CS 594 – Advanced Crypto (Spring 2026)

Alex Block

Lecture 22

April 13, 2026

Memory-hard Functions Continued

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- Both dMHFs and iMHFs are constructed using *directed acyclic graphs* G and a hash function H, modeled as a random oracle.

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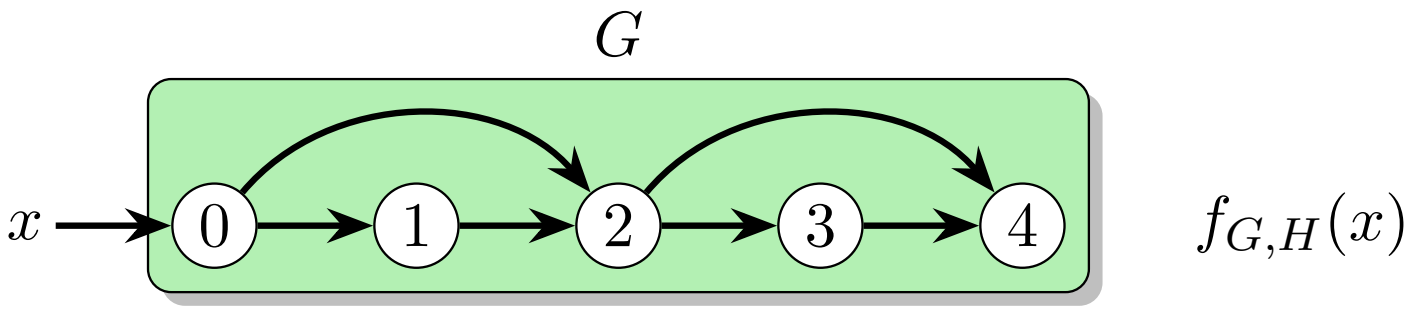

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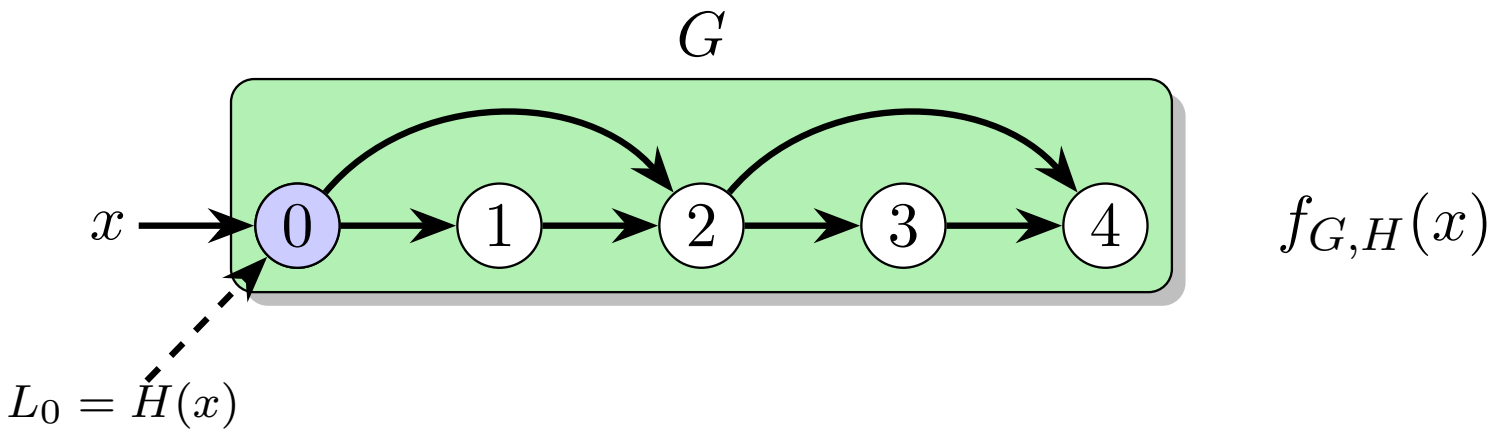

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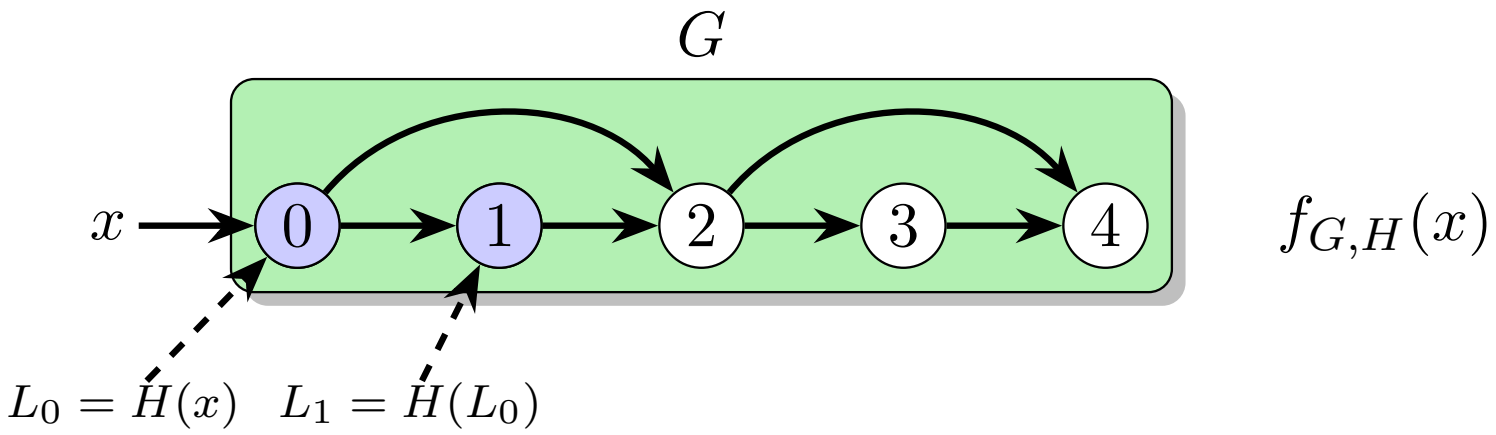

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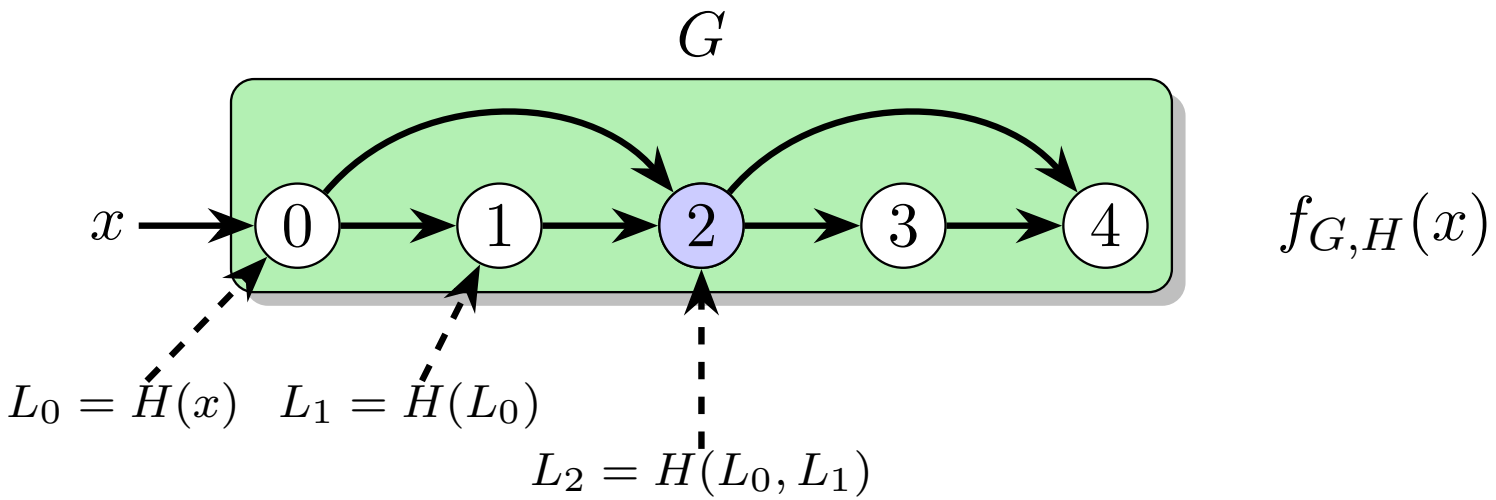

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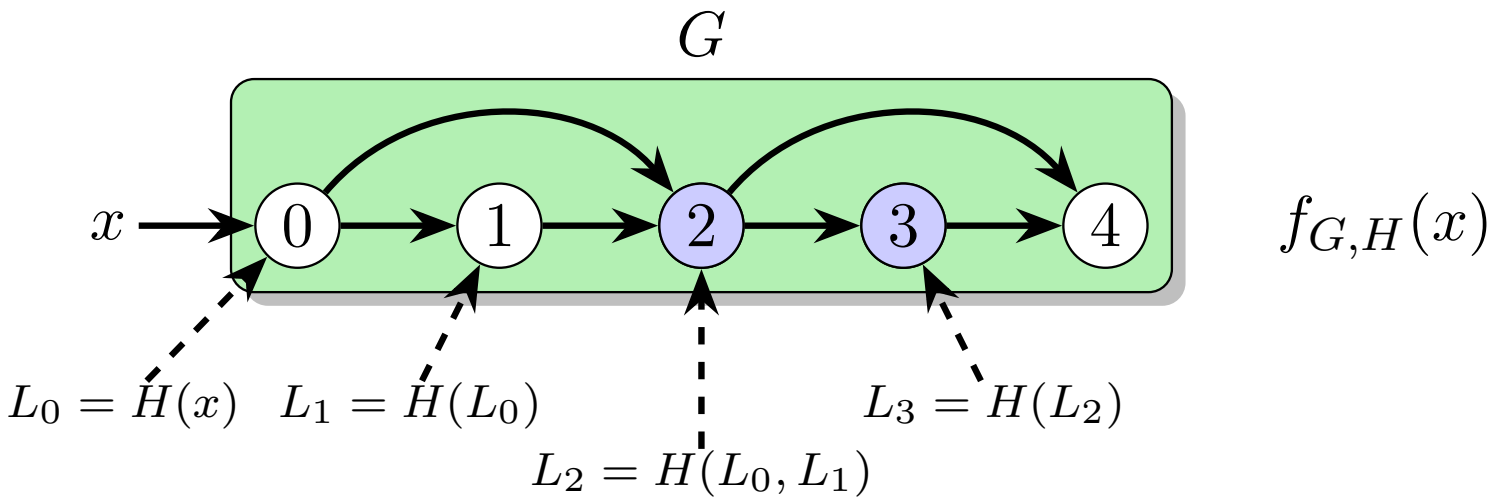

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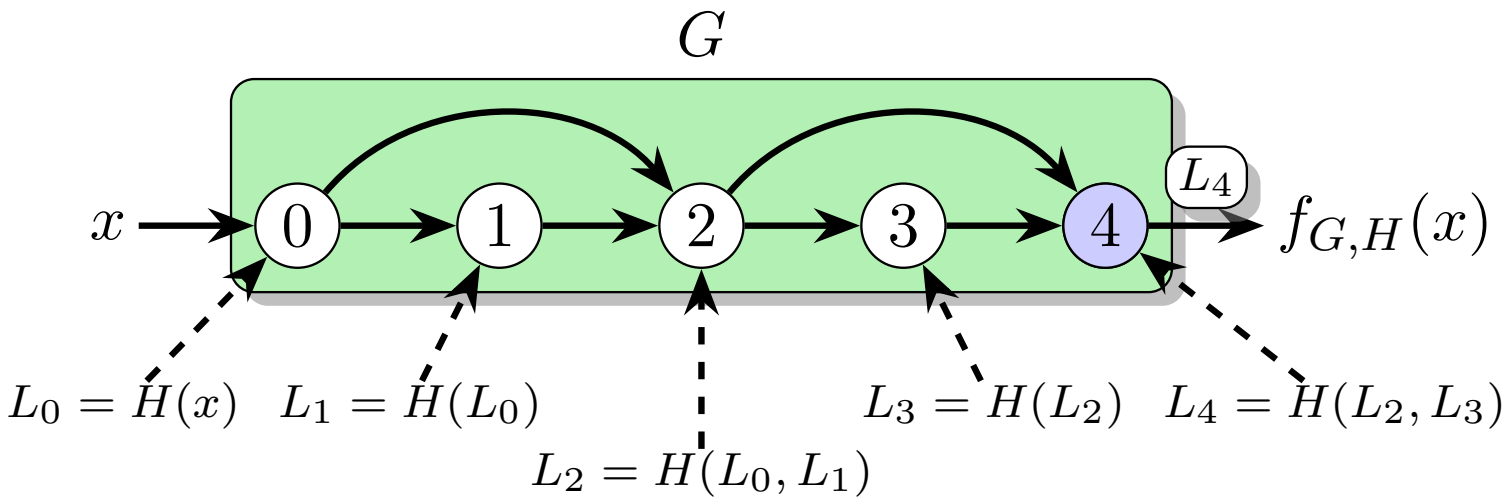

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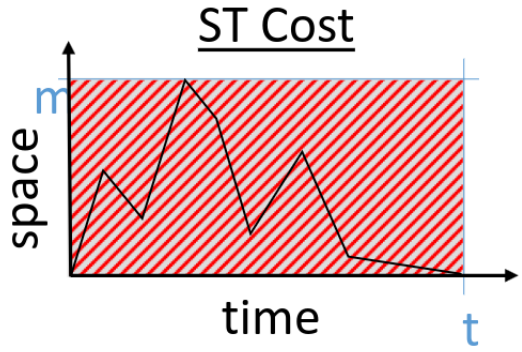

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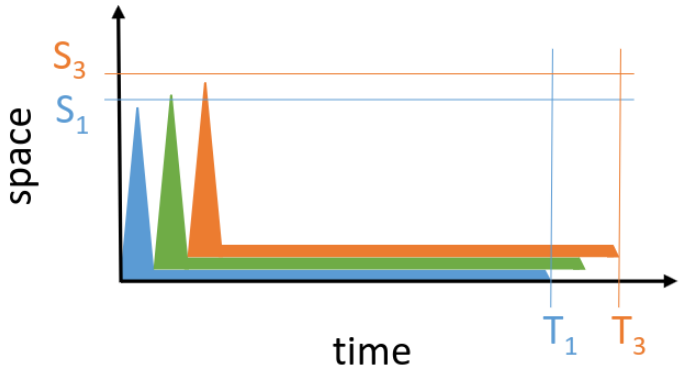

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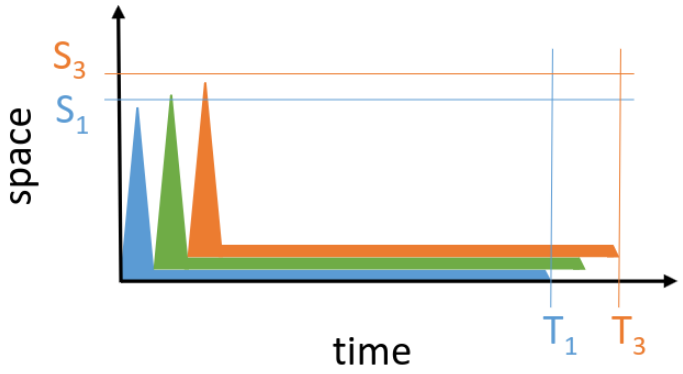

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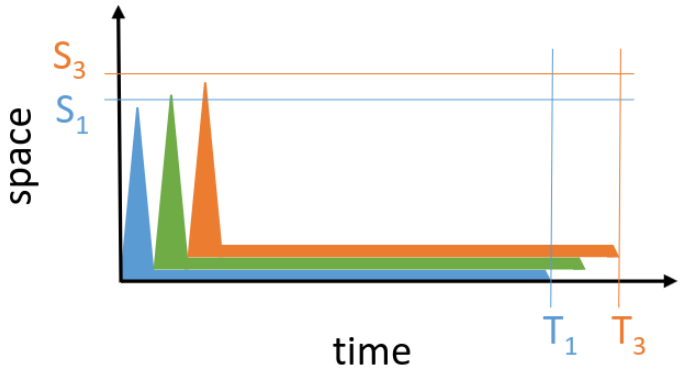

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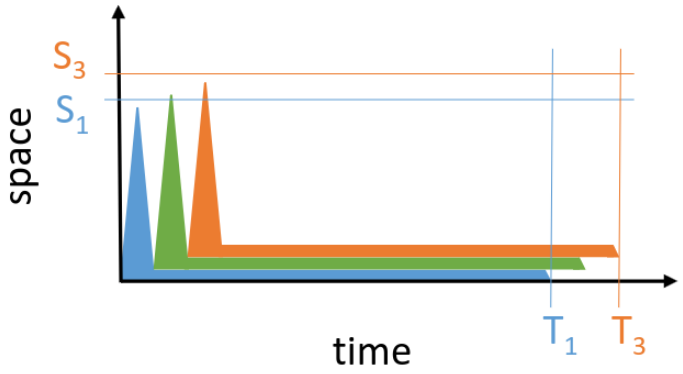

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For measuring memory-hardness, we would like an *amortized* memory cost (amortized area-time complexity)

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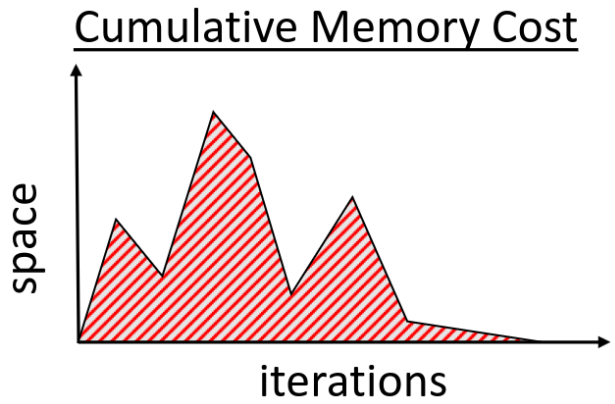

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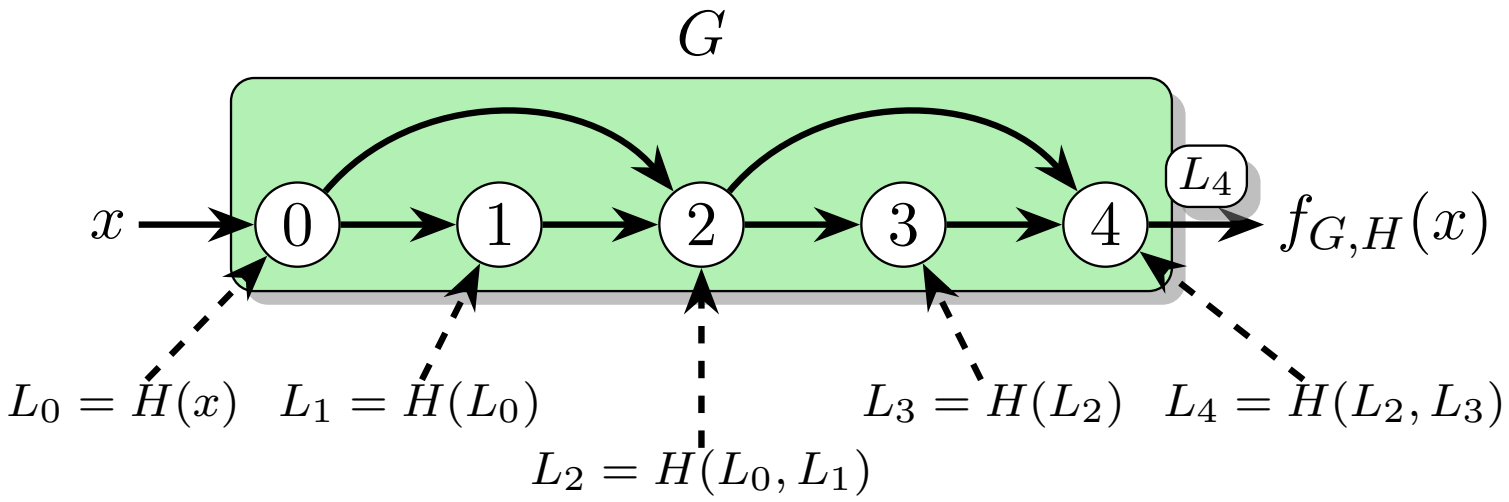

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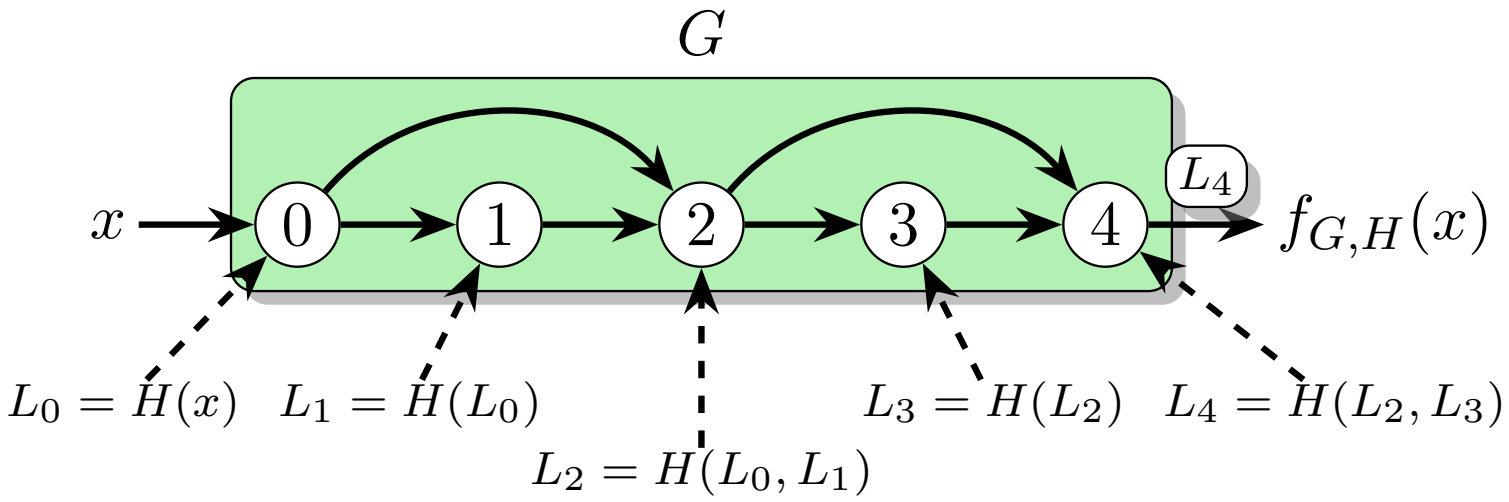

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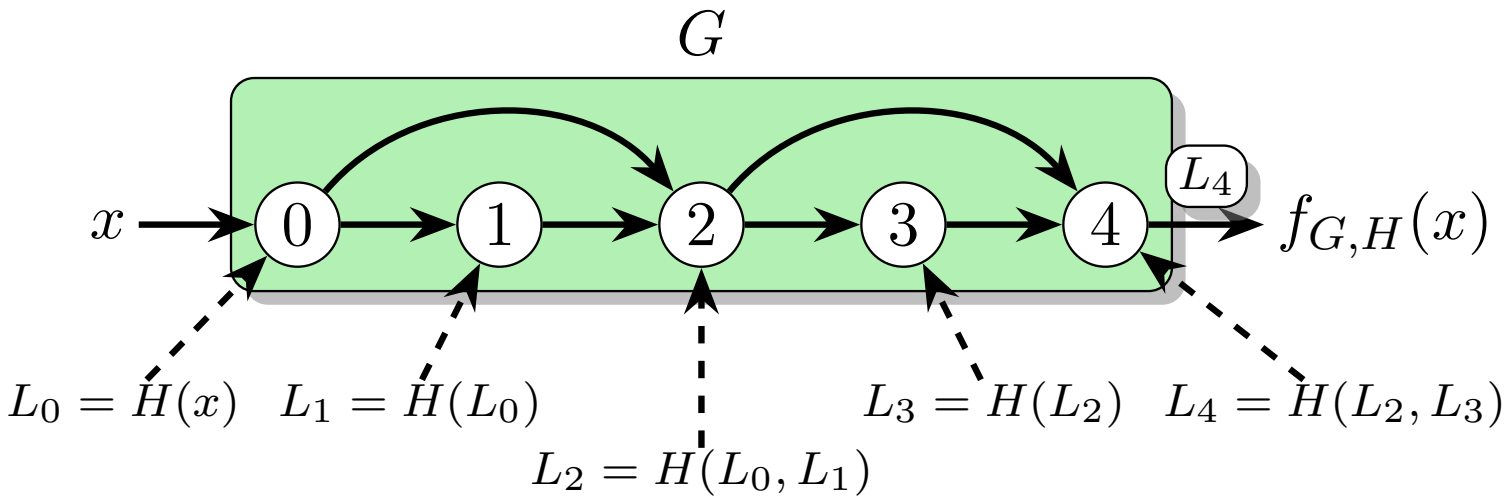

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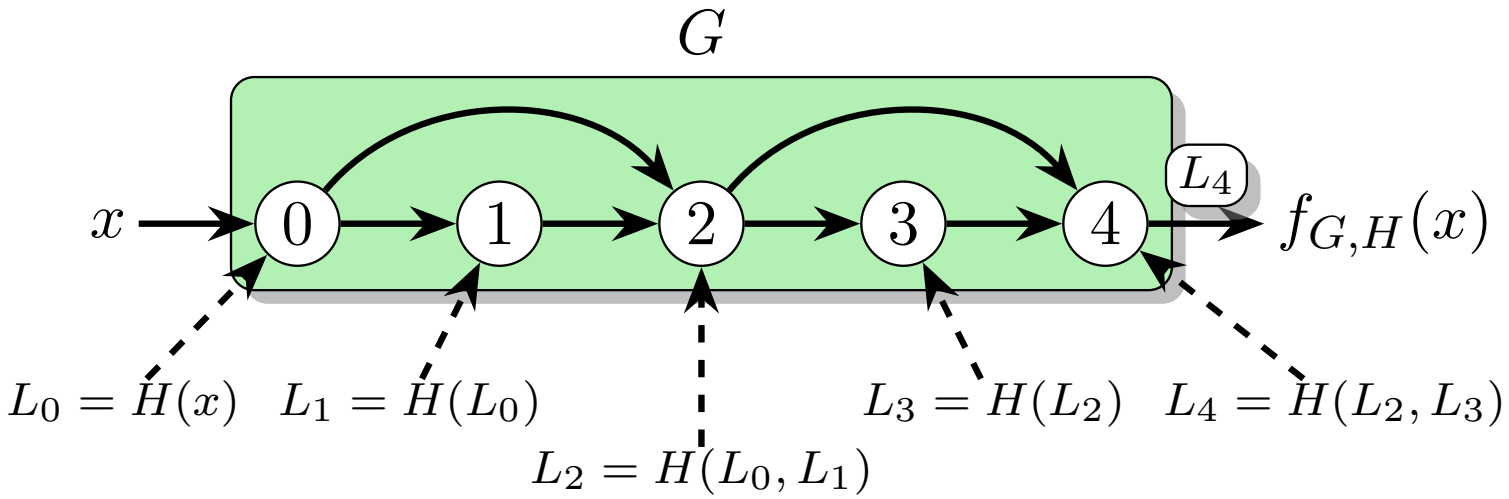

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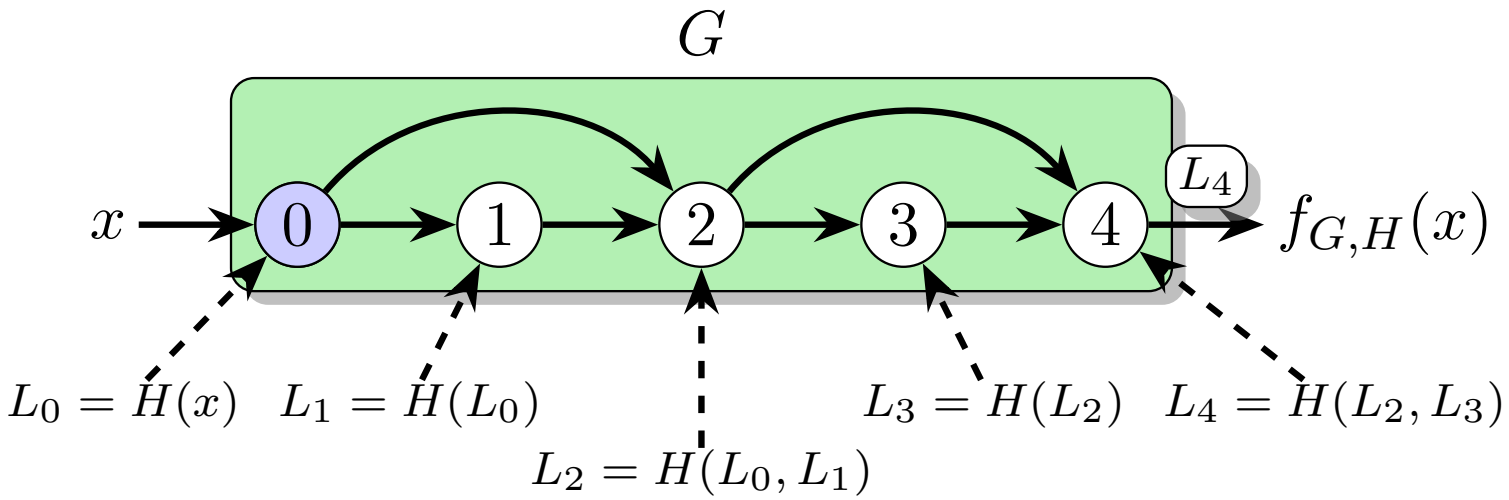

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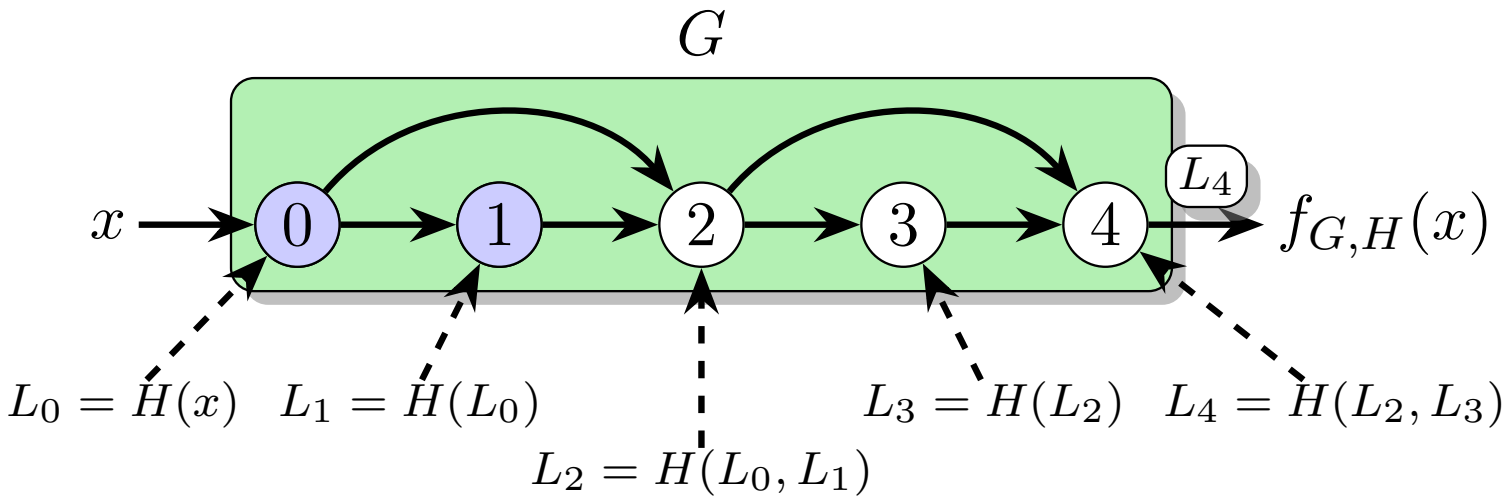

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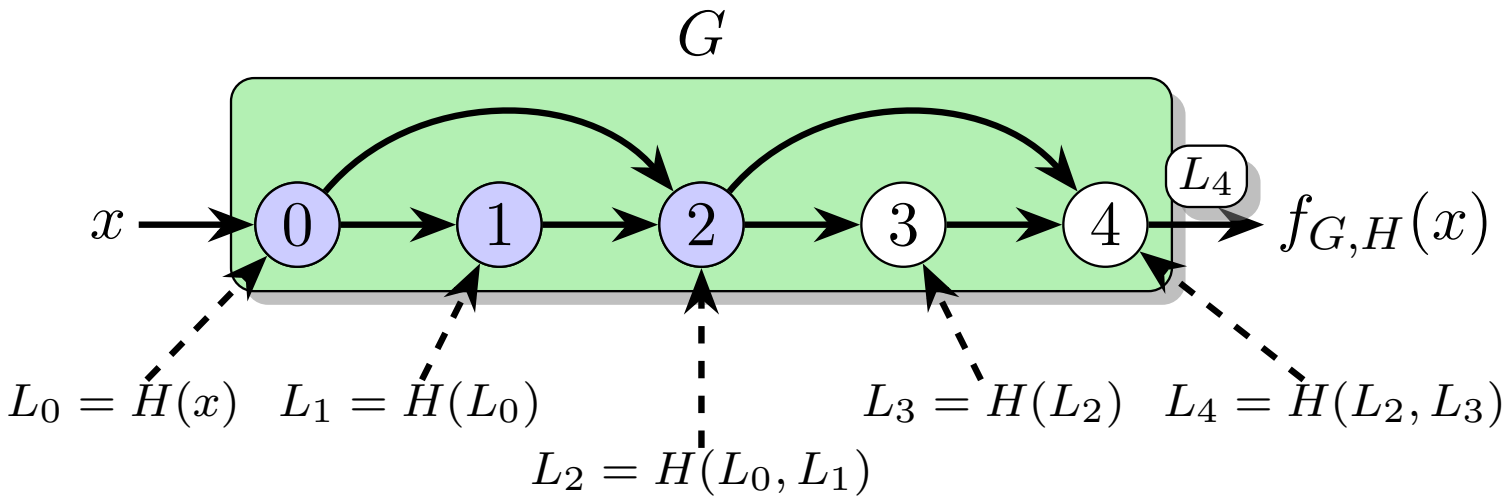

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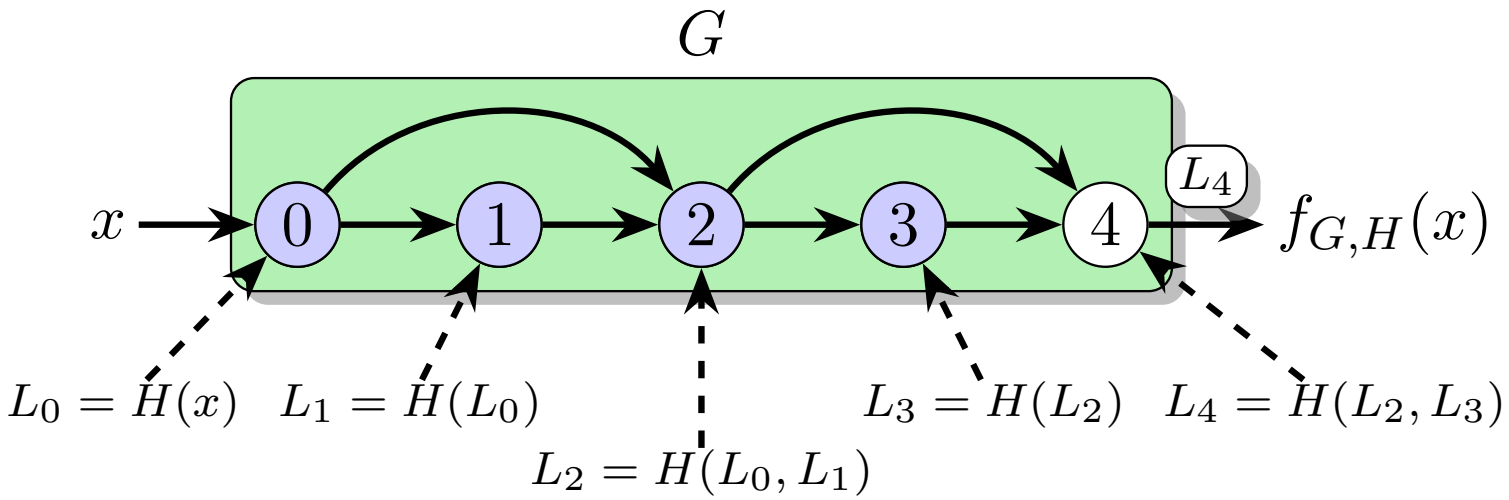

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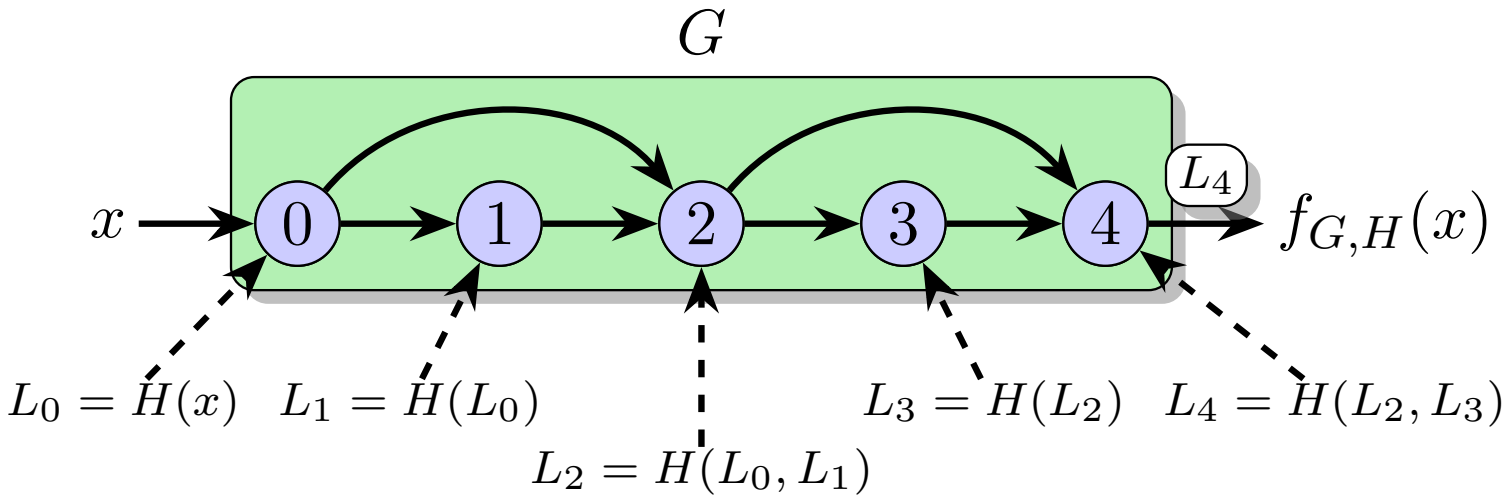

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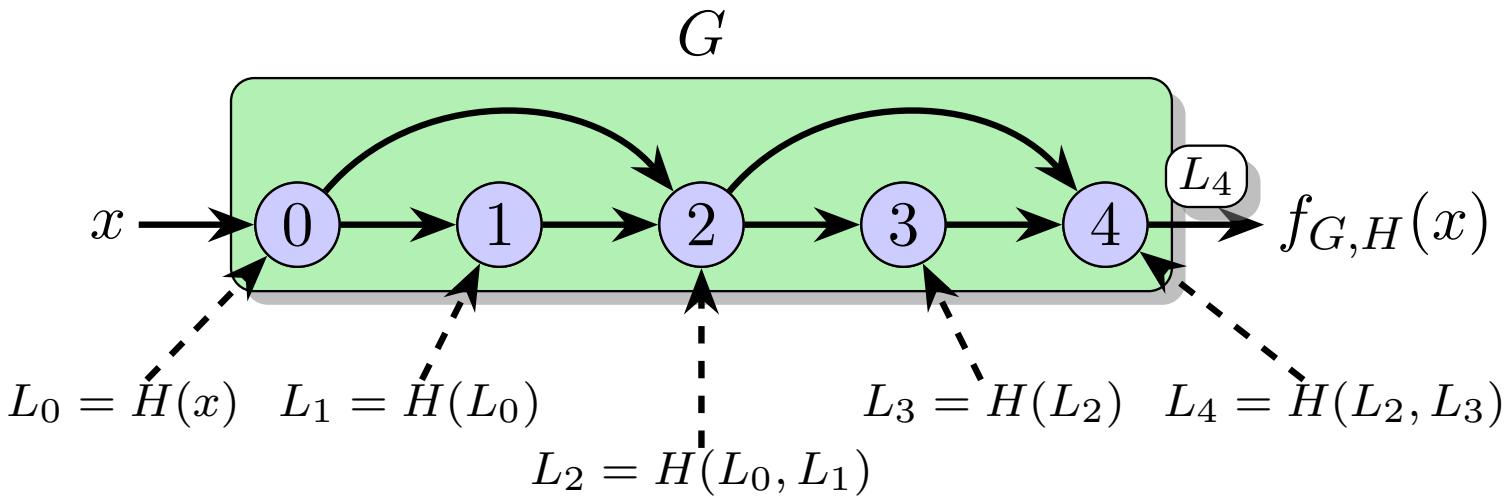

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$$\mathsf{CC}(G, \mathbf{P}) = 1 + 2 + 3 + 4 + 5 = 15$$

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Theorem 2

Any DAG G on n nodes has $\mathsf{CC}(G) \leqslant \sum_{i \leqslant n} i = \frac{n(n+1)}{2}$.

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- Properties of naïve pebbling strategy:
  - Legal for any DAG G.
  - Pebbling time is n.
  - Sequential algorithm: place a single new pebble on the graph during each round of pebbling.

Theorem 2

Any DAG G on n nodes has $\mathsf{CC}(G) \leqslant \sum_{i \leqslant n} i = \frac{n(n+1)}{2}$.

Proof.

The naïve pebbling strategy is legal for all DAGs! □

Graphs with Large CC

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But!

This graph has large in-degree $O(n)$!

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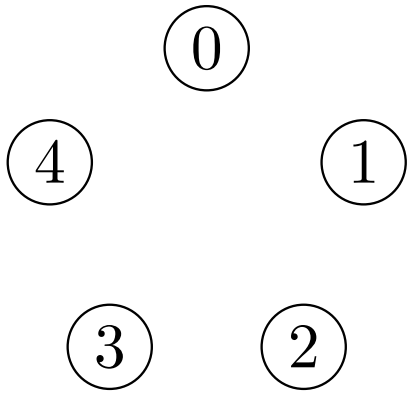

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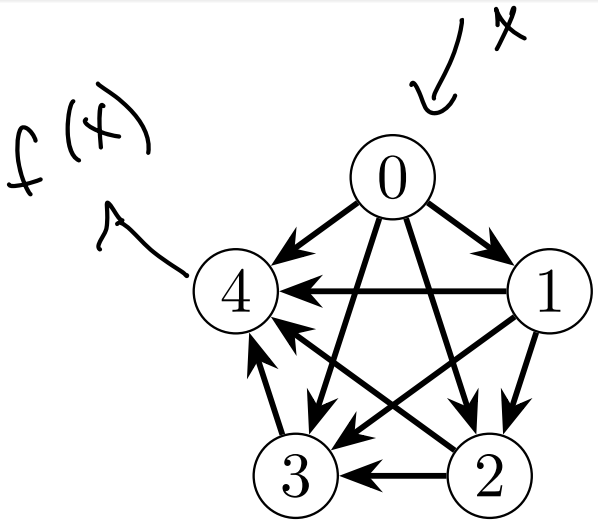

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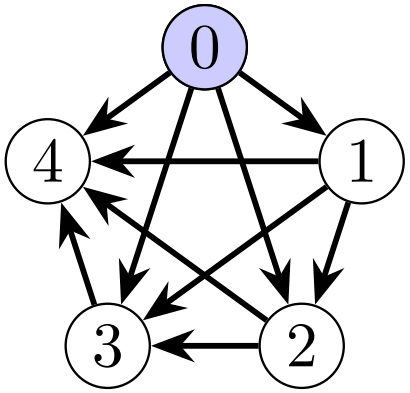

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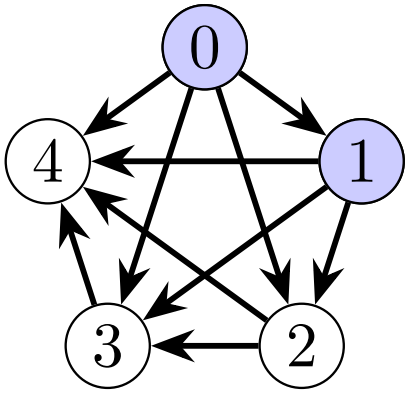

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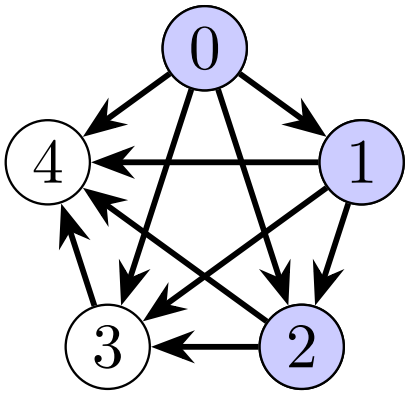

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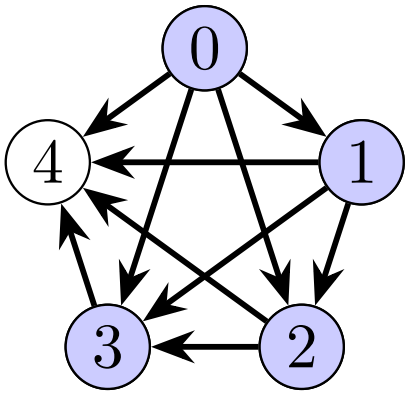

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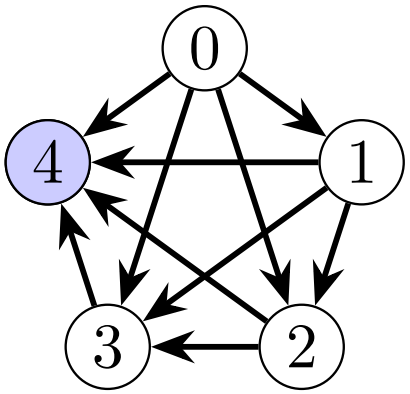

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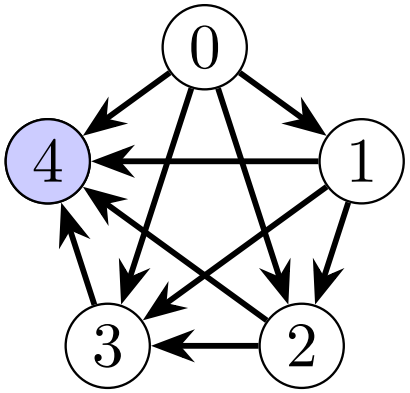

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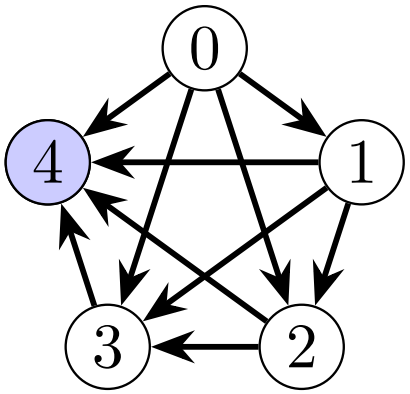

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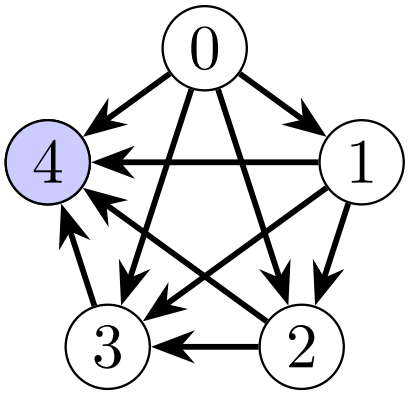

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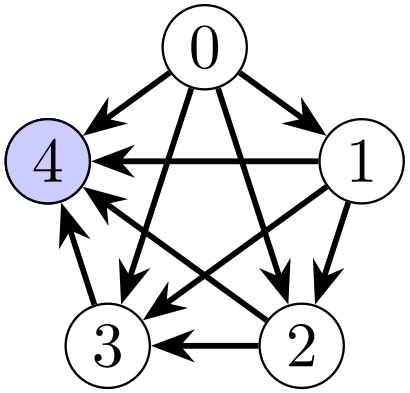

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Label the graph from 0 to $n-1$ in topological order. Then, node i has parent nodes $\{0, \ldots, i-1\}$. So node i needs all of its parents to be pebbled before it can be pebbled. □

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Maximize the costs for fixed running time n (users are impatient!)

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Theorem 4

There exists a DAG G on n vertices with $\delta(G) = O(1)$ such that $\mathsf{CC}(G) \geqslant \Omega\left(\frac{n^2}{\log(n)}\right)$.

CC, CMC, AND THE PROM

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Definition 1 (cmc)

The *cumulative memory complexity (cmc)* of A is defined as

$$\mathsf{cmc}(A_f) = \mathop{\mathbb{E}}_{H}\left[\min_{x,r} \sum_{\sigma \in \text{Trace}(A_f^H(r),x)} |\sigma|\right],$$

where $A_f^H(r)$ denotes that A's random coins are fixed to the string r.

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Let A_f^H be an algorithm computing a function $f_{G,H}$ defined by a DAG G and random oracle $H\colon \{0,1\}^* \to \{0,1\}^\lambda$. Then, in the parallel random oracle model (PROM), we have $\mathsf{cmc}(A_f) = \Omega(\lambda \cdot \mathsf{CC}(G))$.

cmc and CC

- How can we relate cmc to CC?
  - Intuition: if A_f^H is computing $f_{G,H}$ (defined by DAG G and RO H), then the pebbling cost ($\mathsf{CC}(G)$) should translate to $\mathsf{cmc}(A_f)$.

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- We will give a very high-level overview of the proof.

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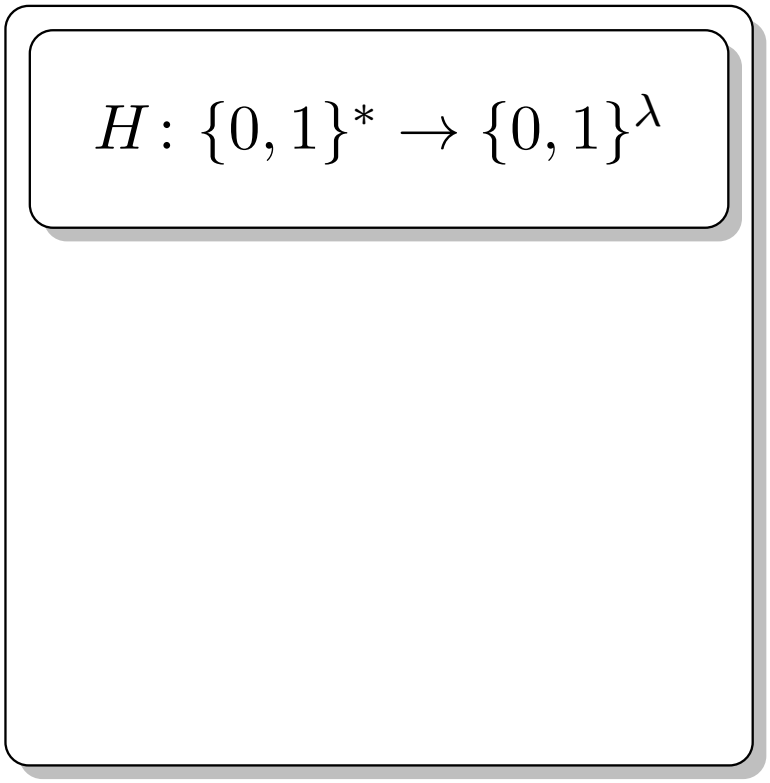

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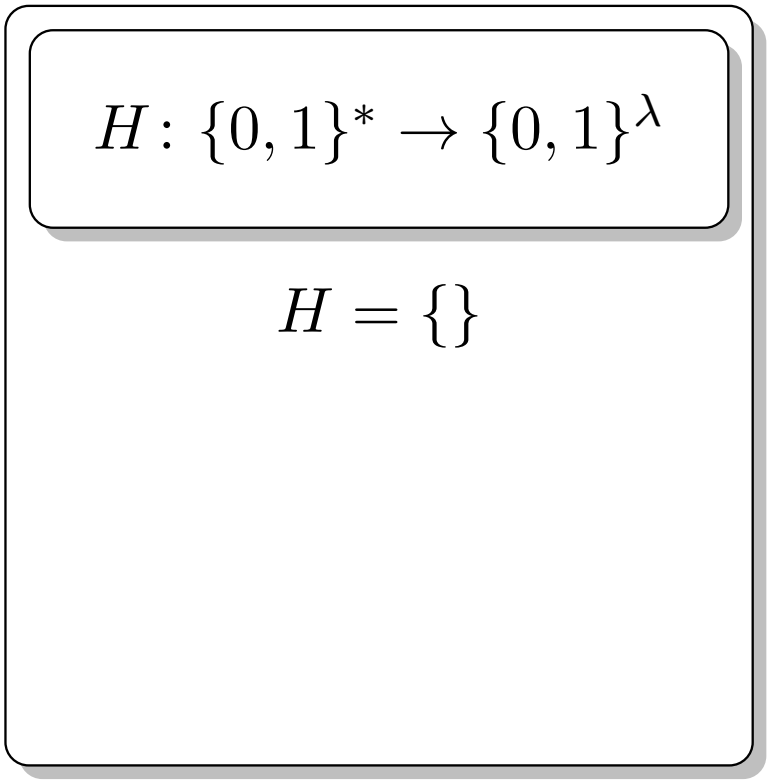

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Fact 2 (ROs are Incompressible)

Any $A^H(h)$ given a hint $h \in \{0,1\}^s$ (which may depend on H) wins the prediction game with probability at most $2^{-k \cdot \lambda + s}$.

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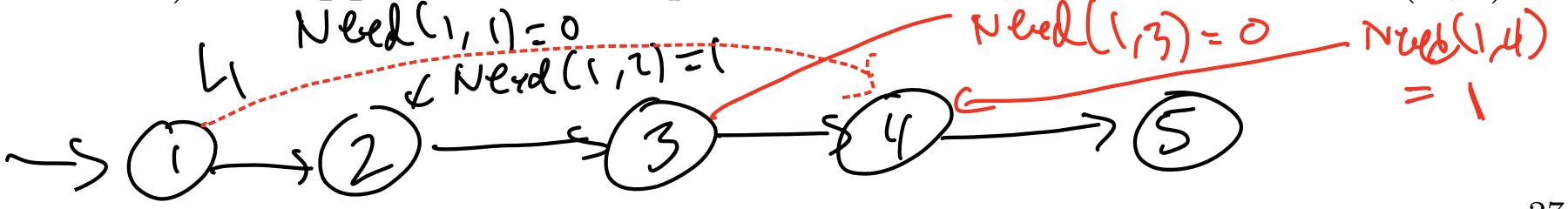
- Define indicator $\text{Need}(v, i) = 1$ if and only if the next time (after round i) L_v appears is an *input* to the RO; otherwise $\text{Need}(v, i) = 0$.



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↑

Relate $|P_i|$ to $|O_i| + |q_i|$

Next Time: wrap up the proof + PQ Crypto Start