CS 594 – Advanced Crypto (Spring 2026)

Alex Block

Lecture 21

April 08, 2026

Memory-hard Functions

Motivation: Password Storage

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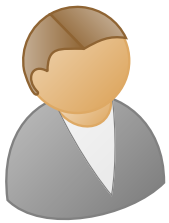

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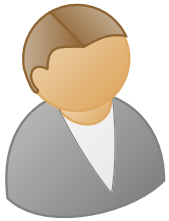

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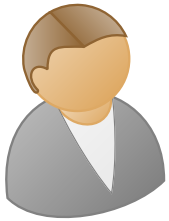

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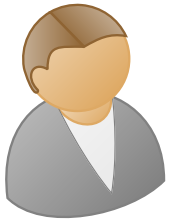

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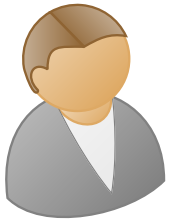

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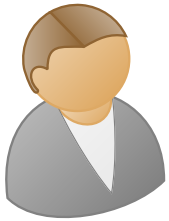

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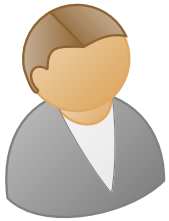

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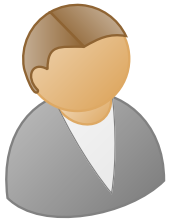

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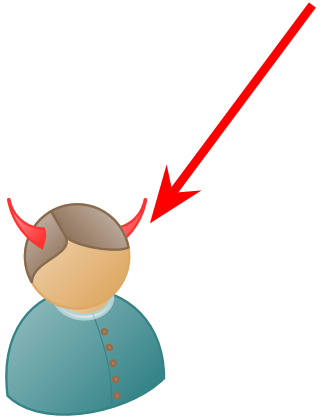

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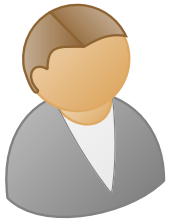

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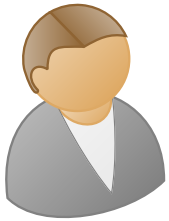

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$x \xrightarrow{H(x)} \underbrace{\circ \xrightarrow{H(\cdot)} \circ \xrightarrow{H(\cdot)} \cdots \circ}_{32 \text{ times}}$

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https://haveibeenpwned.com/

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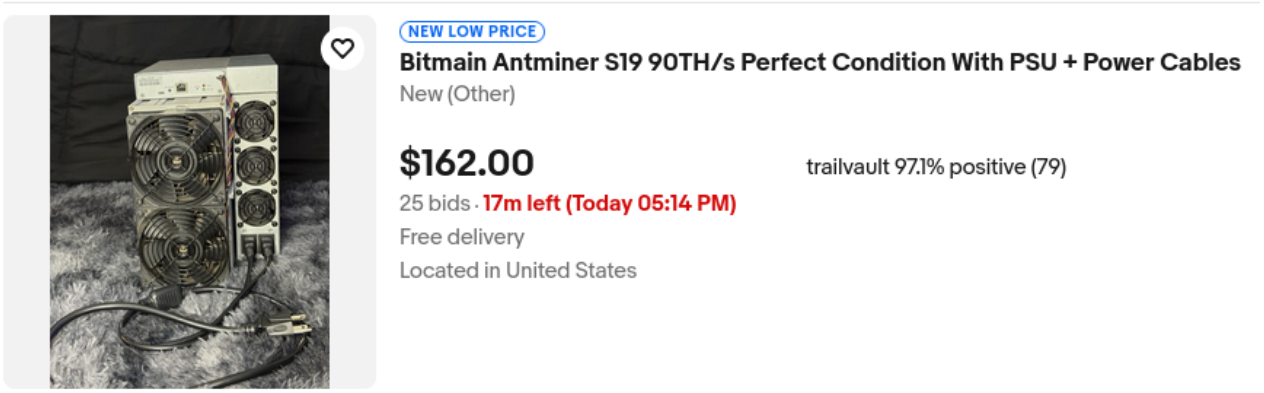

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Goal

Force attacker to use *large amounts of memory* for duration of computation, even on customized hardware.

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↳ "Fancy" hash-chaining

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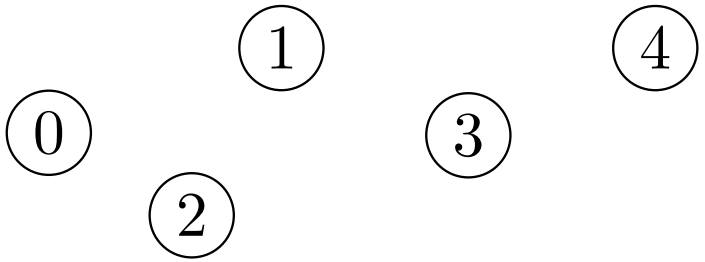

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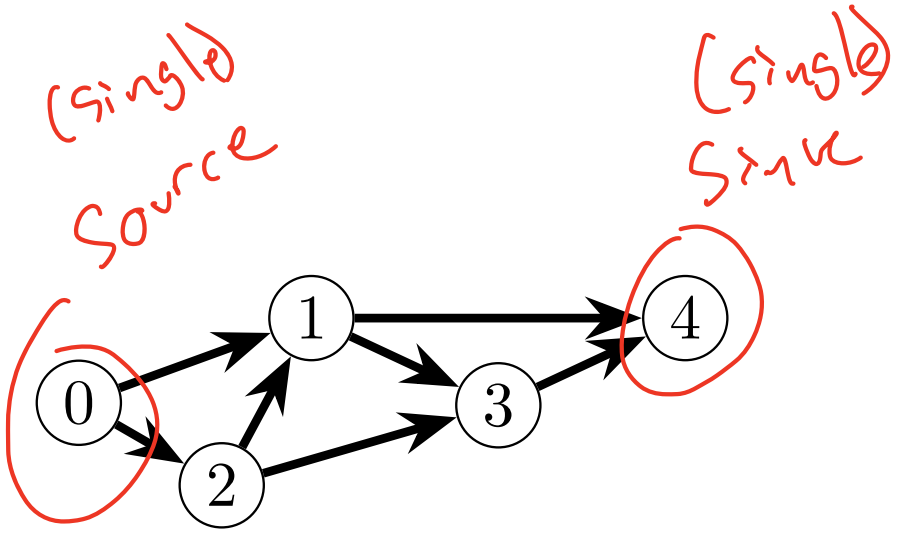

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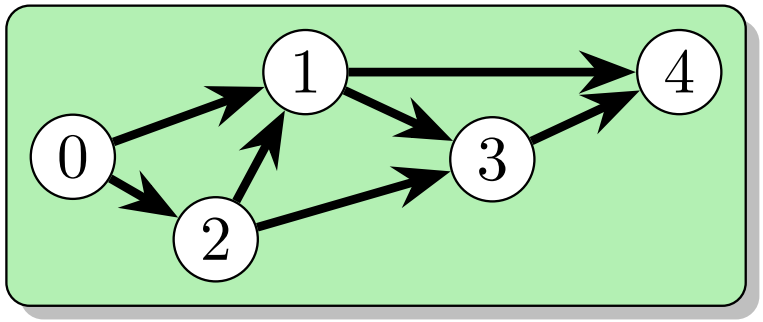

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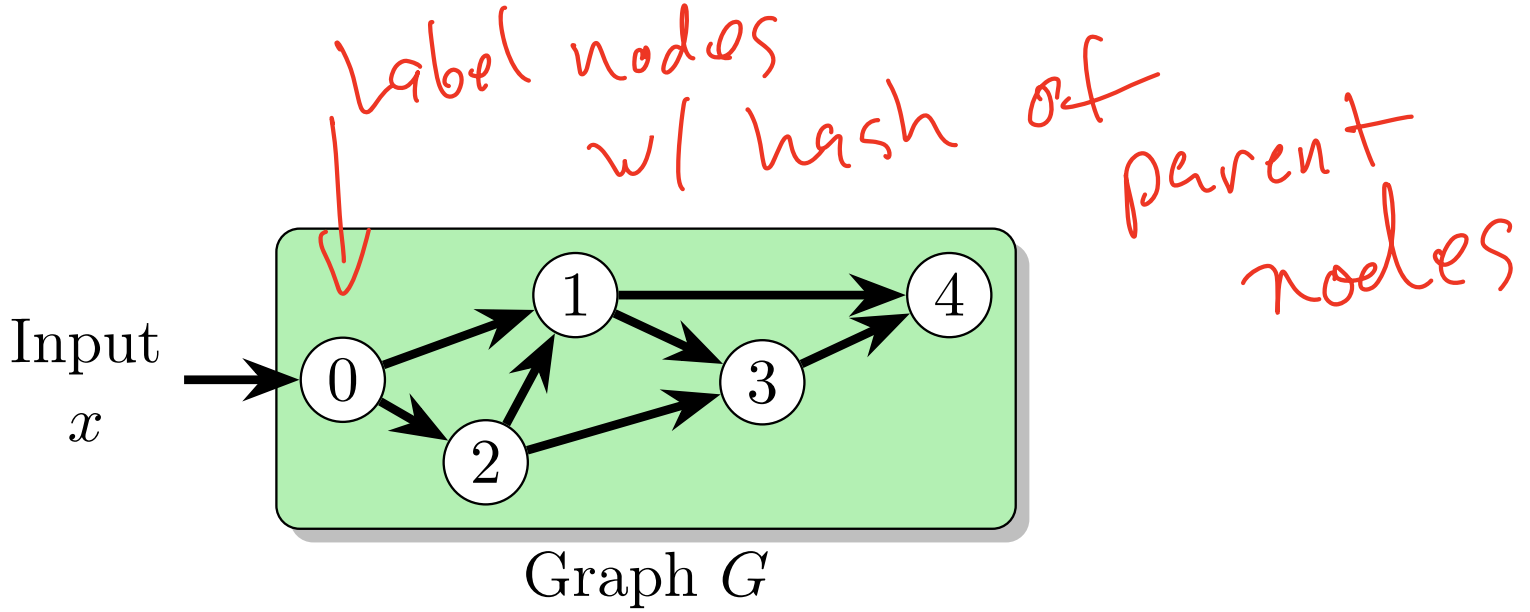

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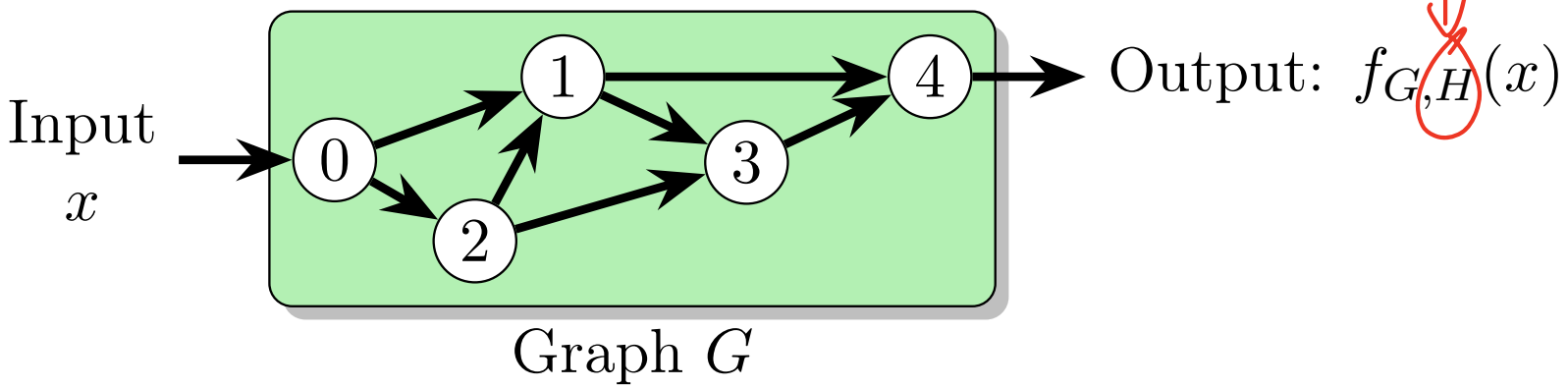

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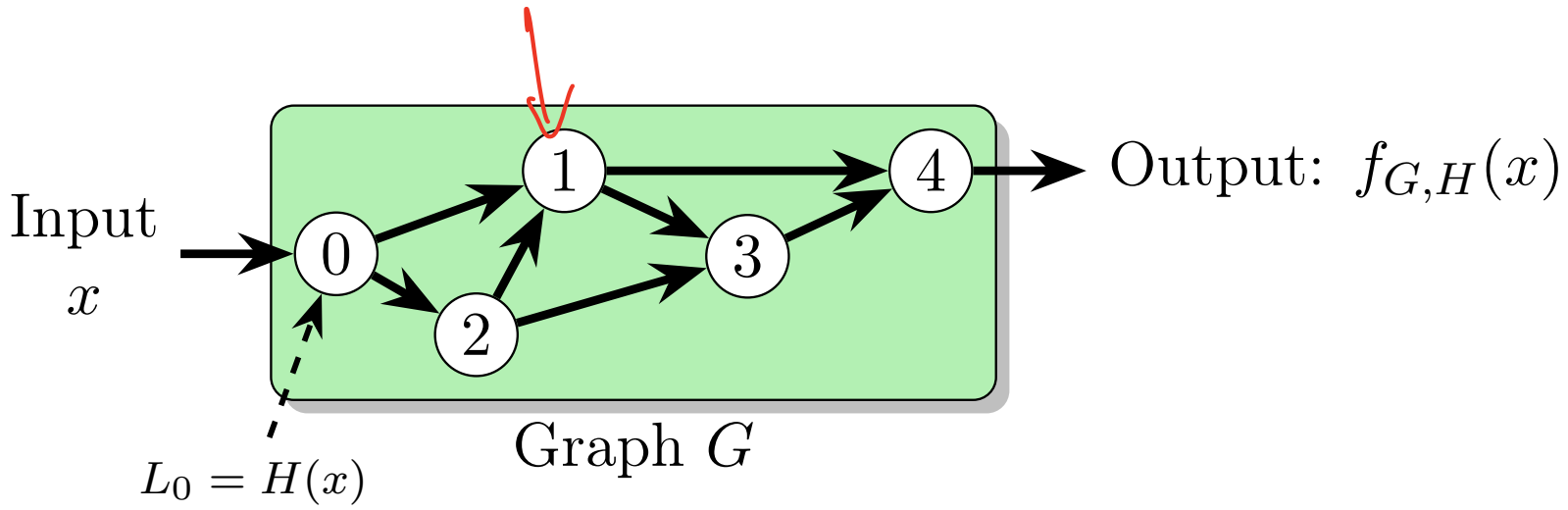

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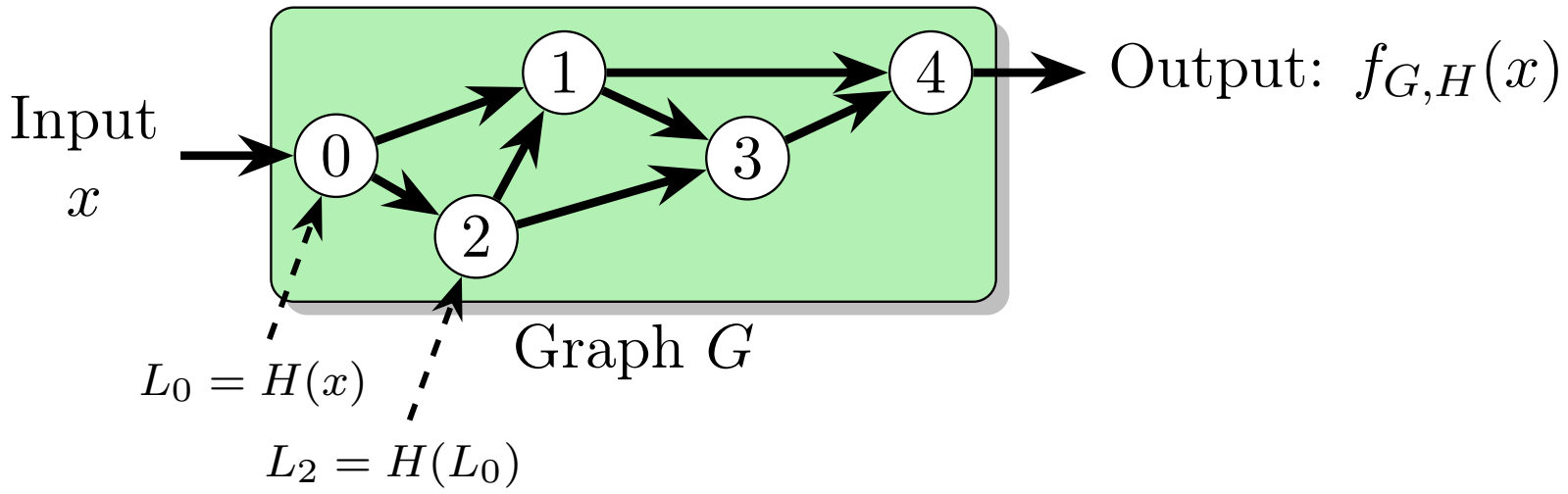

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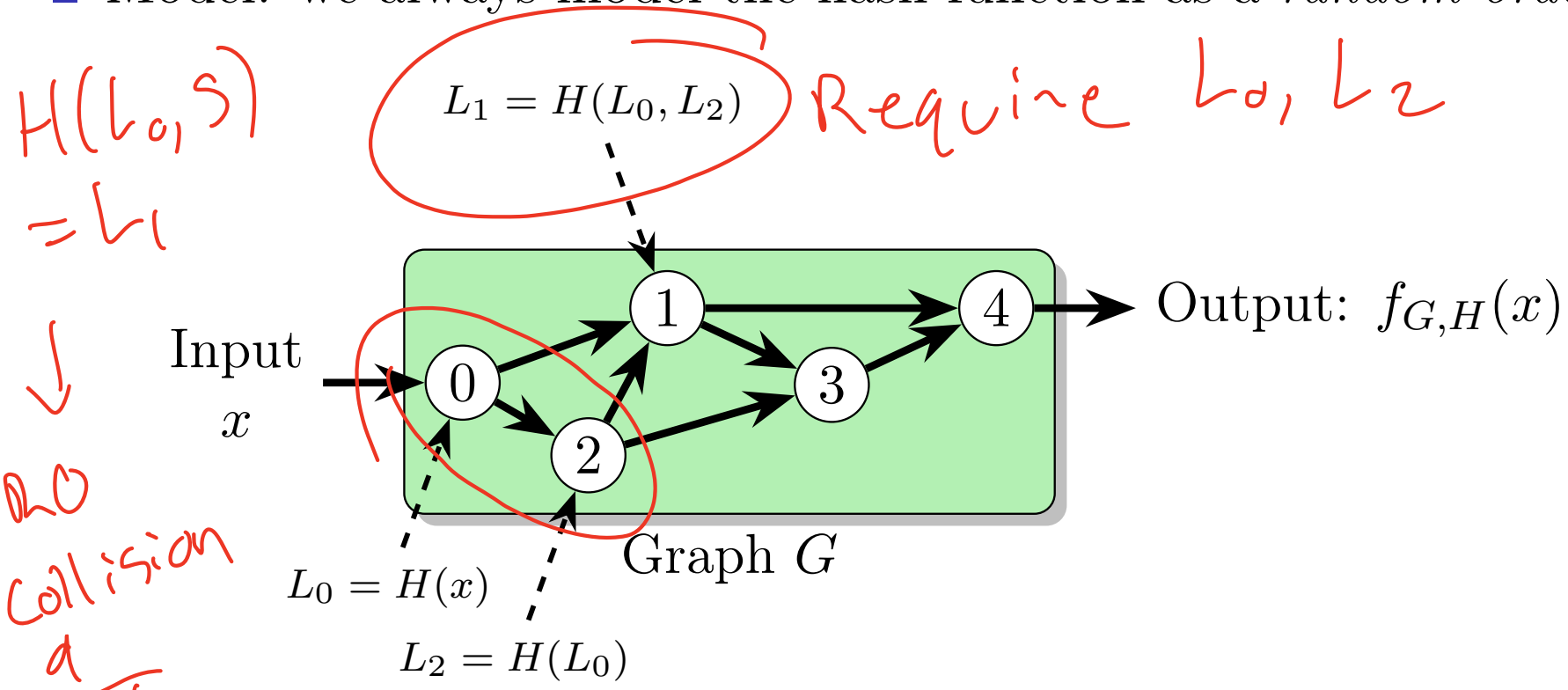

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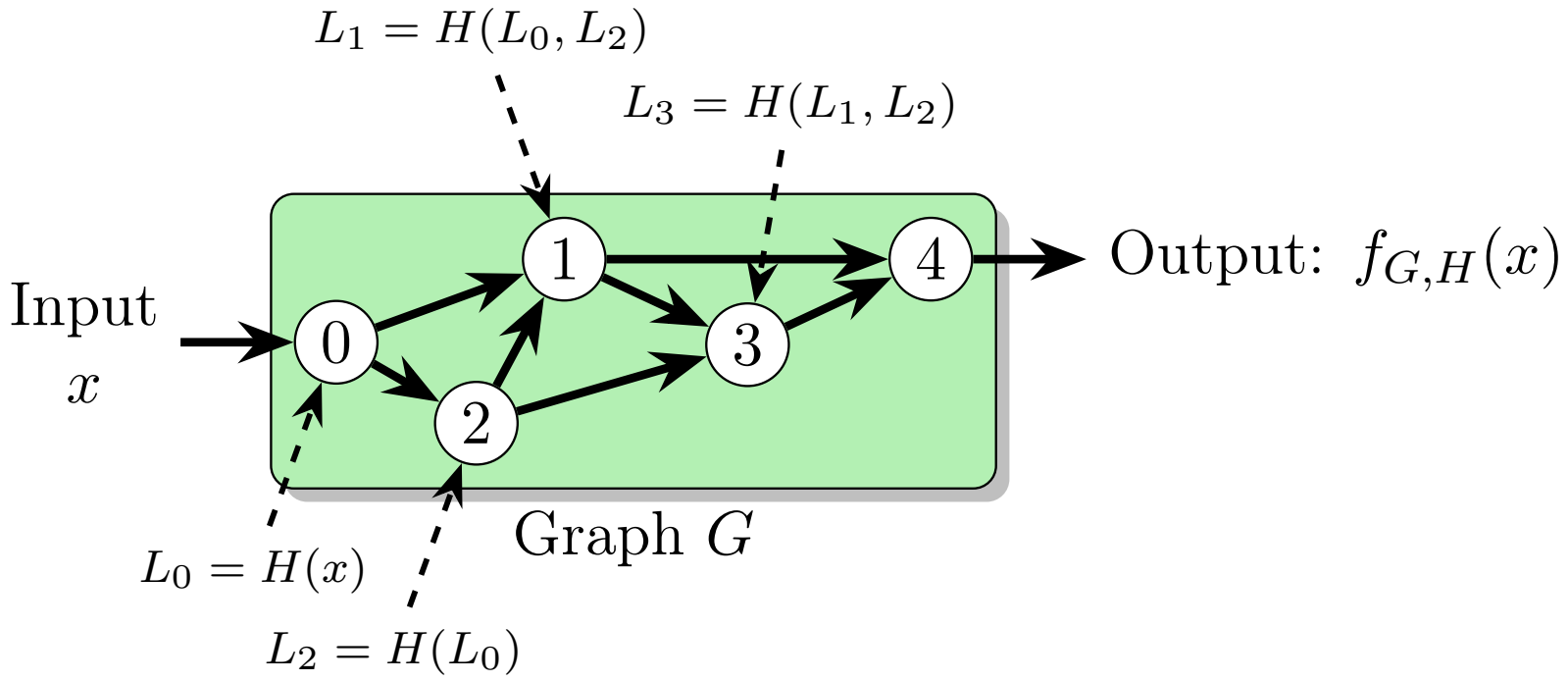

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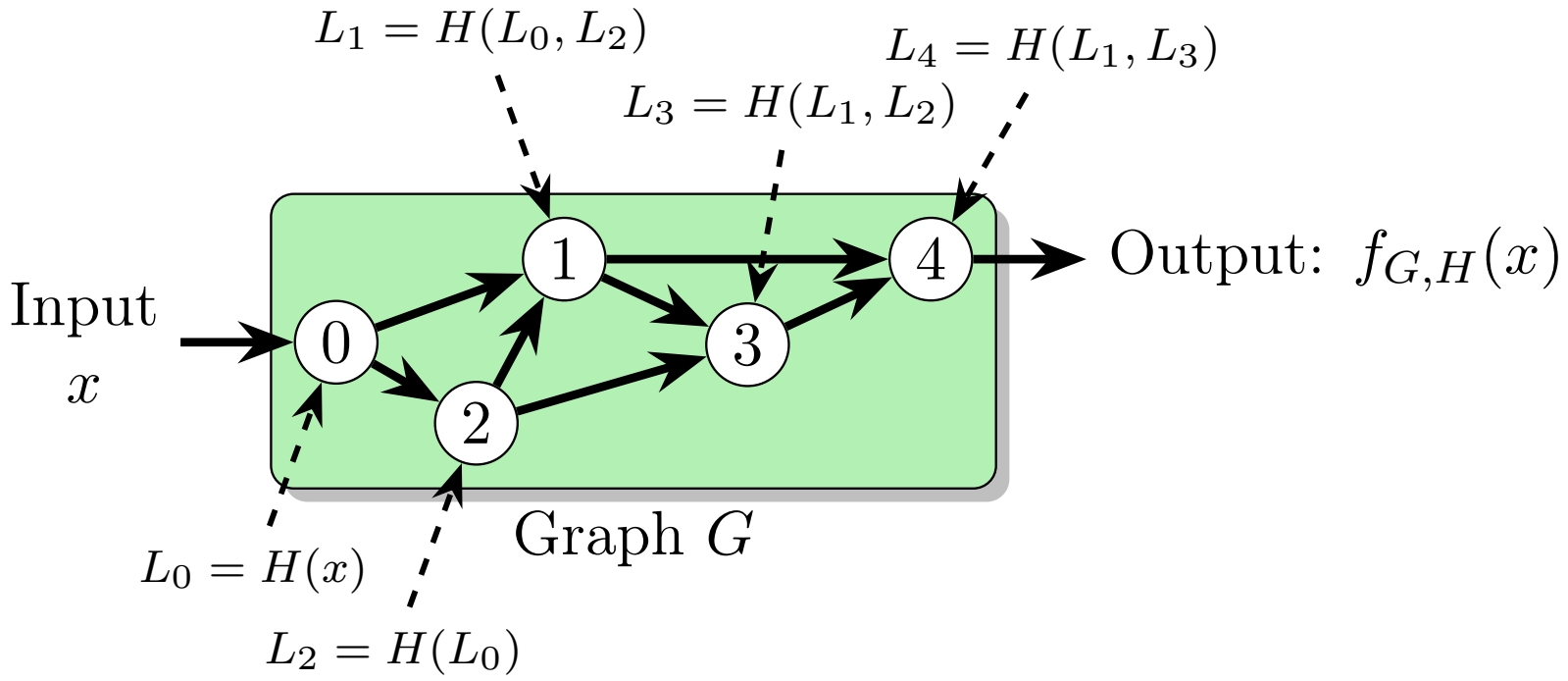

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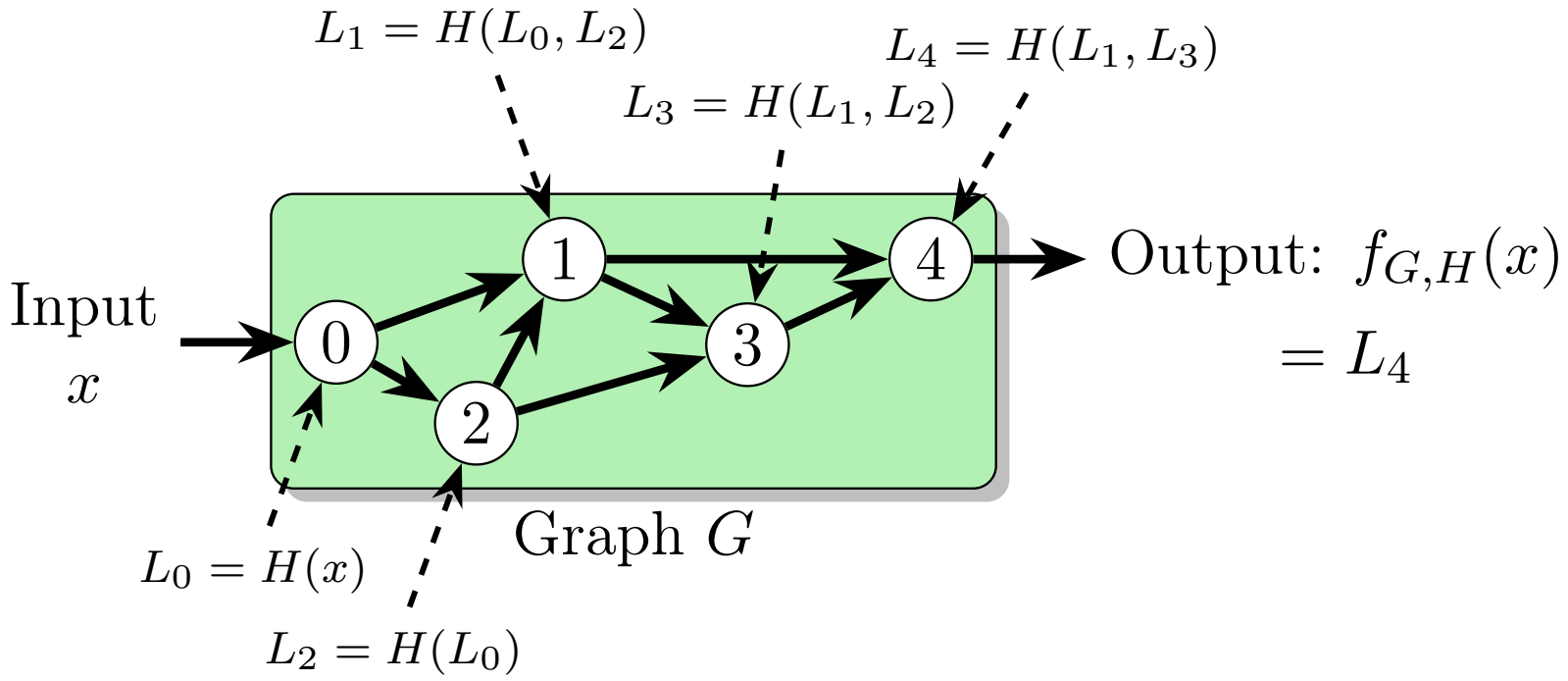

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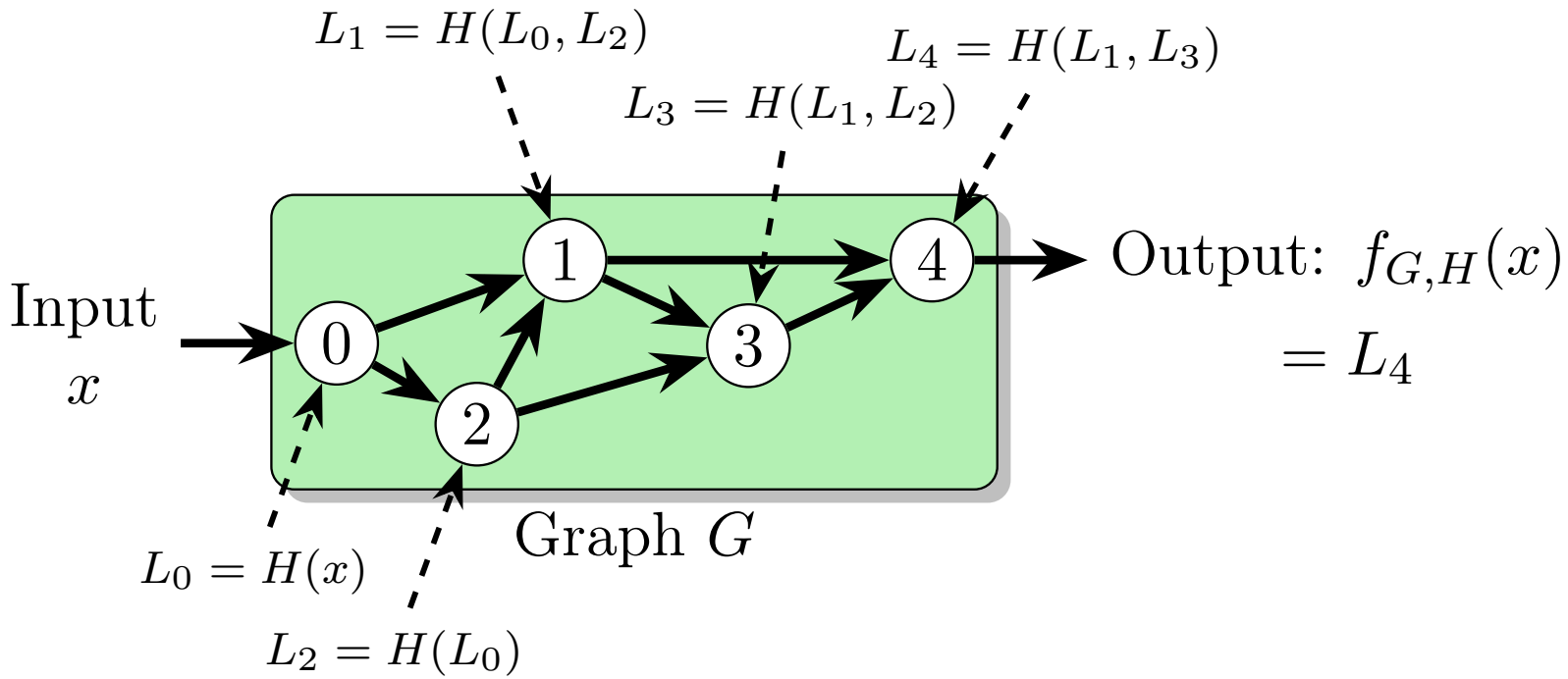

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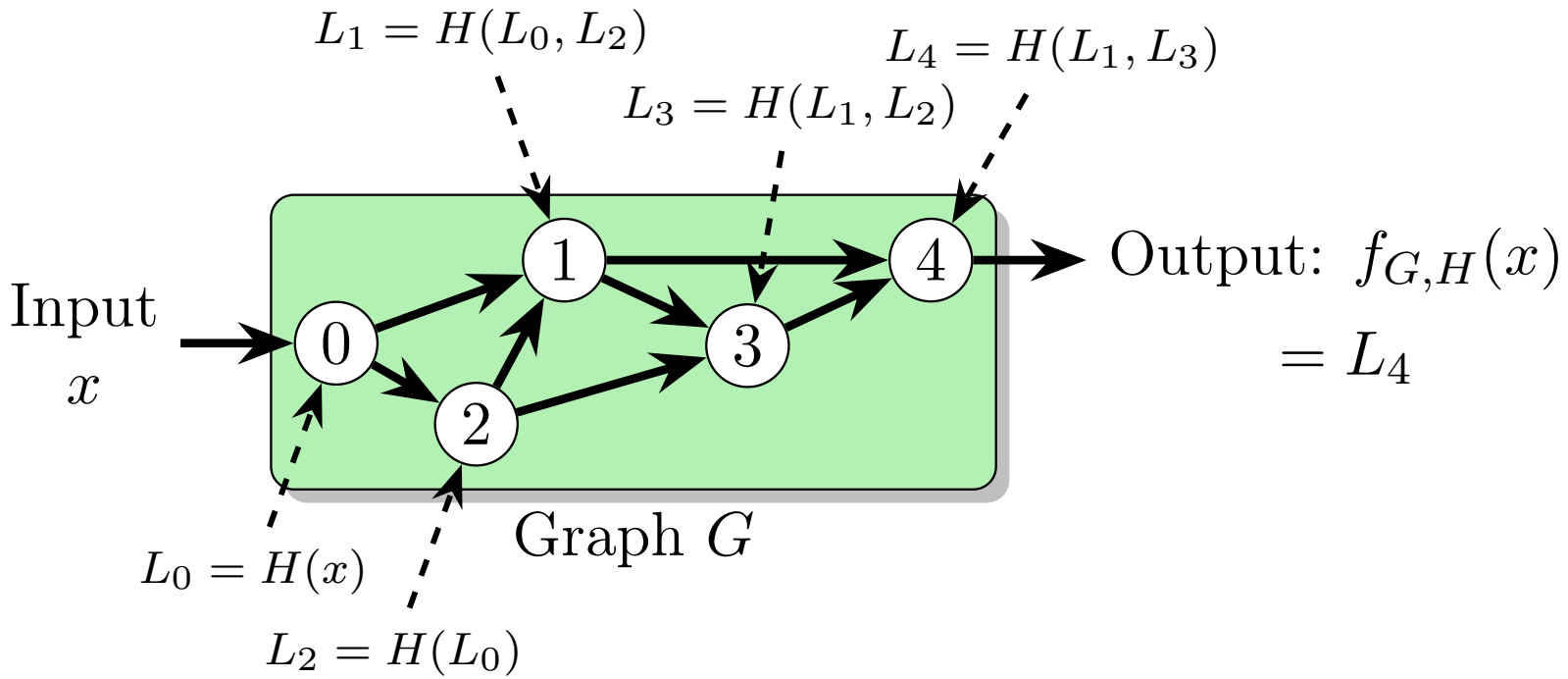

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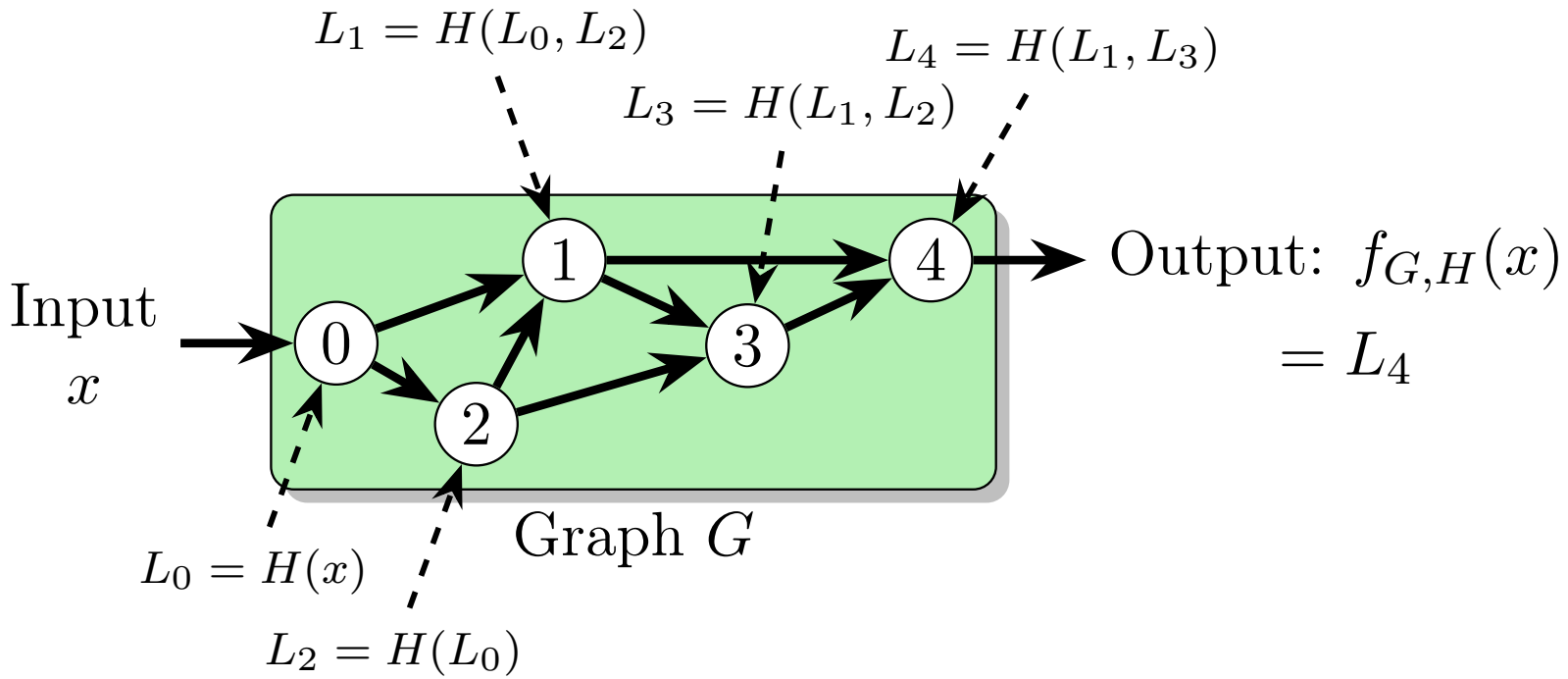

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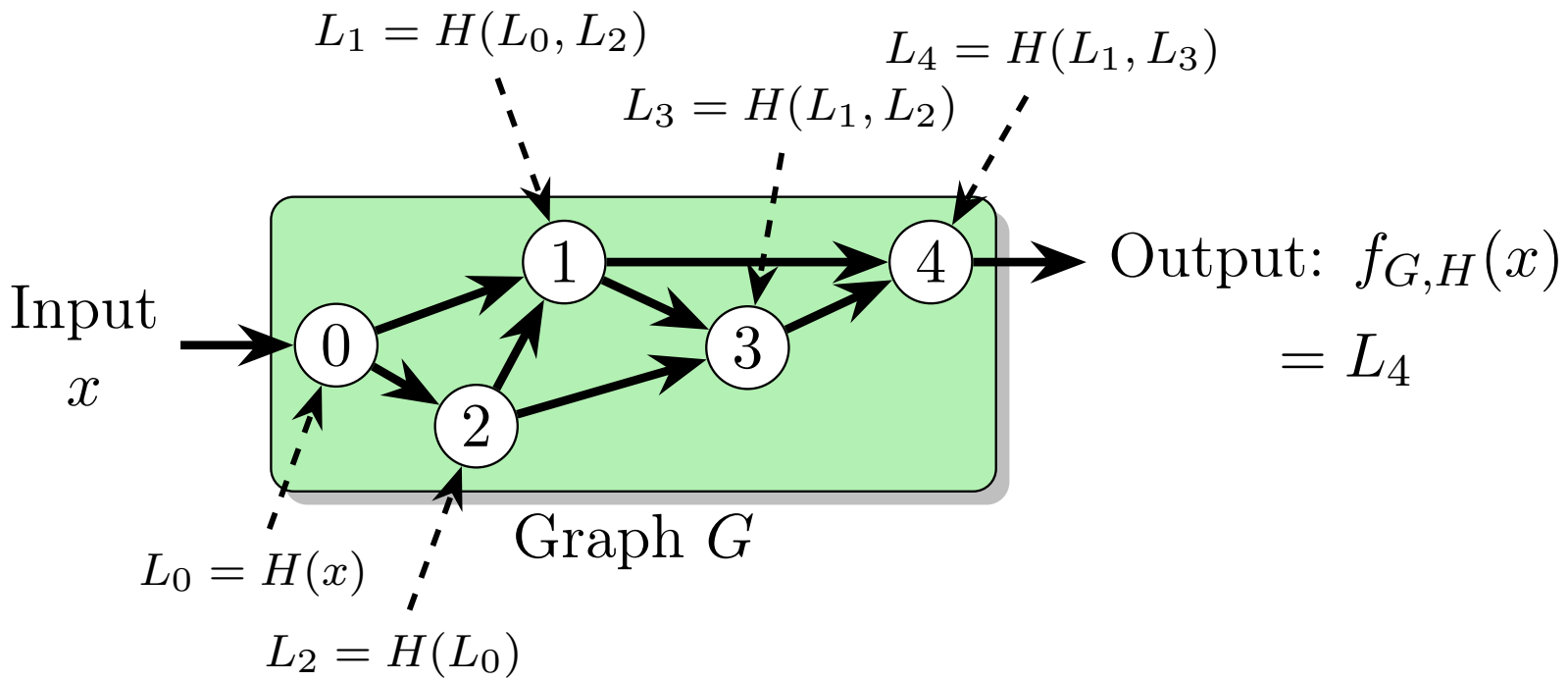

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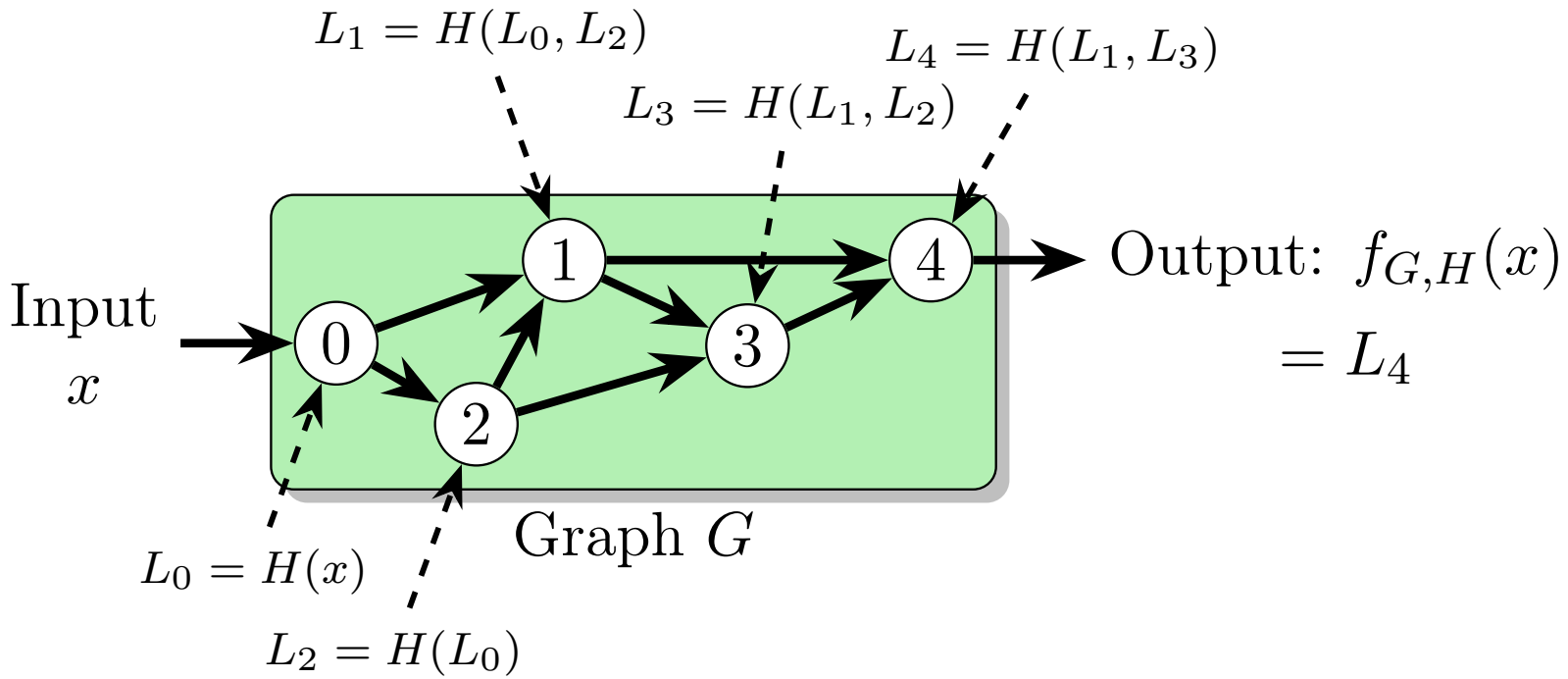

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  - $|V| = N = 2^n$ nodes/vertices (not in this example).

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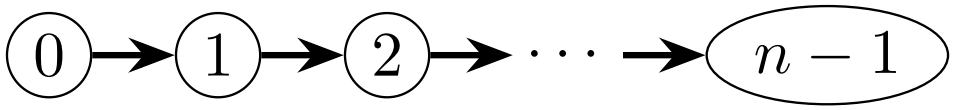

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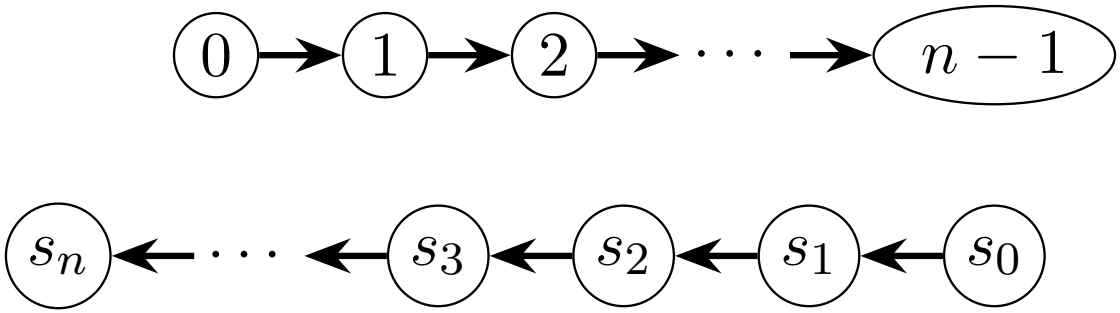

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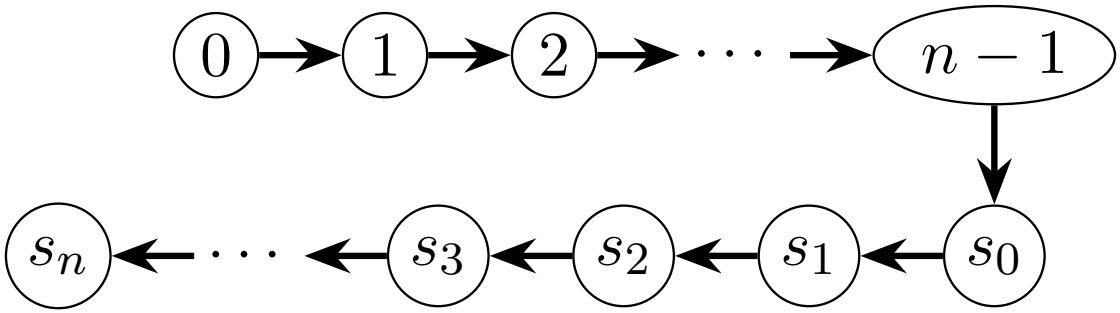

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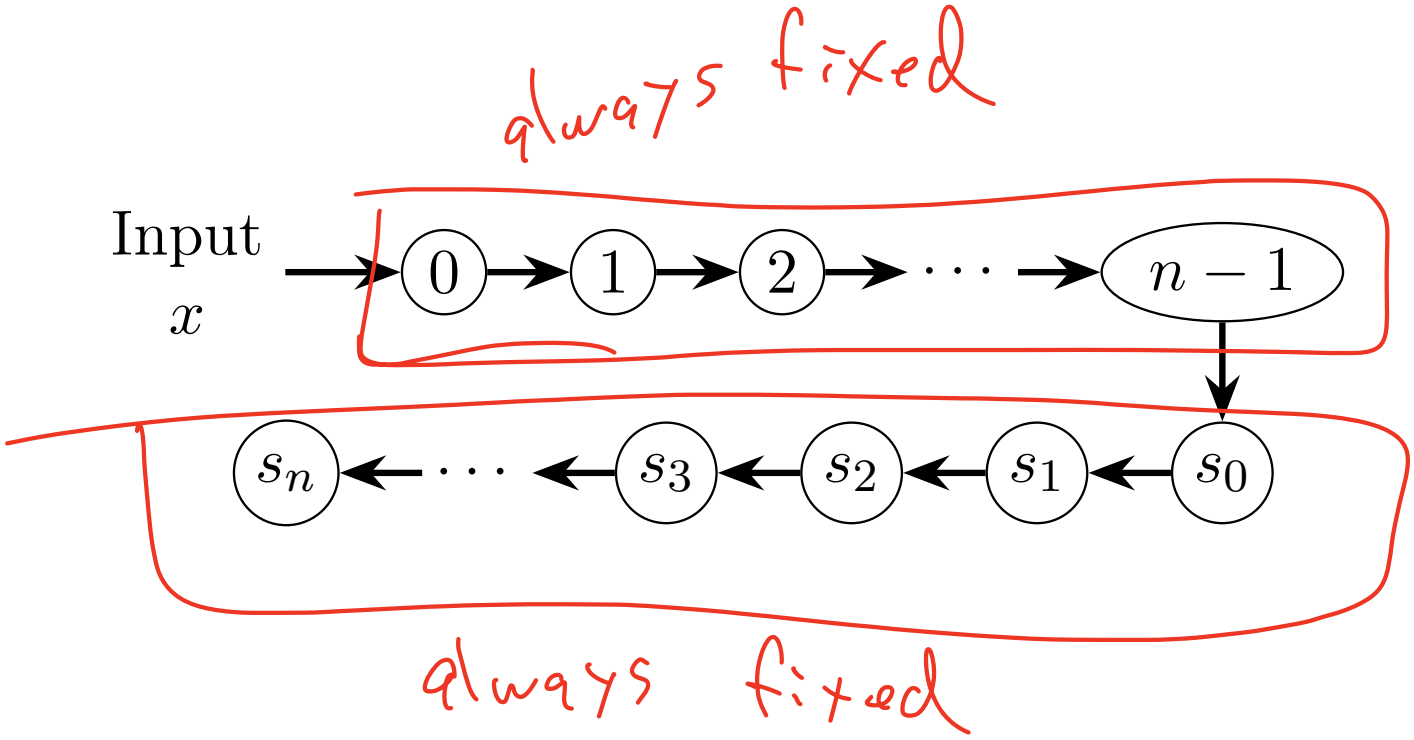

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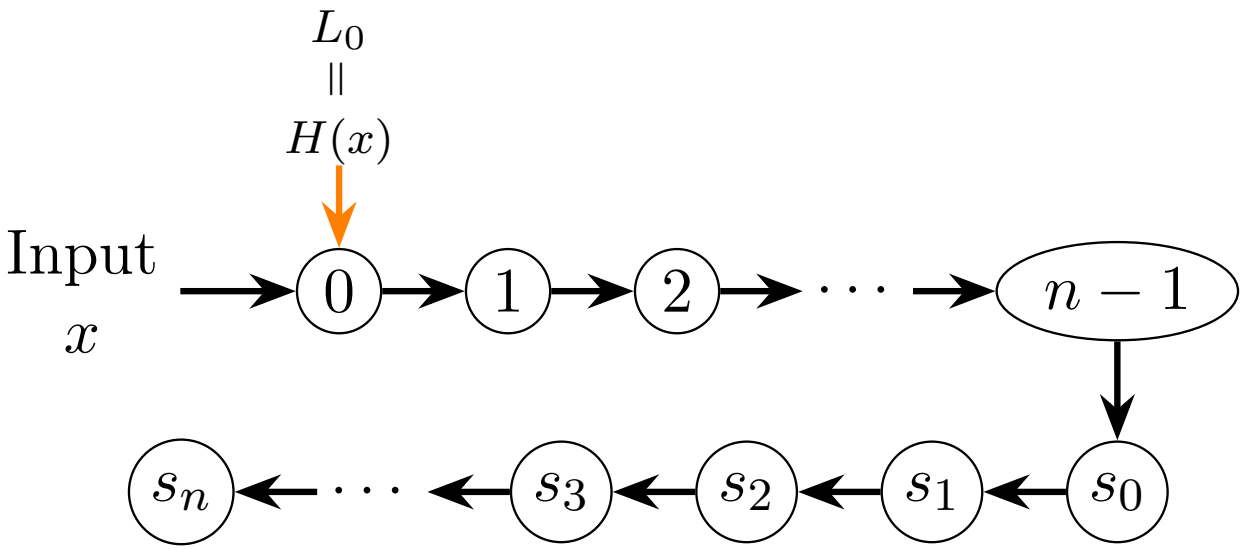

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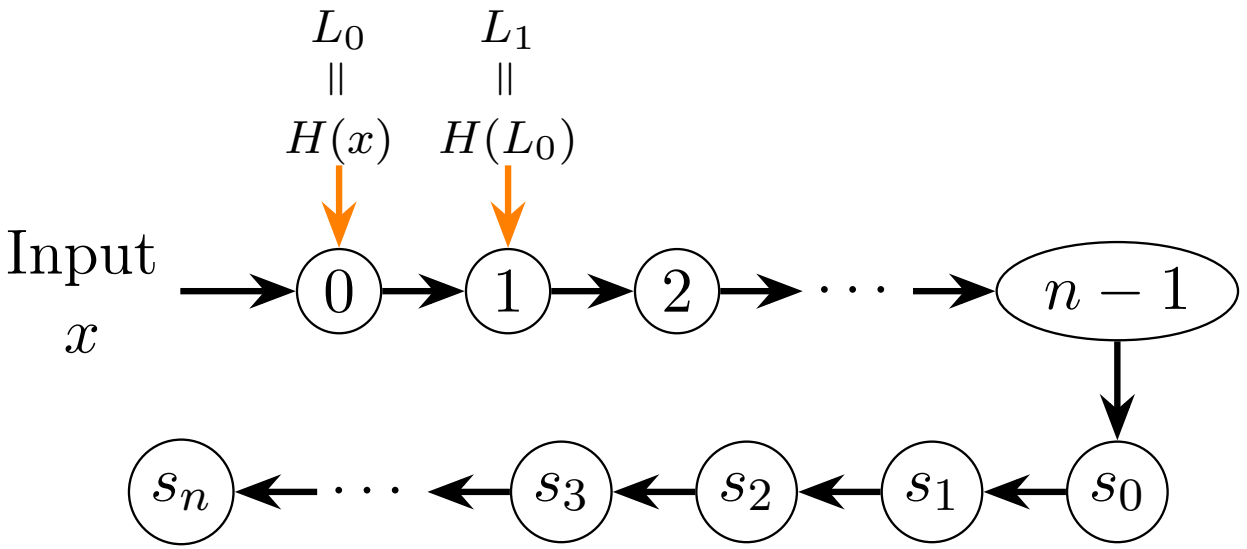

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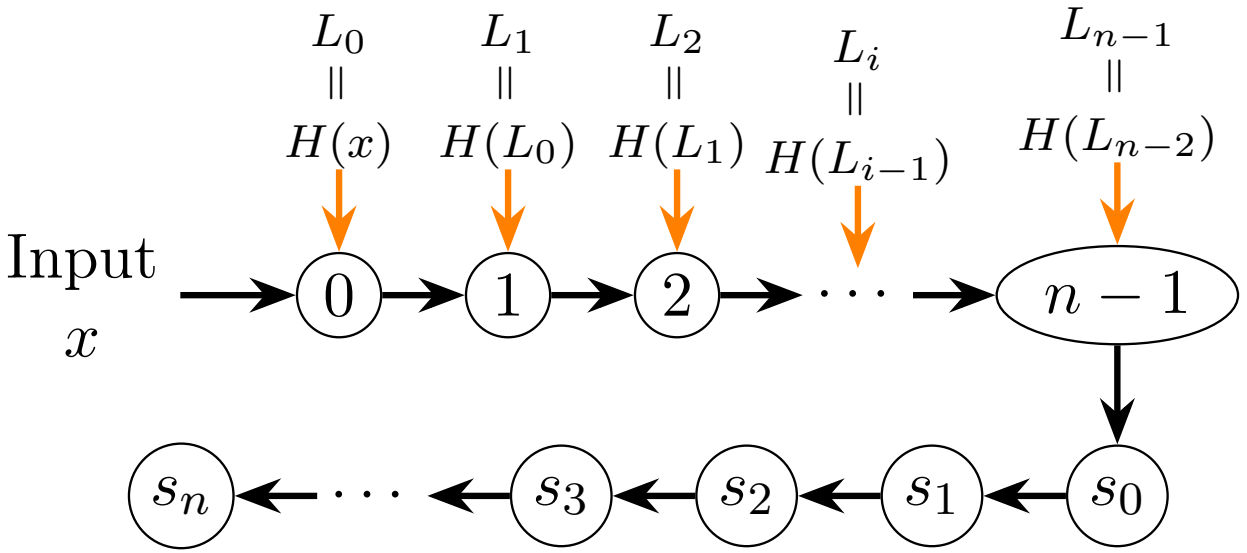

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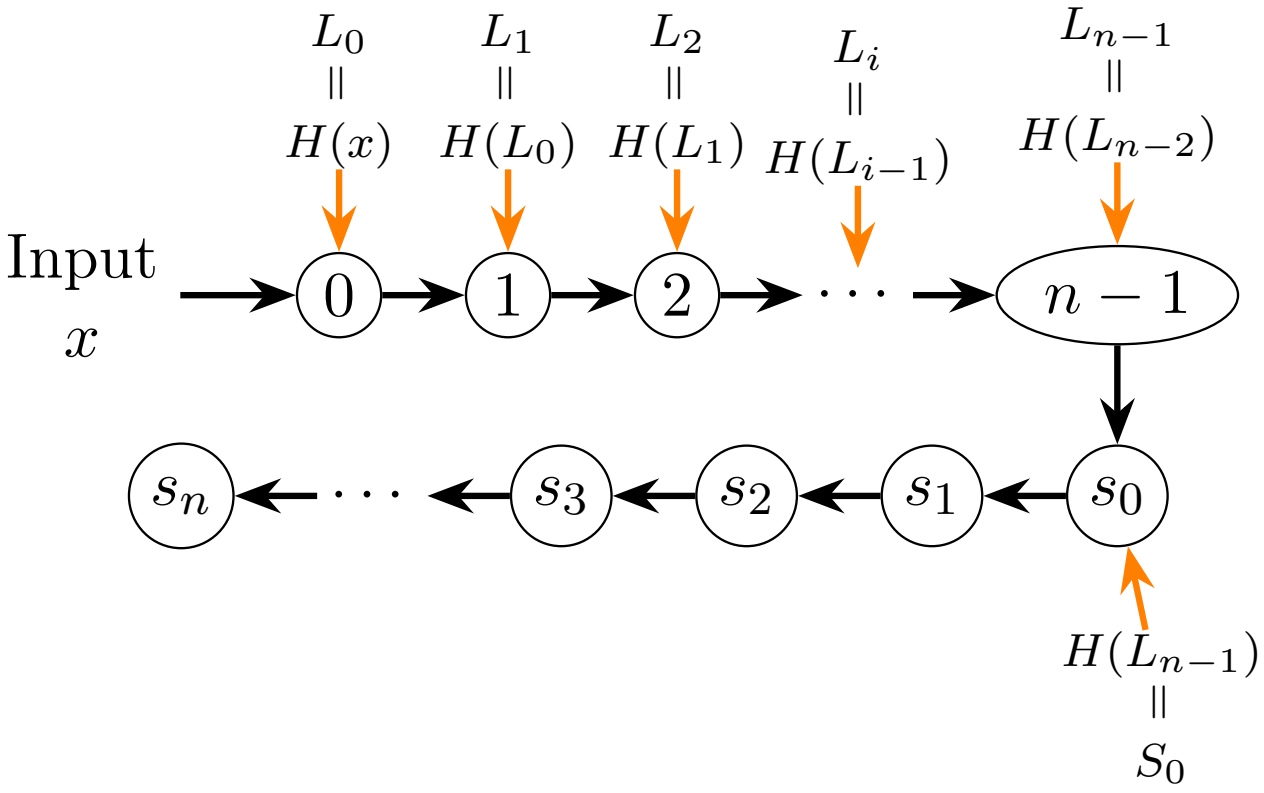

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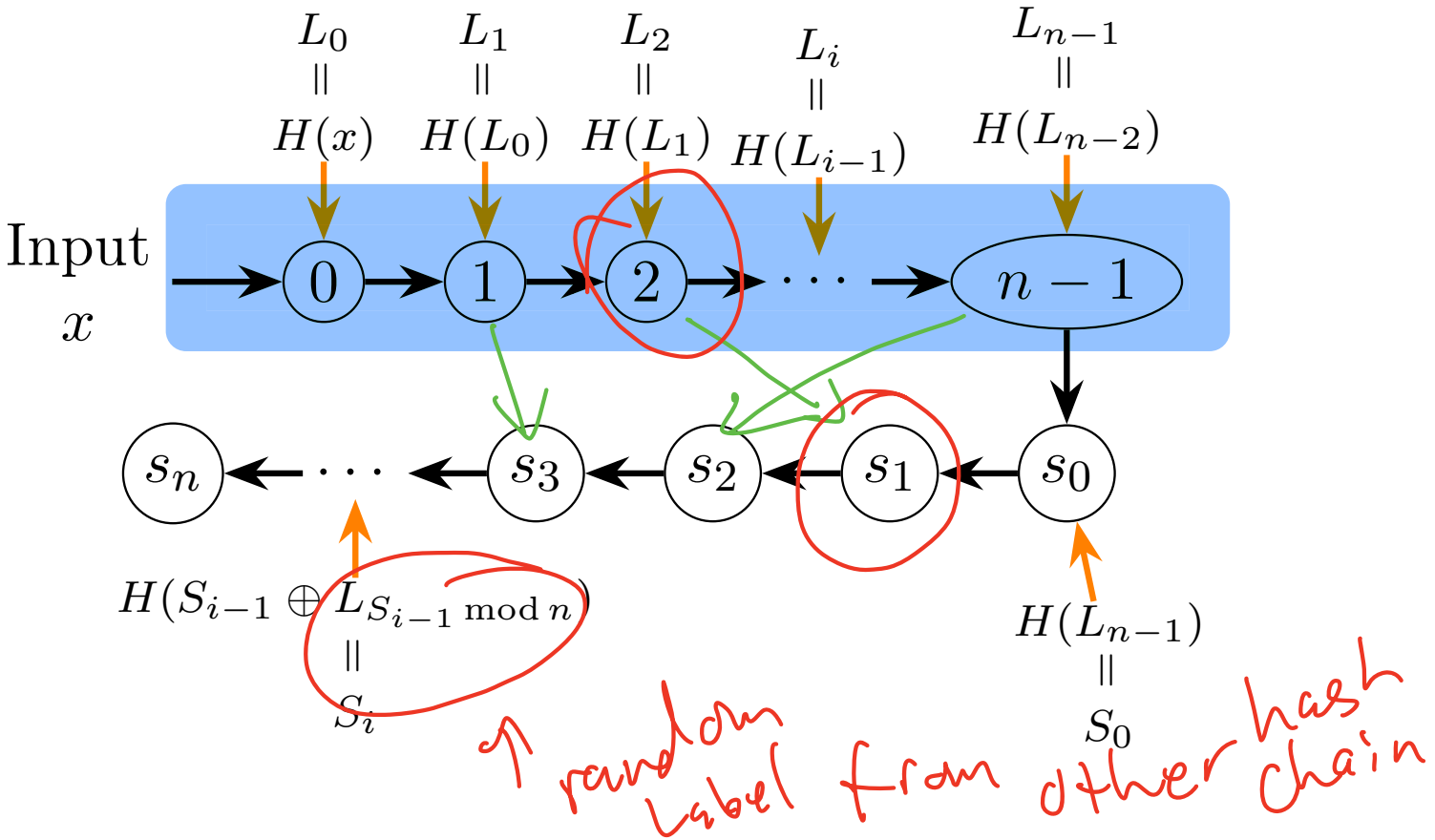

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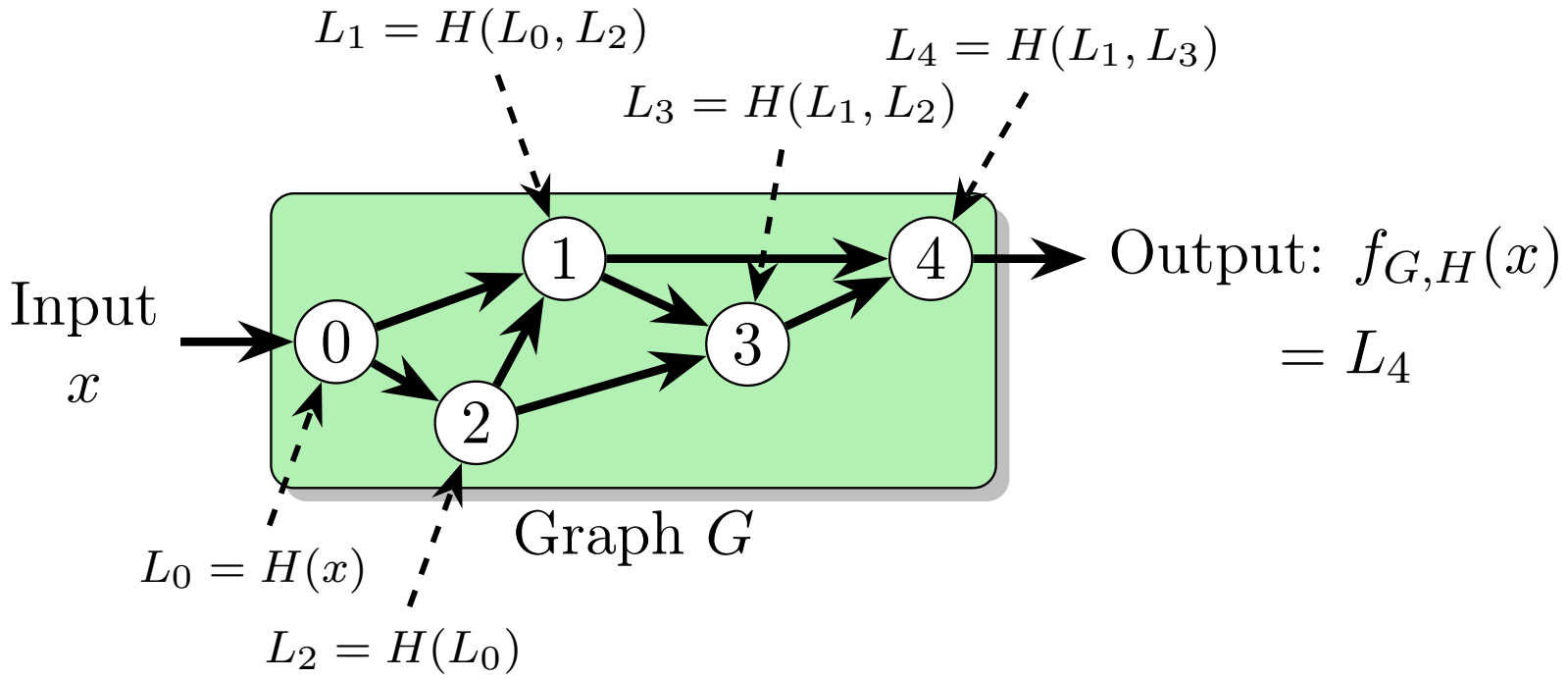

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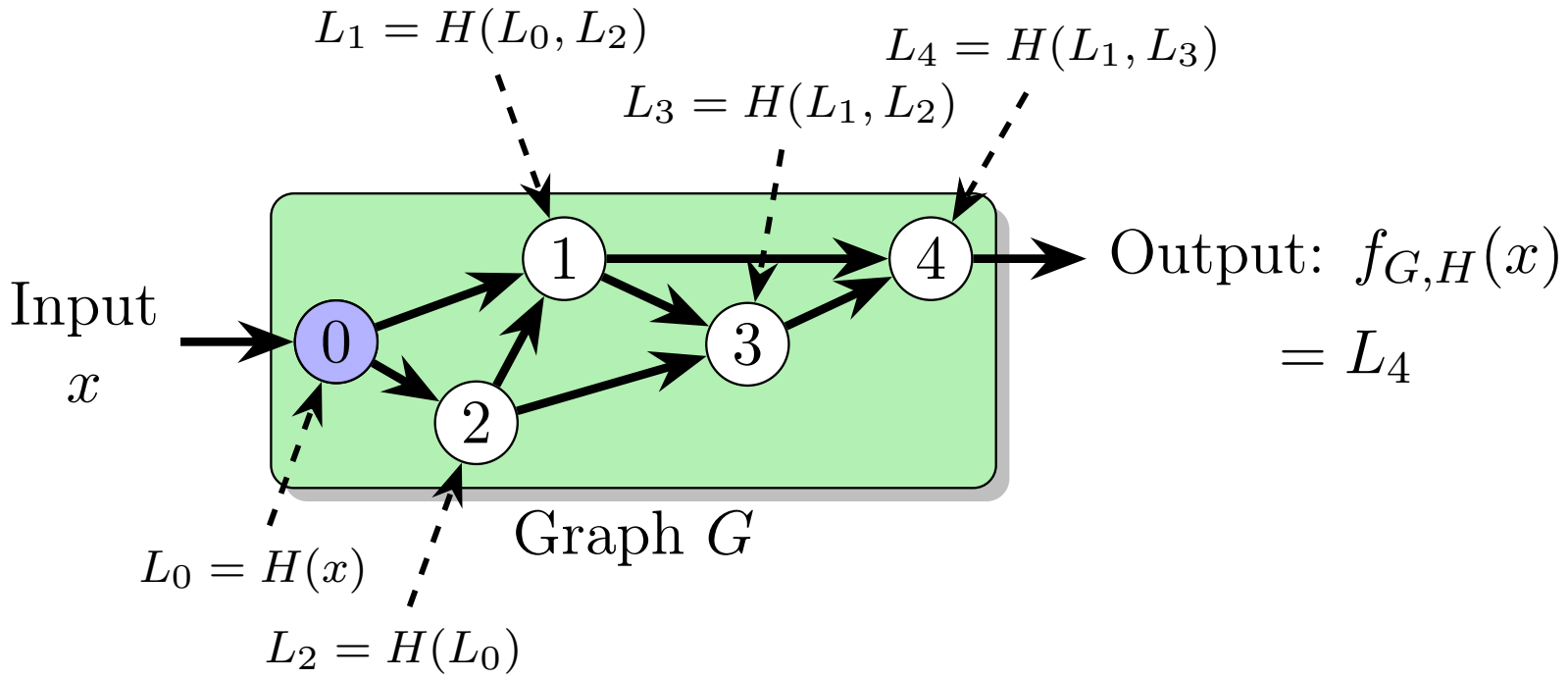

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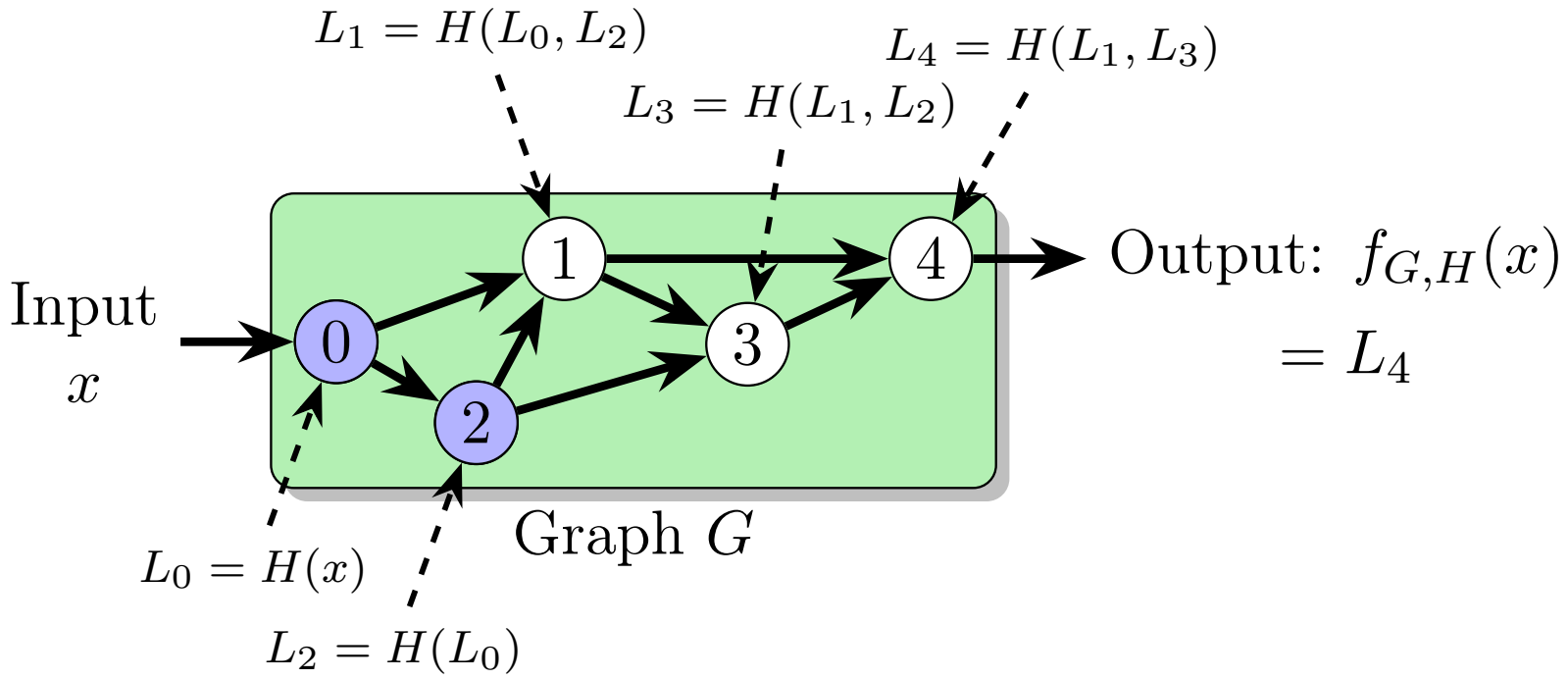

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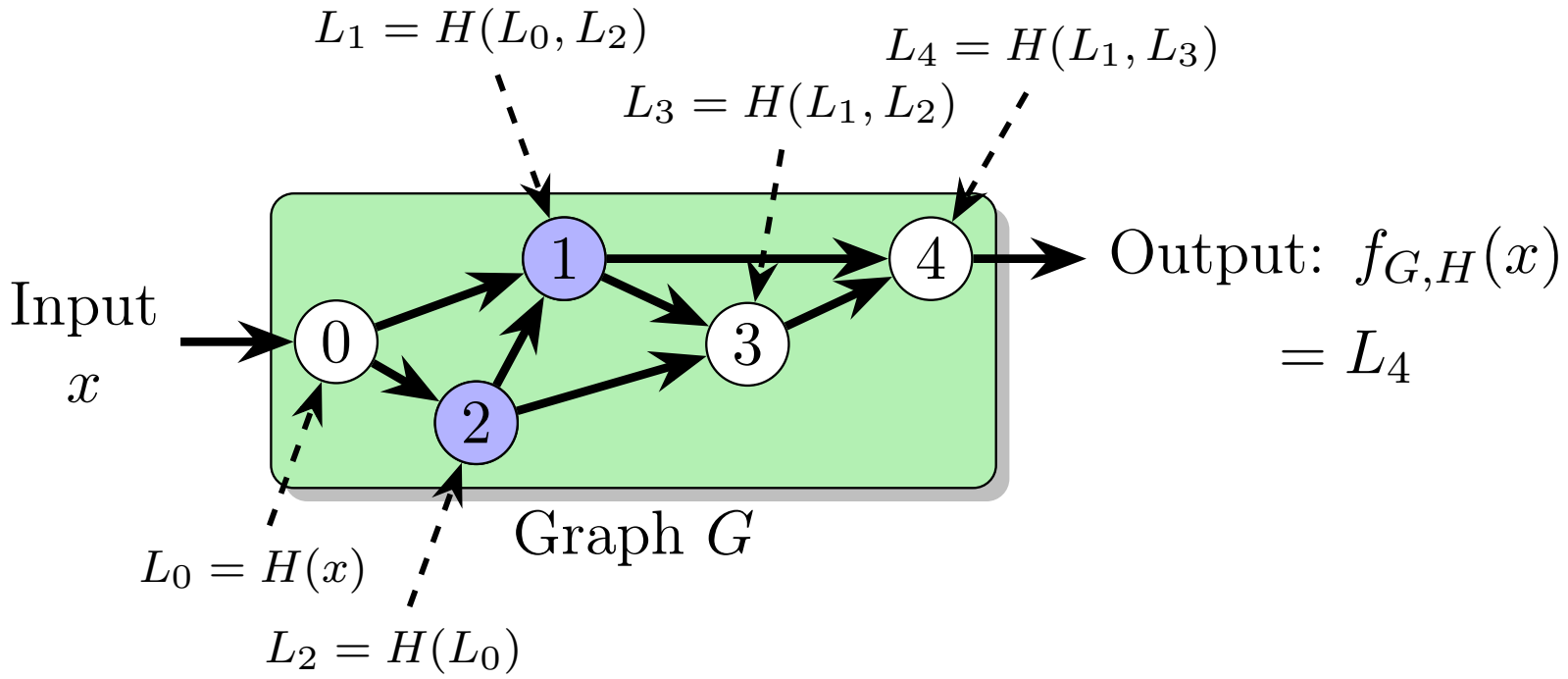

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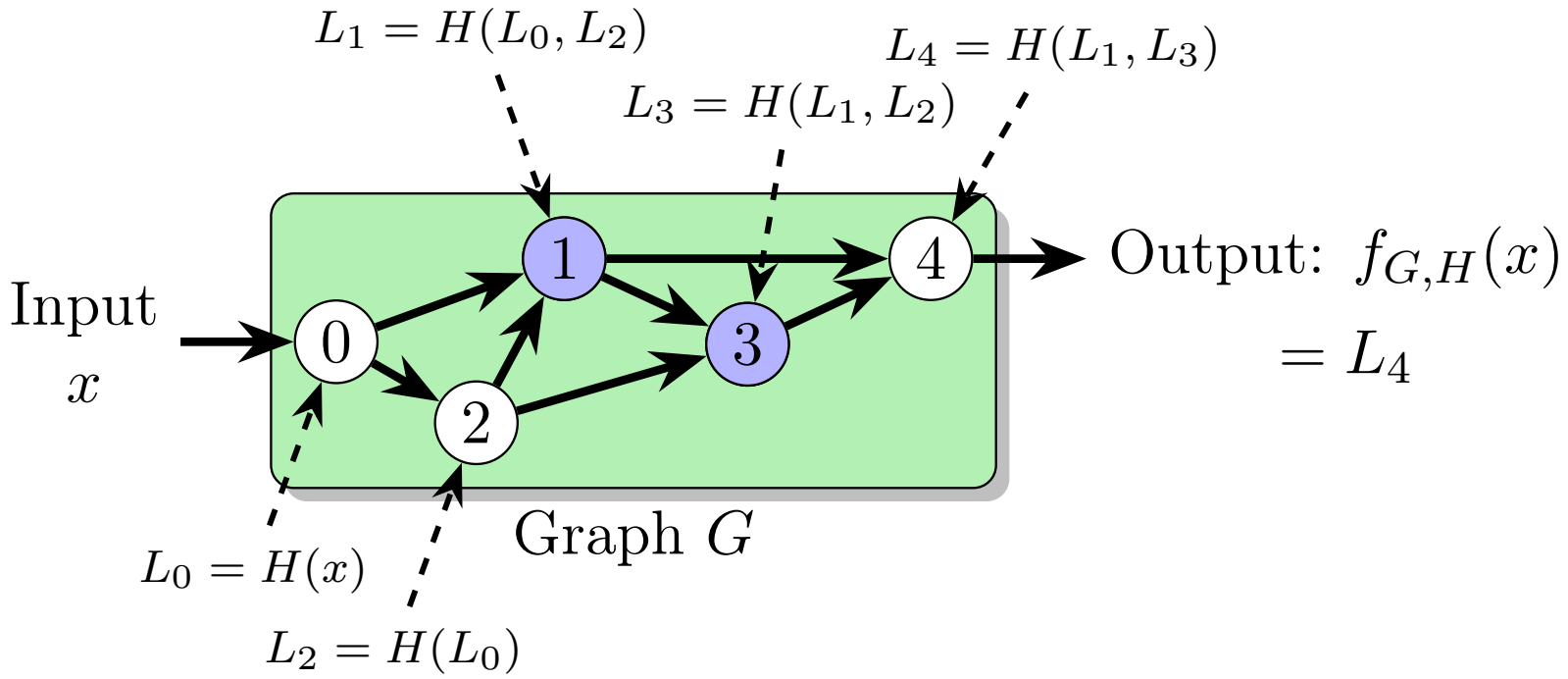

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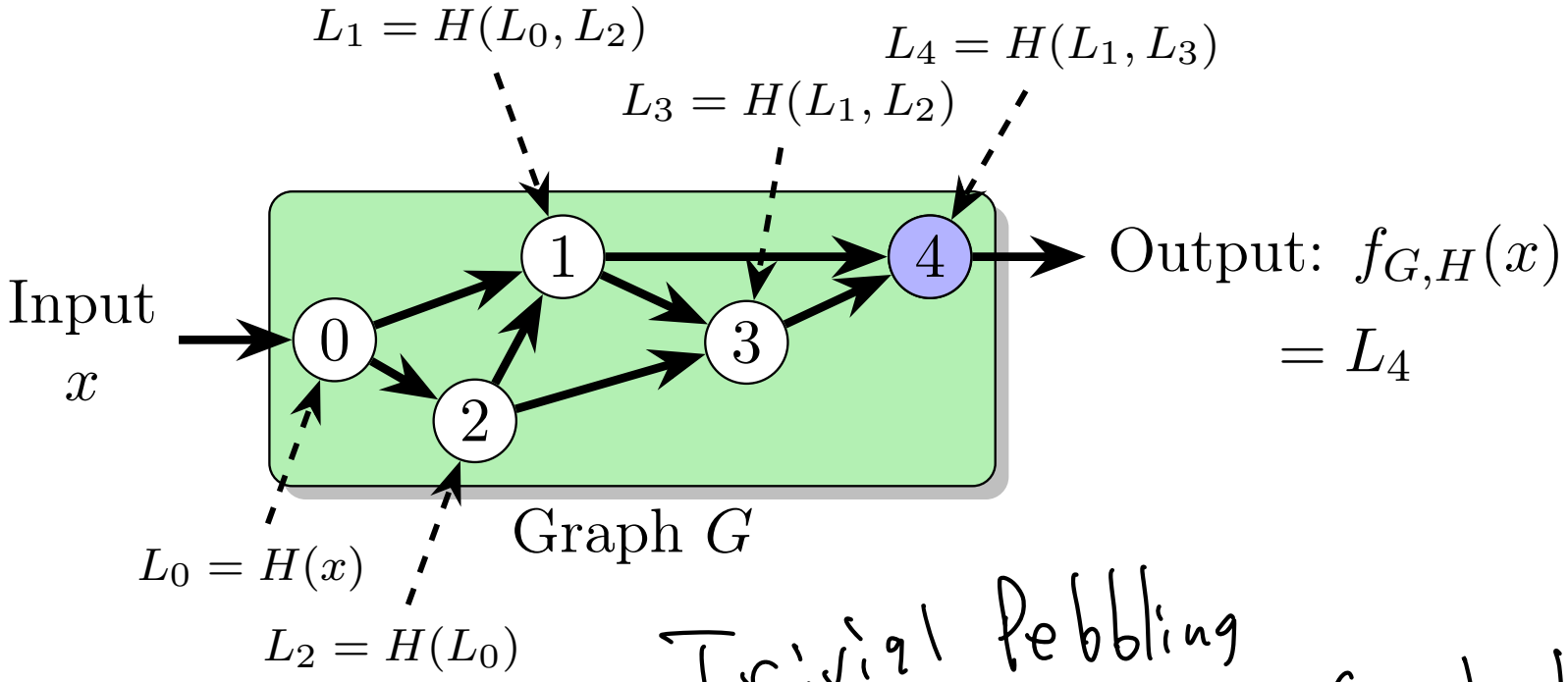

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Trivial Pebbling

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- Idea: translate memory costs to the cost of *pebbling* the underlying DAG of a MHF.

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ST-Complexity is not appropriate for password hashing!

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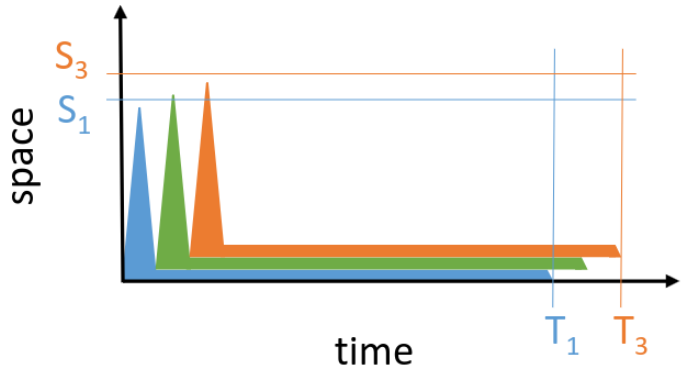

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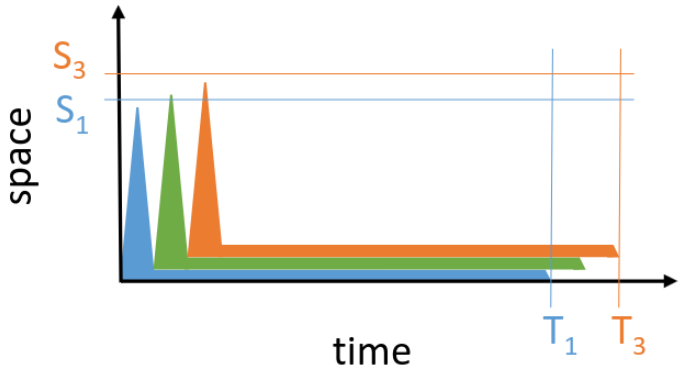

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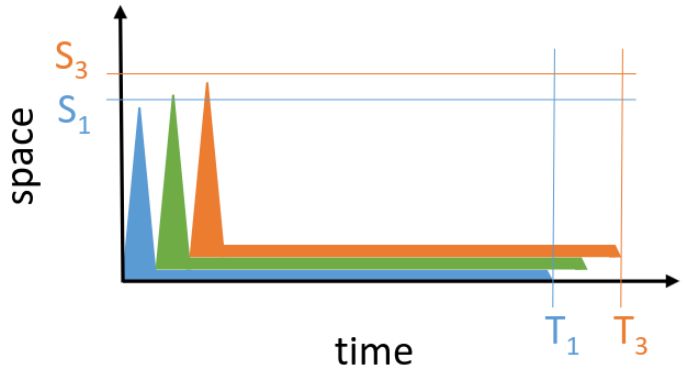

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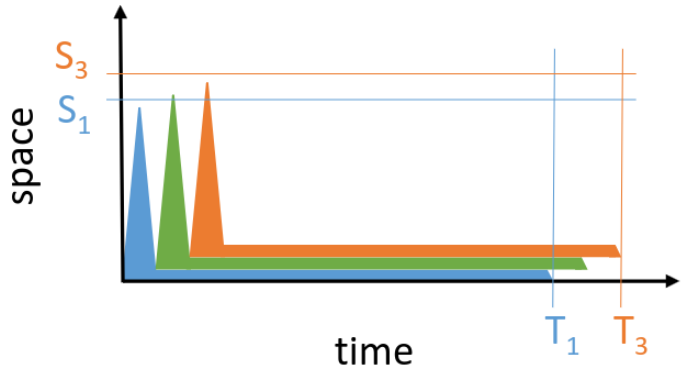

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For measuring memory-hardness, we would like an *amortized* memory cost (amortized area-time complexity)

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Next Time: CC, the Parallel ROM, and Depth-Robustness