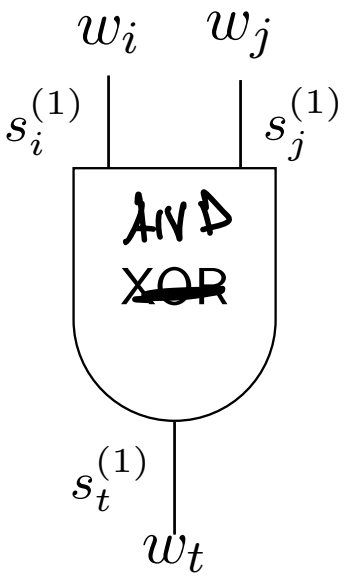

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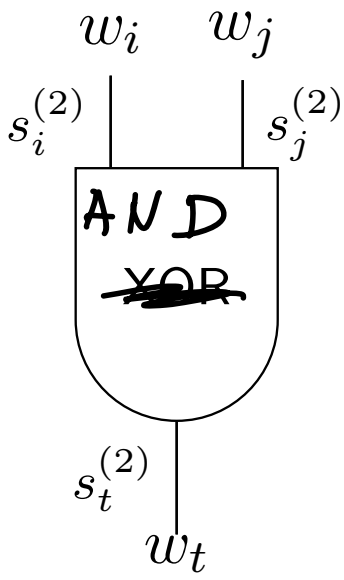

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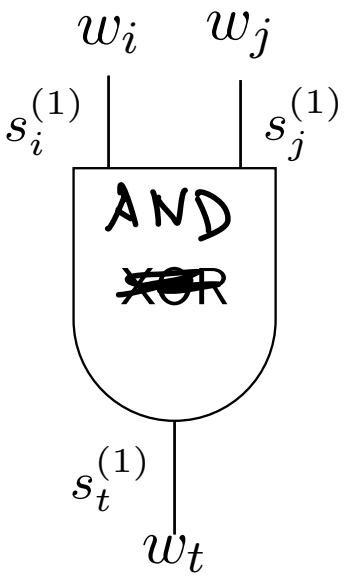
GMW Protocol: Evaluation of AND



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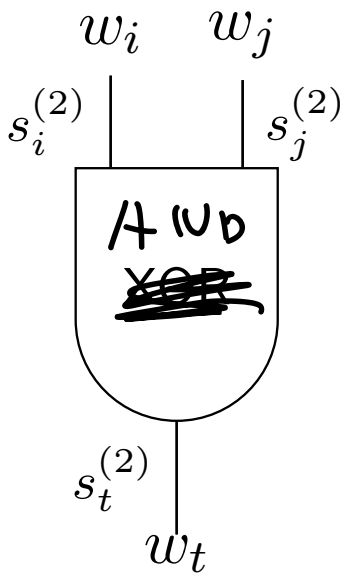

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- Need $s_t^{(1)}$ and $s_t^{(2)}$ such that $s_t^{(1)} \oplus s_t^{(2)} = w_i \wedge w_j$.
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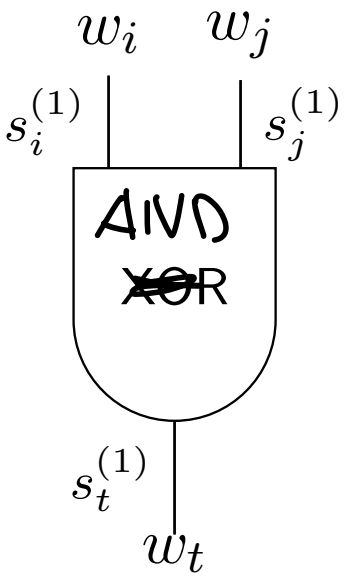

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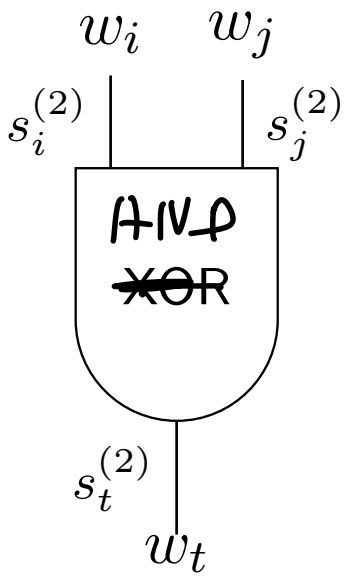

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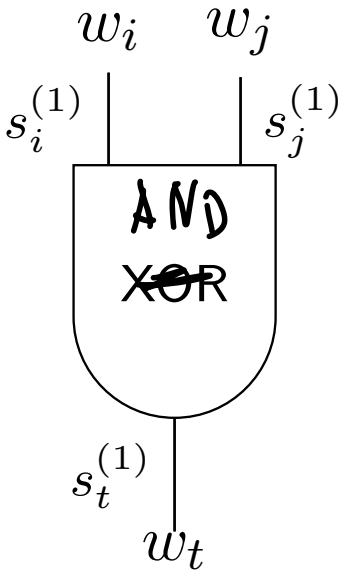

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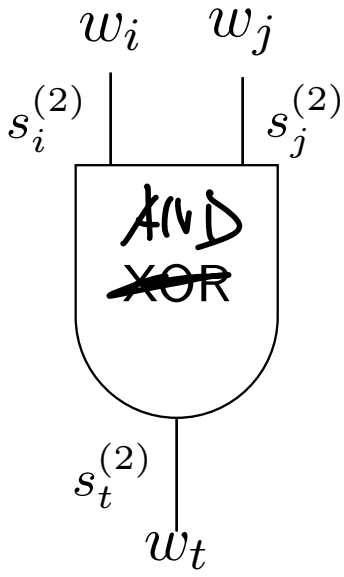

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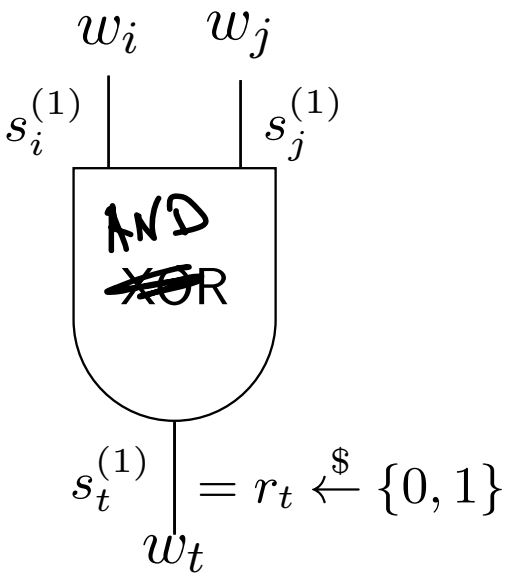

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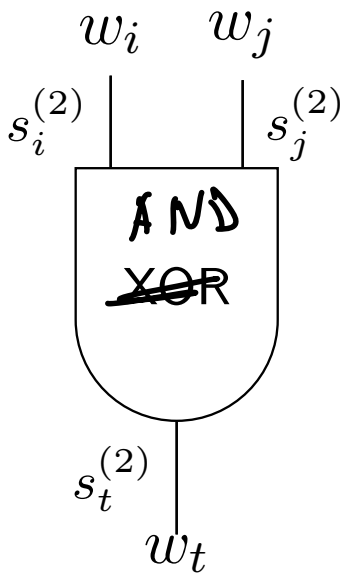

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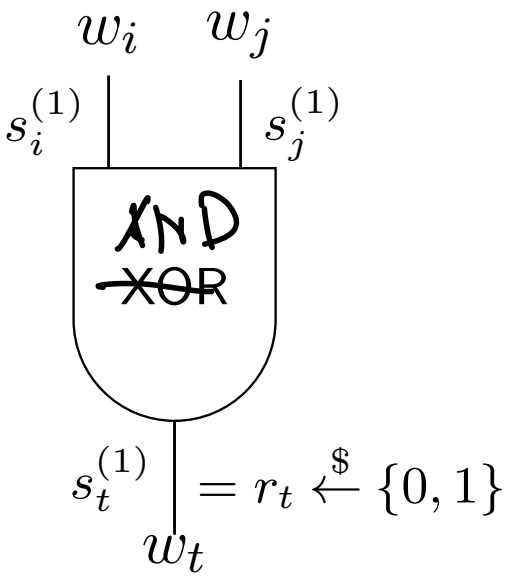

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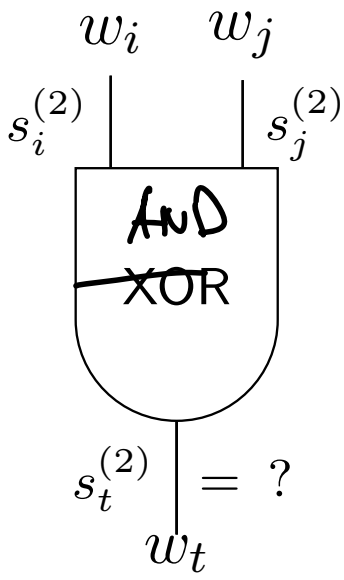

P_2's view

- Need $s_t^{(1)}$ and $s_t^{(2)}$ such that $s_t^{(1)} \oplus s_t^{(2)} = w_i \wedge w_j$.
  - Know that: $s_i^{(1)} \oplus s_i^{(2)} = w_i$ and $s_j^{(1)} \oplus s_j^{(2)} = w_j$.

- Idea: need to construct new secret sharing of w_t without knowing its value.
  - P_1 samples $r_t \xleftarrow{\$} \{0,1\}$, sets this as share for $s_t^{(1)}$.

GMW Protocol: Evaluation of AND

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P_1 Defines this function

- Let $S := S_{s_i^{(1)}, s_j^{(1)}}(s_i^{(2)}, s_j^{(2)}) = (s_i^{(1)} \oplus s_i^{(2)}) \wedge (s_j^{(1)} \oplus s_j^{(2)})$.

T — fixed for P_1

variables

$S : \{0,1\}^2 \to \{0,1\}$

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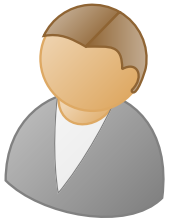

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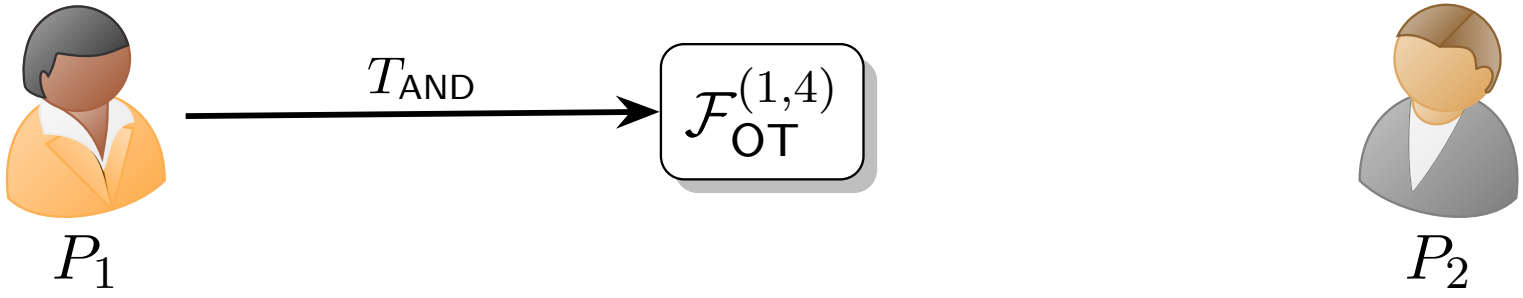

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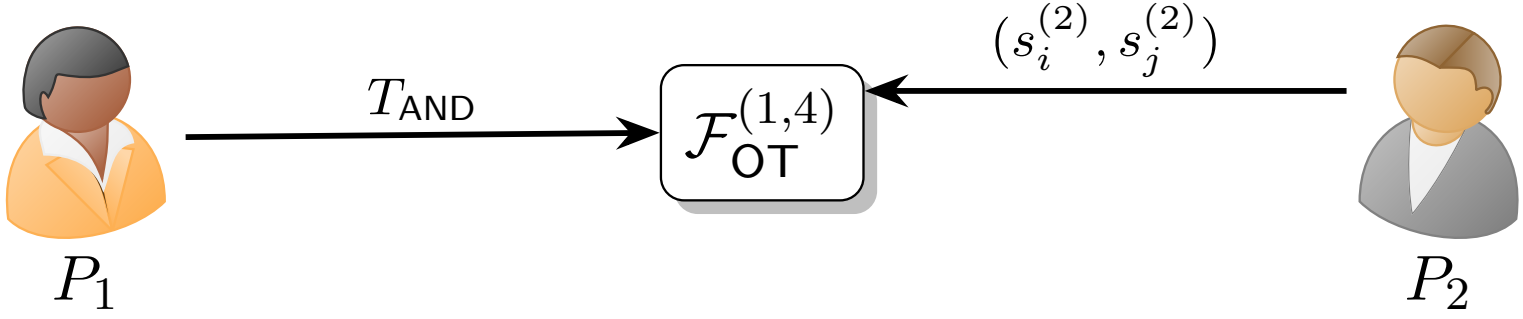

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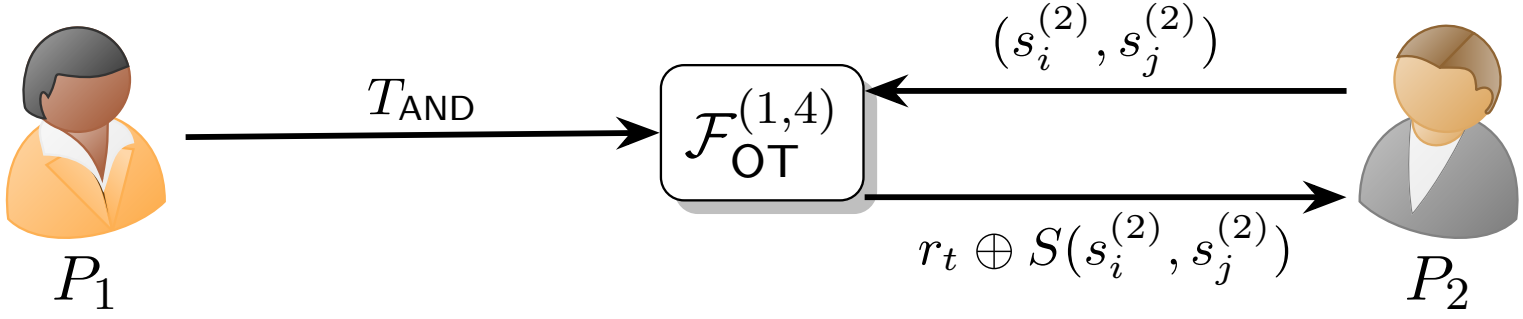

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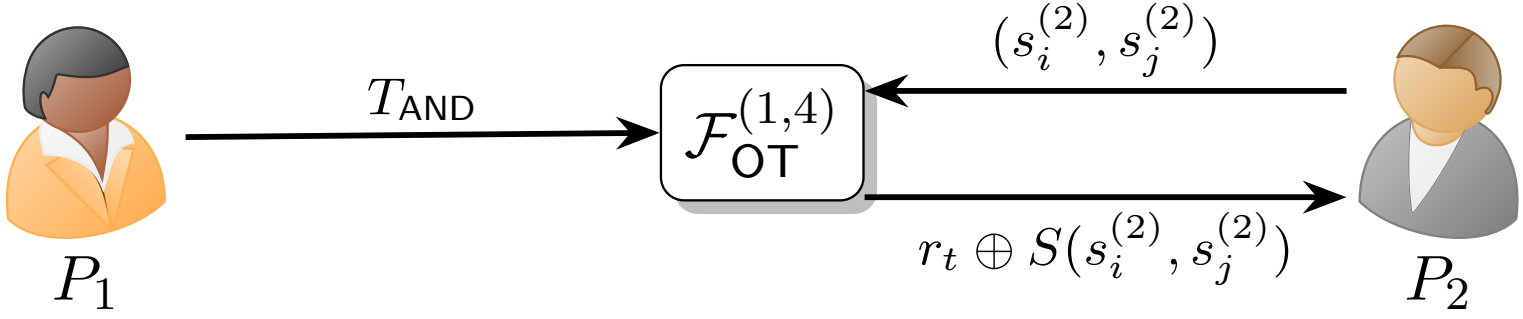

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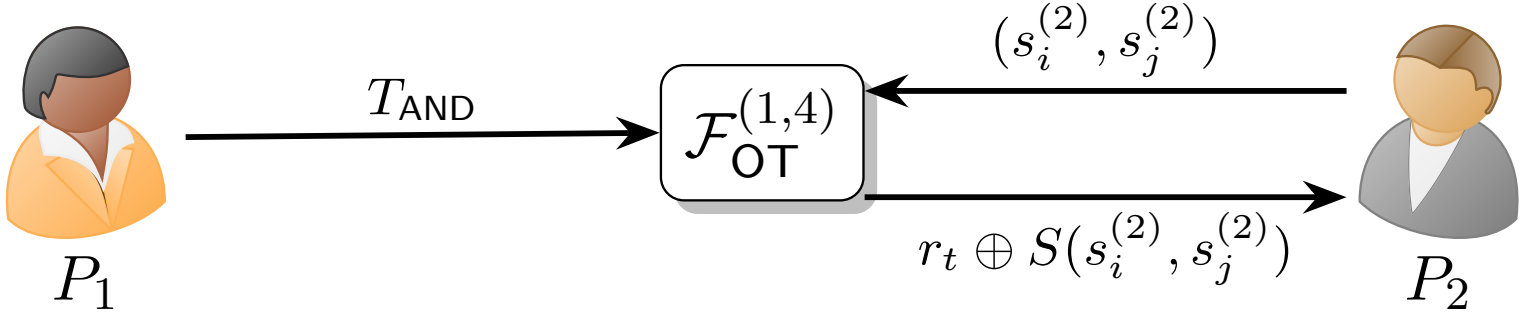

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- Parties have learned $\mathcal{F}(\mathbf{x}, \mathbf{y})$!

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- Evaluating AND gates is more complicated; we'll come back to this.

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P_2

P_3

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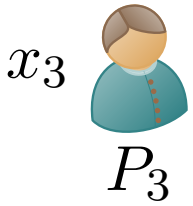

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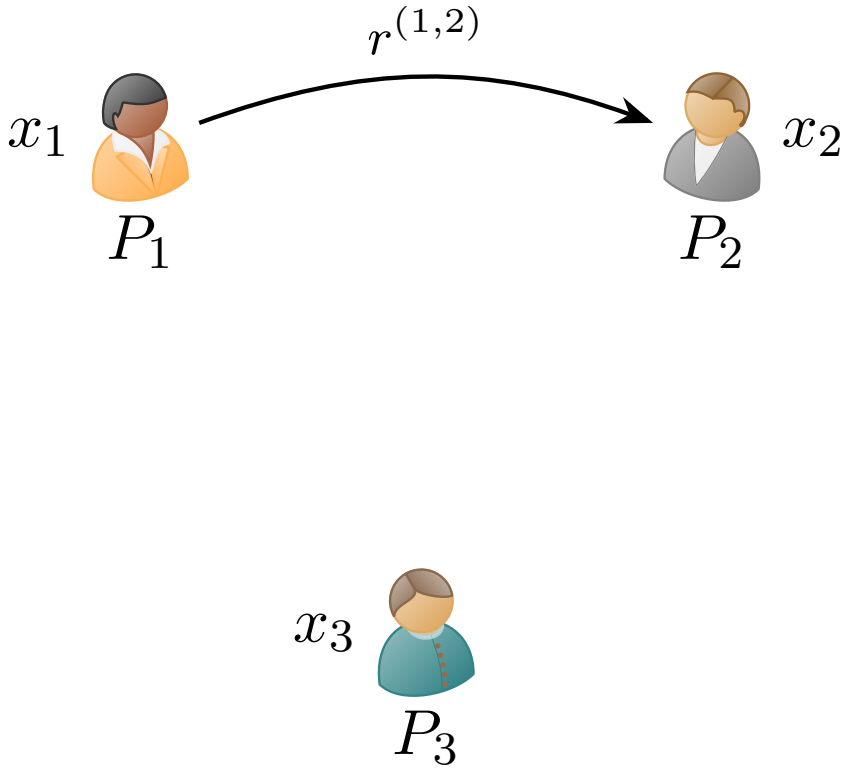

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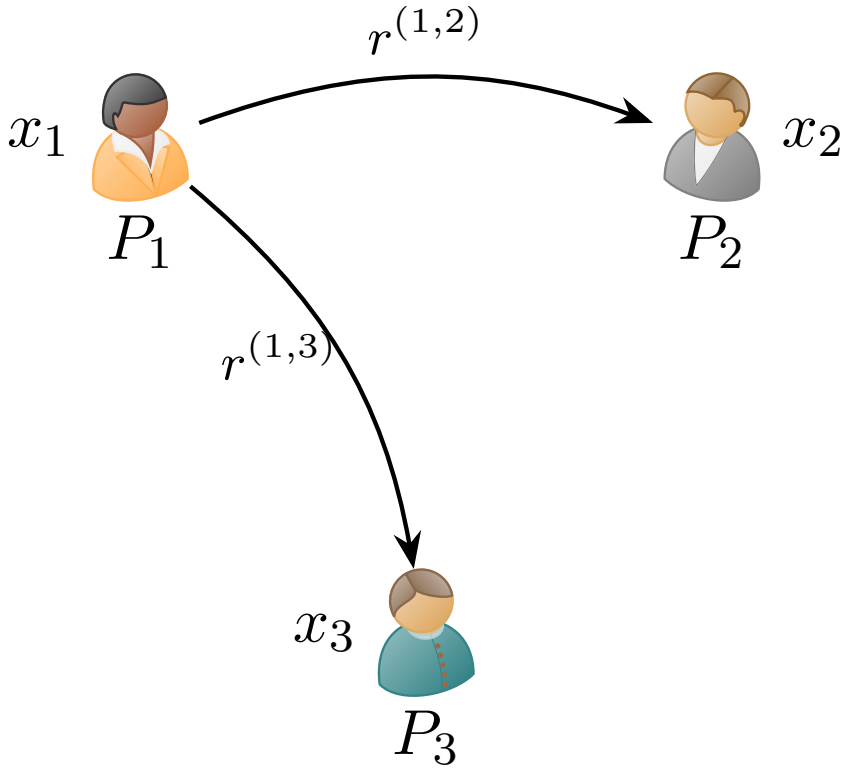

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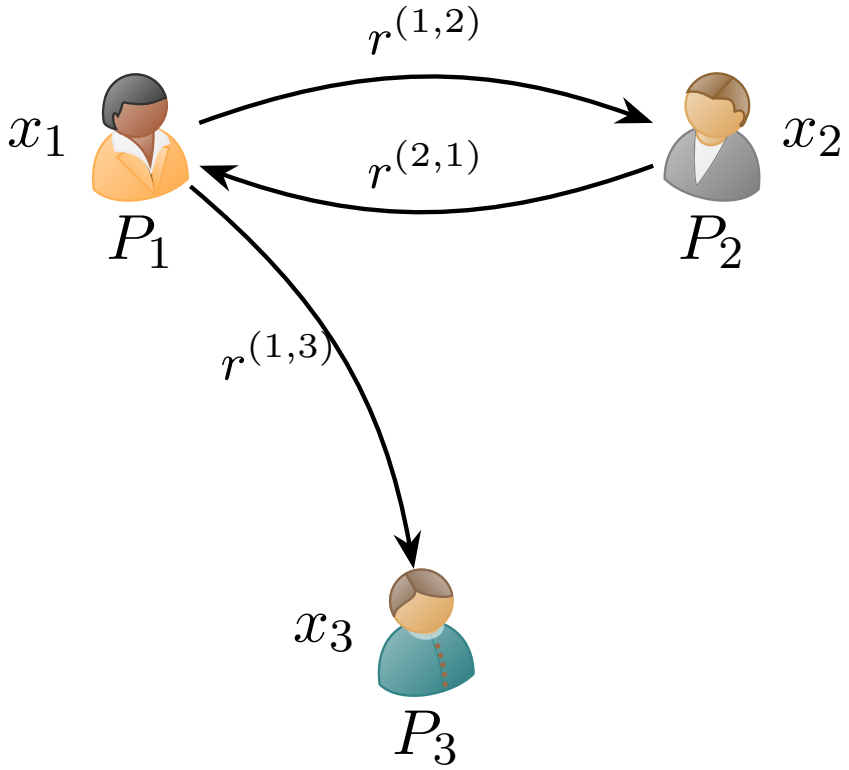

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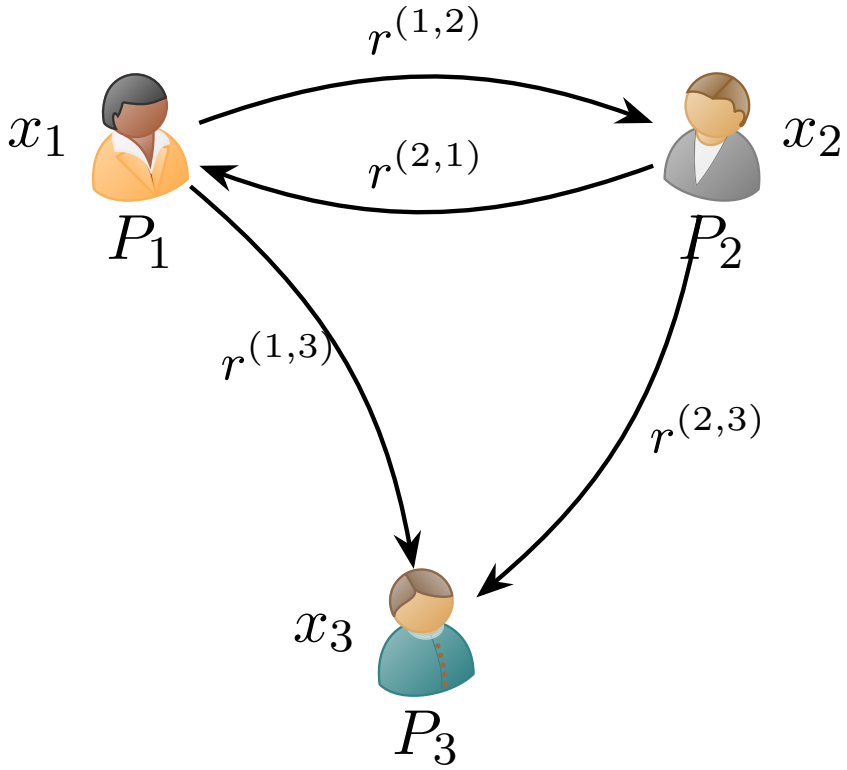

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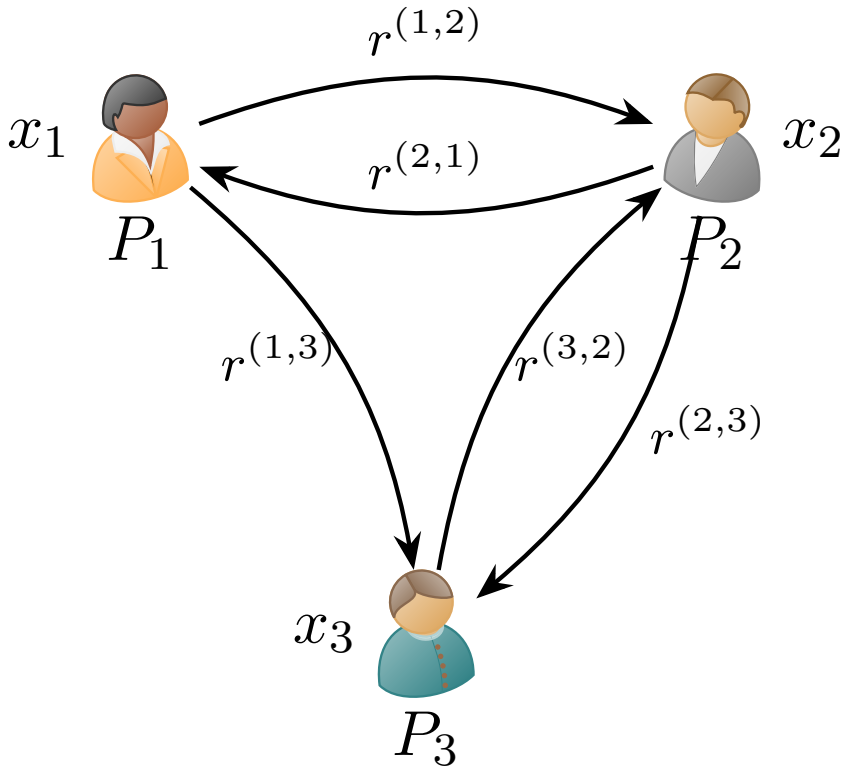

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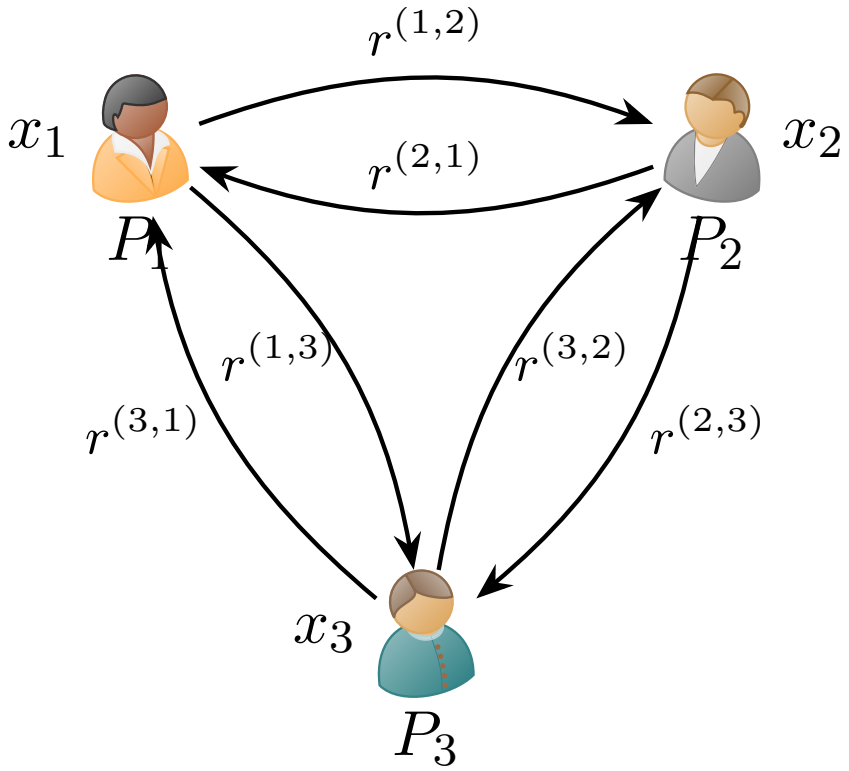

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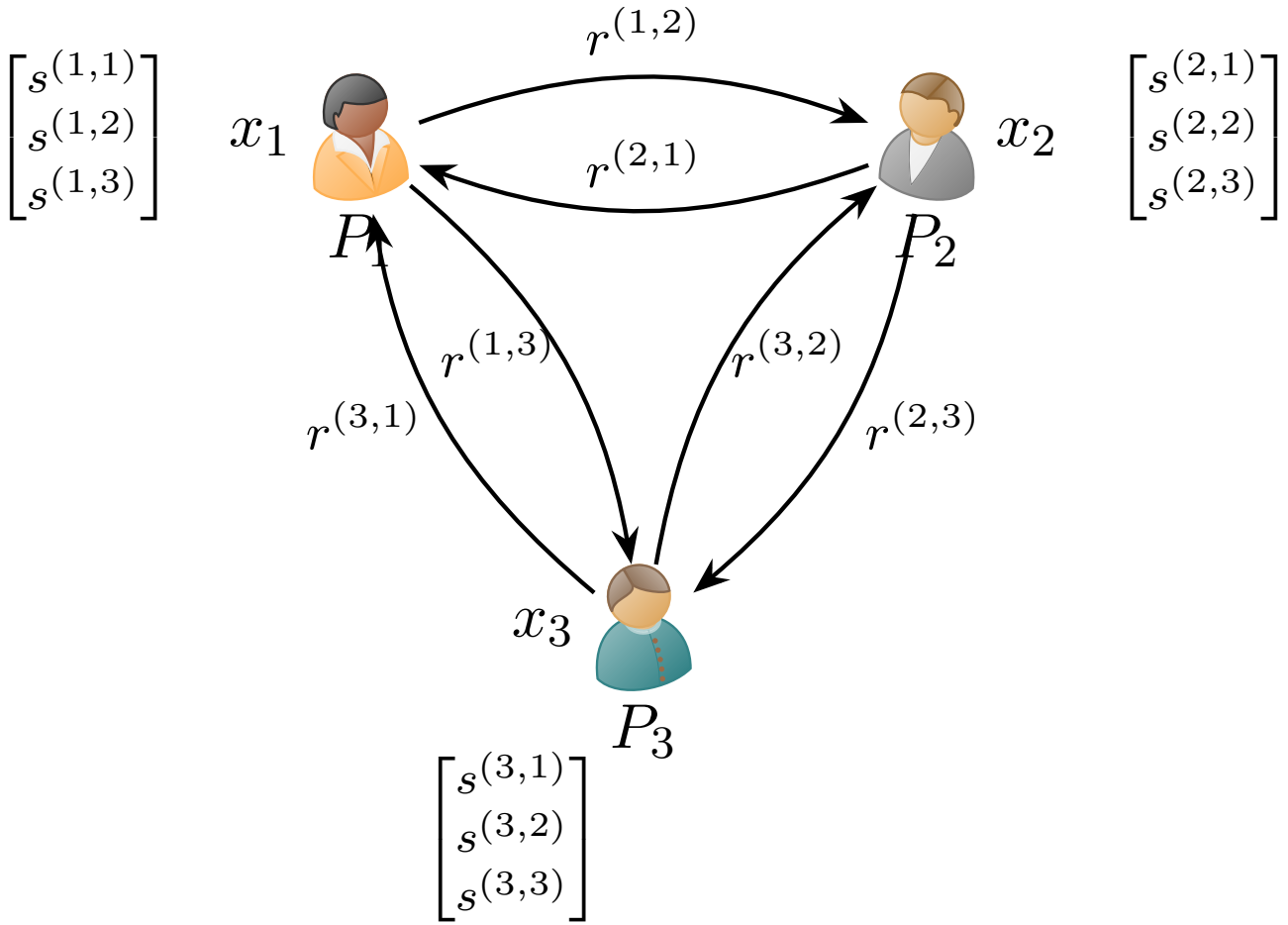

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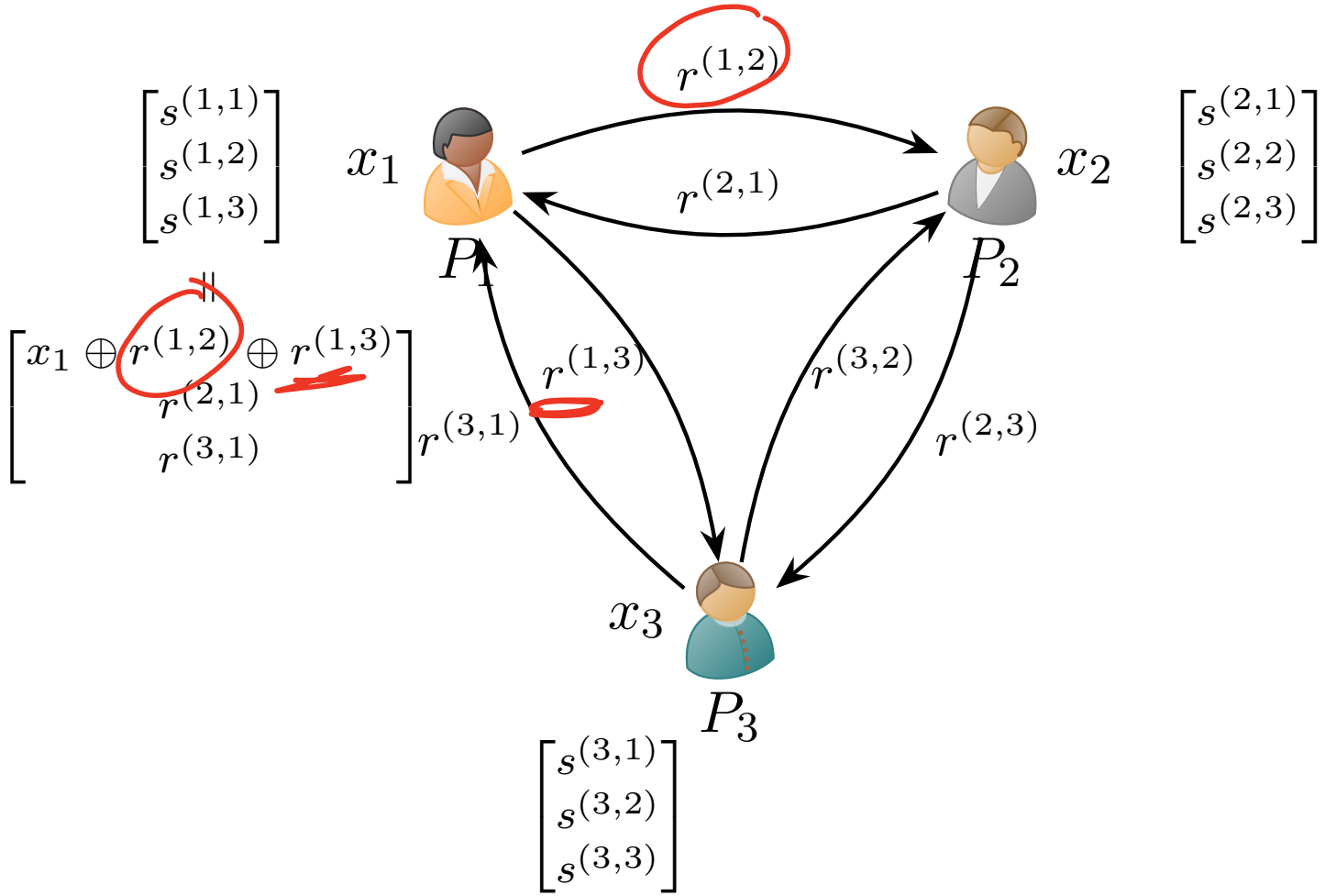

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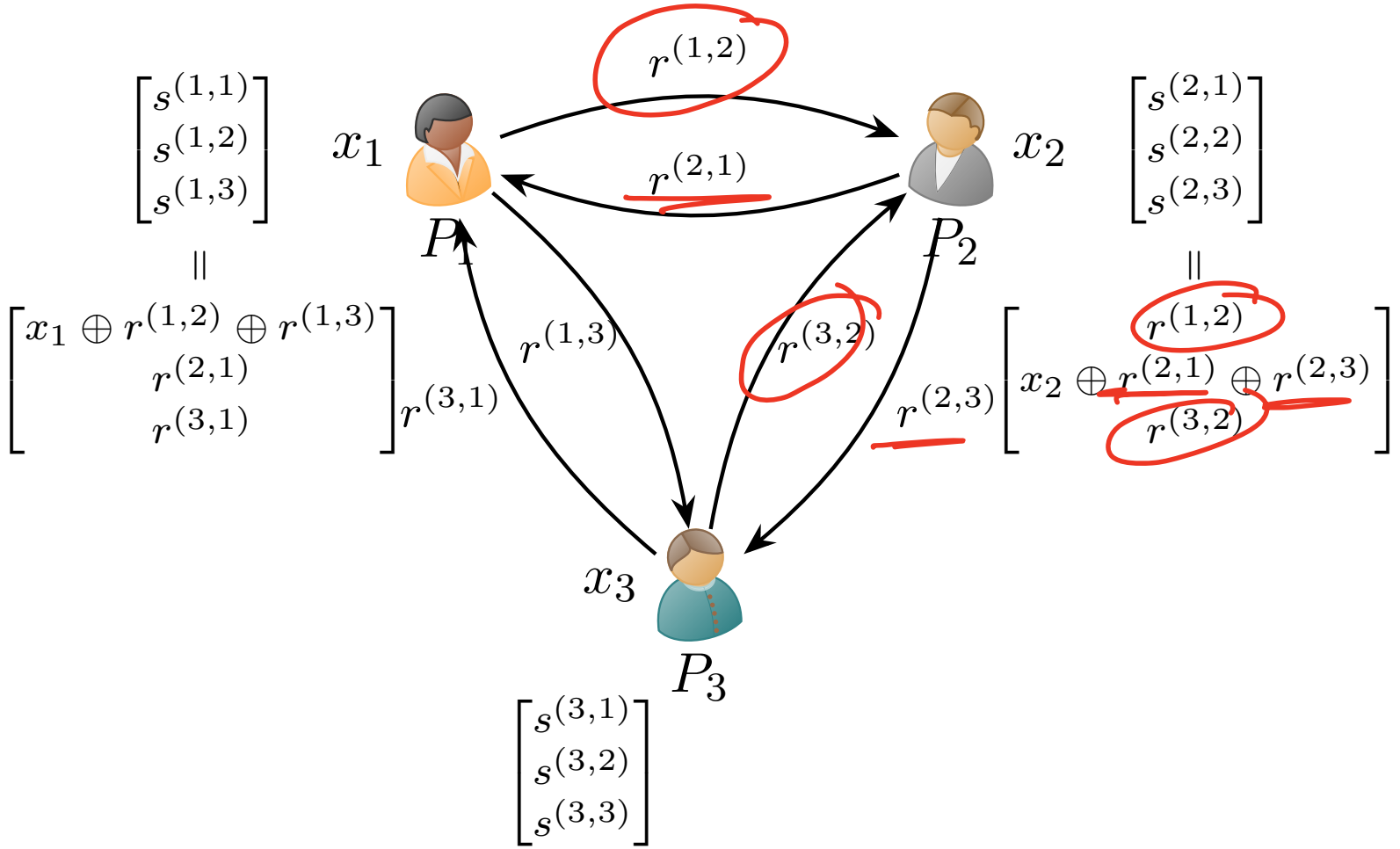

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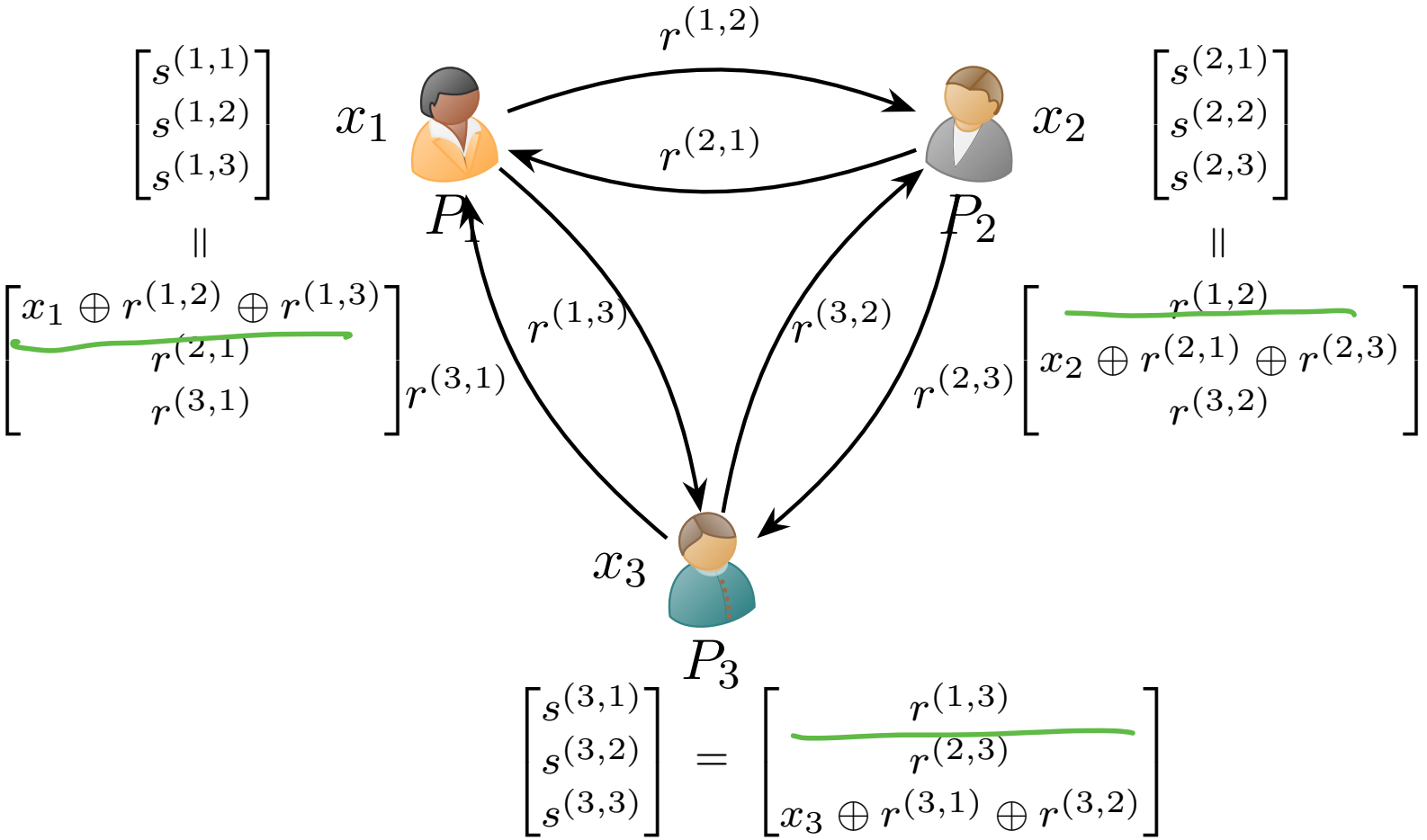
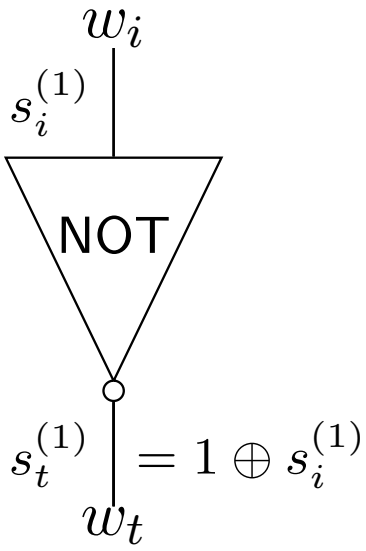
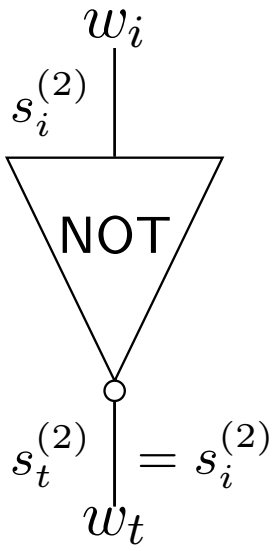


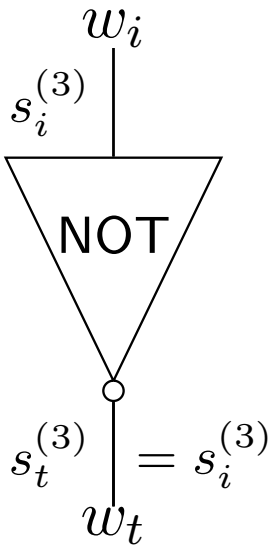
GMW Protocol: 3-Player NOT Gate



P_1's view



P_2's view

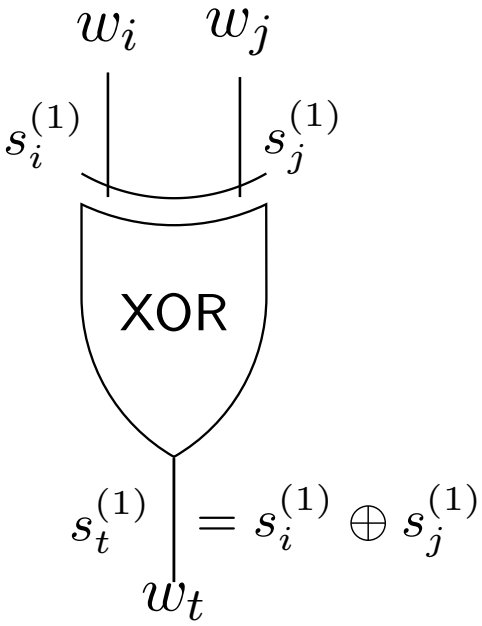


P_3's view

$$s_i^{(1)} \oplus s_i^{(2)} \oplus s_i^{(3)} = w_i$$

$$s_t^{(1)} \oplus s_t^{(2)} \oplus s_t^{(3)} = 1 \oplus (s_i^{(1)} \oplus s_i^{(2)} \oplus s_i^{(3)}) = 1 \oplus w_i$$

GMW Protocol: 3-Player XOR Gate



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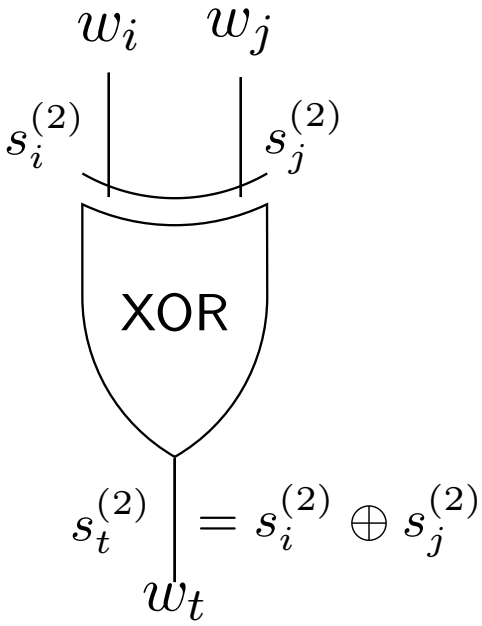

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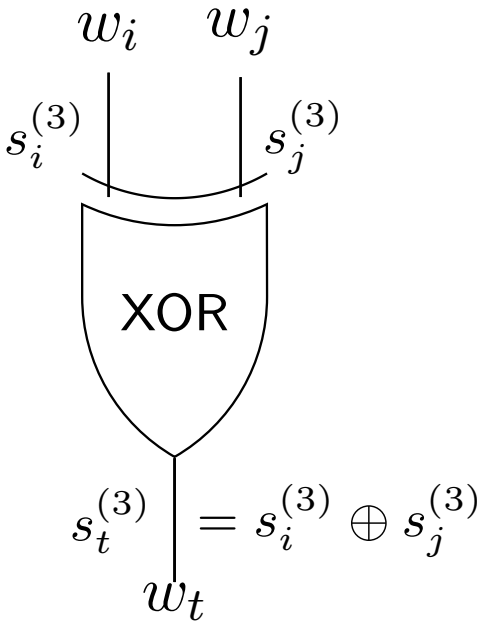

P_3's view

$s_i^{(1)} \oplus s_i^{(2)} \oplus s_i^{(3)} = w_i \quad s_j^{(1)} \oplus s_j^{(2)} \oplus s_j^{(3)} = w_j$

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$$= \left(\bigoplus_{k=1}^{m} s_i^{(k)} \wedge s_j^{(k)}\right) \oplus \left(\bigoplus_{\substack{k,k' \in [m] \\ k \neq k'}} s_i^{(k)} \wedge s_j^{(k')}\right).$$

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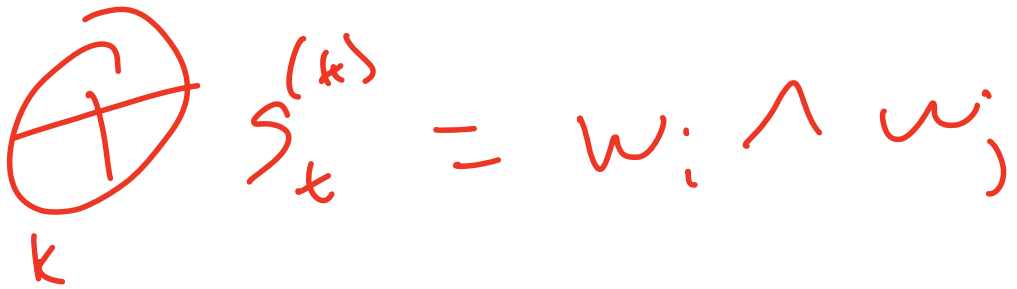
- Then for all $k \in [m]$, P_k sets $s_t^{(k)} = (s_i^{(k)} \wedge s_j^{(k)}) \oplus \bigoplus_{k' \neq k} s_i^{(k)} \wedge s_j^{(k')}$.



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  - Parties exchange $n + s_{\mathsf{out}} + s_{\mathsf{AND}} \cdot O(m^2 \cdot \mathsf{bits}(\mathcal{F}_{\mathsf{OT}}^{(1,4)}))$ bits.

Next Time: MPC Wrap-up