CS 594 – Advanced Crypto (Spring 2026)

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Lecture 23

April 15, 2026

Wrapping up MHFs

Last Time: Cumulative Memory Complexity

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Definition 1 (cmc)

The *cumulative memory complexity (cmc)* of A is defined as

$$\mathsf{cmc}(A_f) = \mathop{\mathbb{E}}_{H}\left[\min_{x,r} \sum_{\sigma \in \text{Trace}(A_f^H(r),x)} |\sigma|\right],$$

where $A_f^H(r)$ denotes that A's random coins are fixed to the string r.

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- We will give a very high-level overview of the proof.

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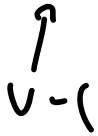

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- Define indicator $\text{Need}(v, i) = 1$ if and only if the next time (after round i) L_v appears is an *input* to the RO; otherwise $\text{Need}(v, i) = 0$.

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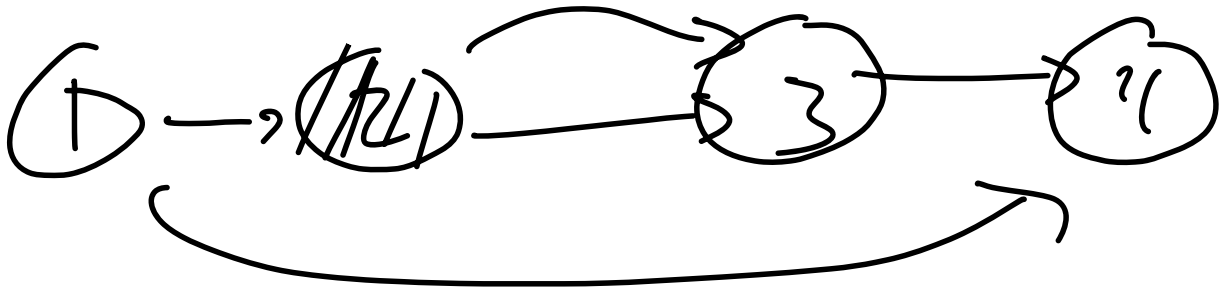

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  - Moreover, this implies that P_i is *not* a valid pebbling step!

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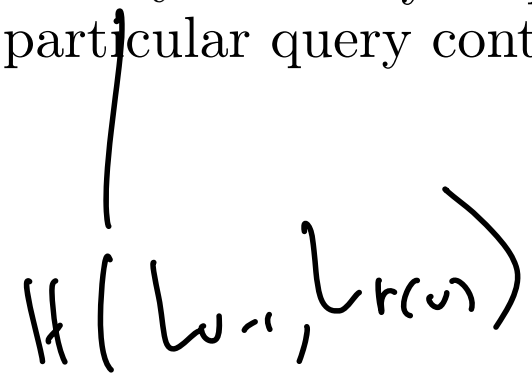

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  - Taking a Union bound over all nodes $v \in [n]$ gives us the result.
    - I.e., $\Pr[\exists v : \text{Order}(v)] \leqslant n \cdot \Pr[\text{Order}(1)] \leqslant nq2^{-\lambda}$.

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  - Any algorithm A_f computing $f_{G,H}$ has $\mathsf{cmc}(A_f) = \Omega(\lambda \cdot \mathsf{CC}(G))$.

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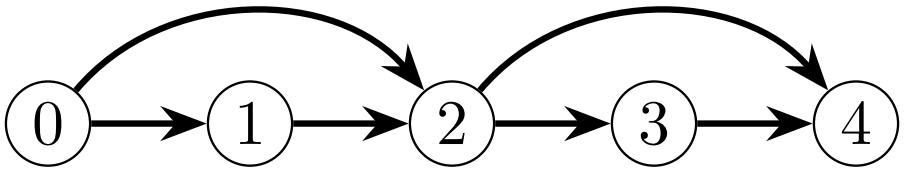

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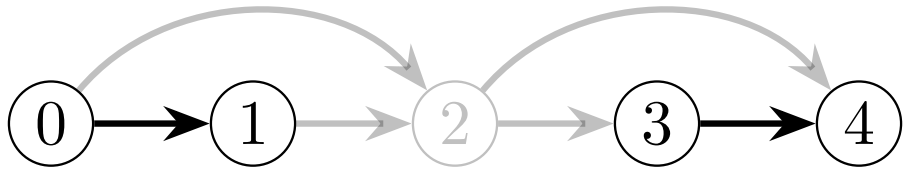

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Theorem 4 (Erdős, Graham, and Szemeredi 1975)

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Post-quantum Cryptography

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- Note: you should think of any such F as having only a sparse number of x such that $F(x) = 1$.

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- Note: you should think of any such F as having only a sparse number of x such that $F(x) = 1$.

Question

Quantum Computers vs. Symmetric-Key Crypto

- Quantum Computers can speed up brute-force searches over $\{0,1\}^\lambda$ from 2^λ time to $2^{\lambda/2}$ time.

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How do we relate Grover's Algorithm to breaking Symmetric-Key Cryptography?

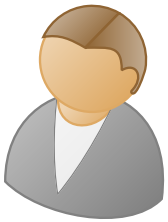
Quantum Computers vs. Symmetric-Key Crypto

Quantum Computers vs. Symmetric-Key Crypto

Alice

Quantum Computers vs. Symmetric-Key Crypto

Alice

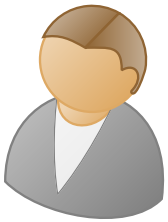


Bob

Quantum Computers vs. Symmetric-Key Crypto

$$k \in \{0,1\}^\lambda$$

Alice



Bob

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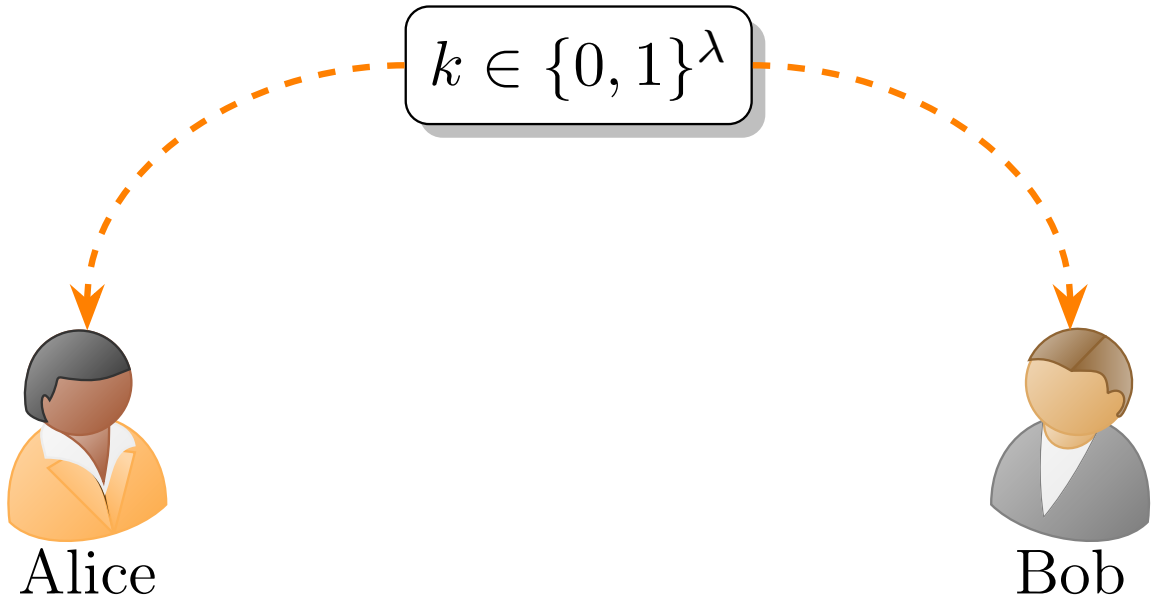

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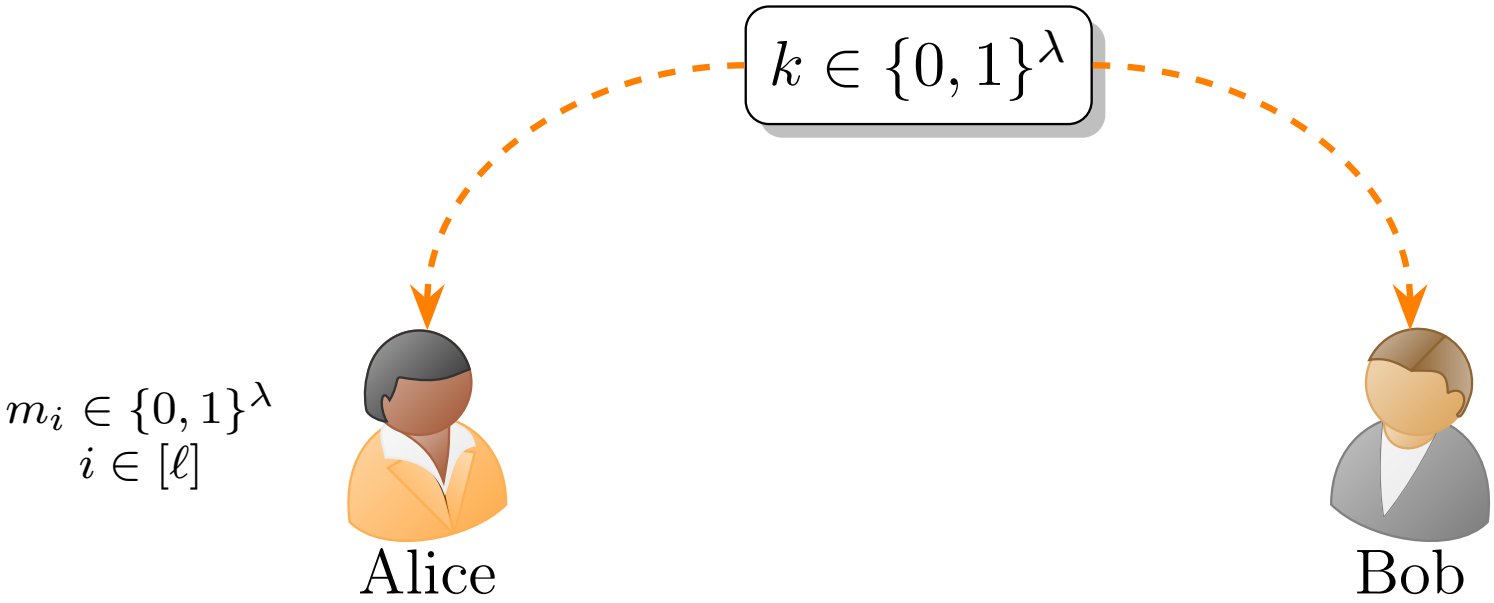

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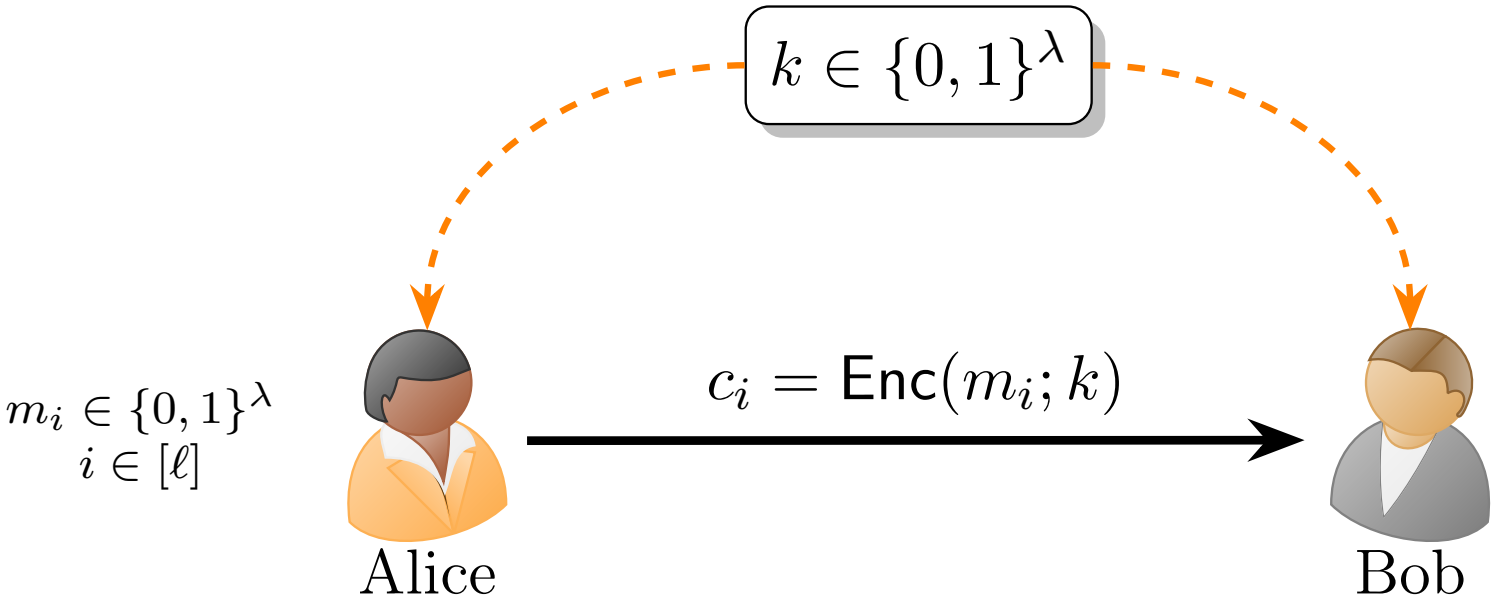

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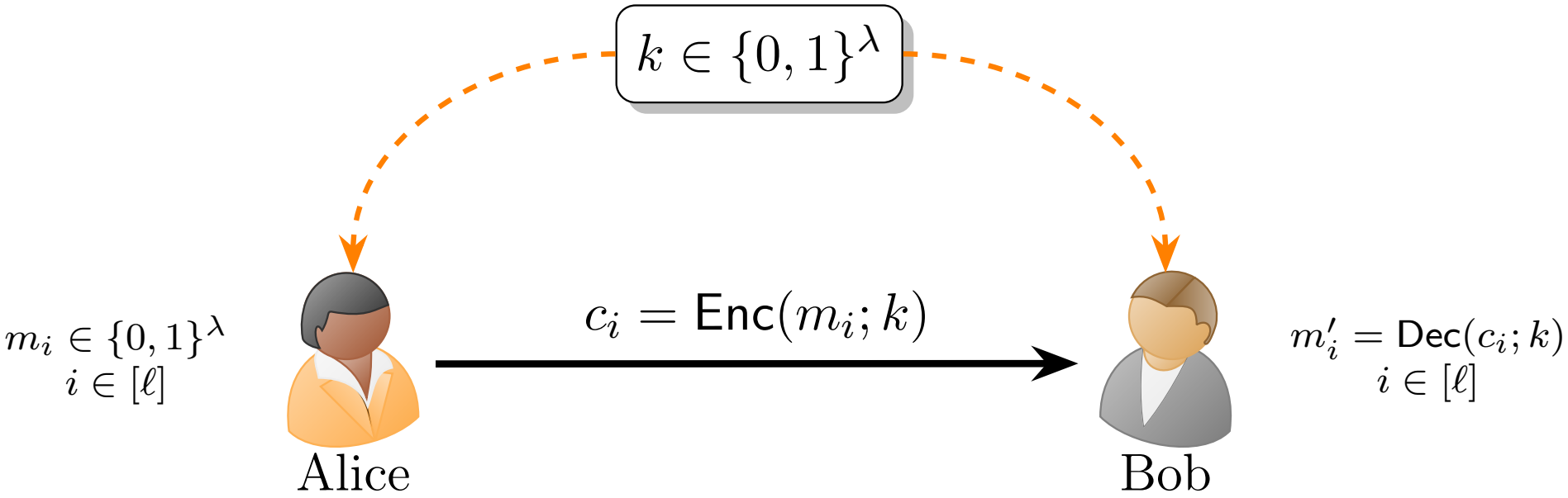

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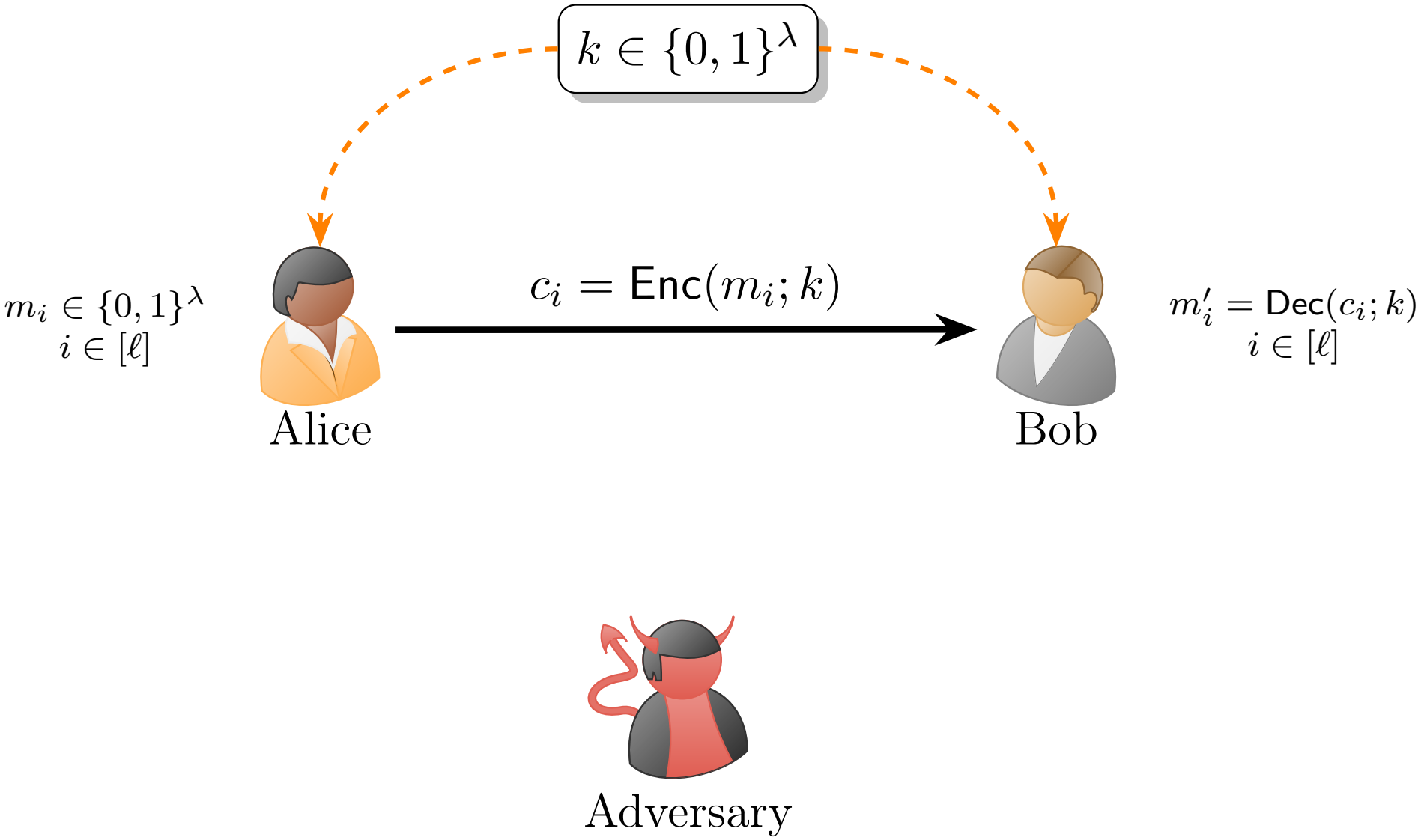

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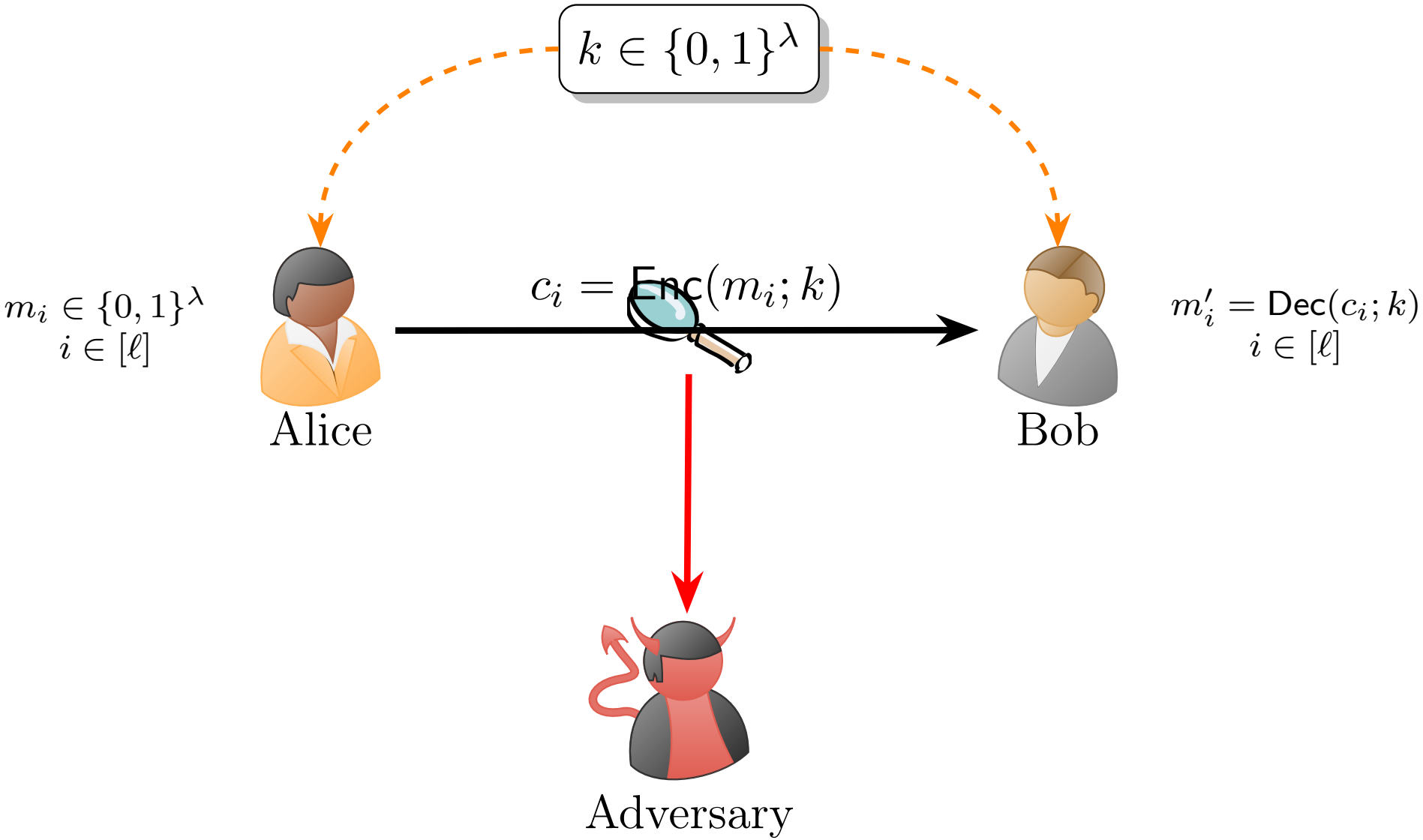

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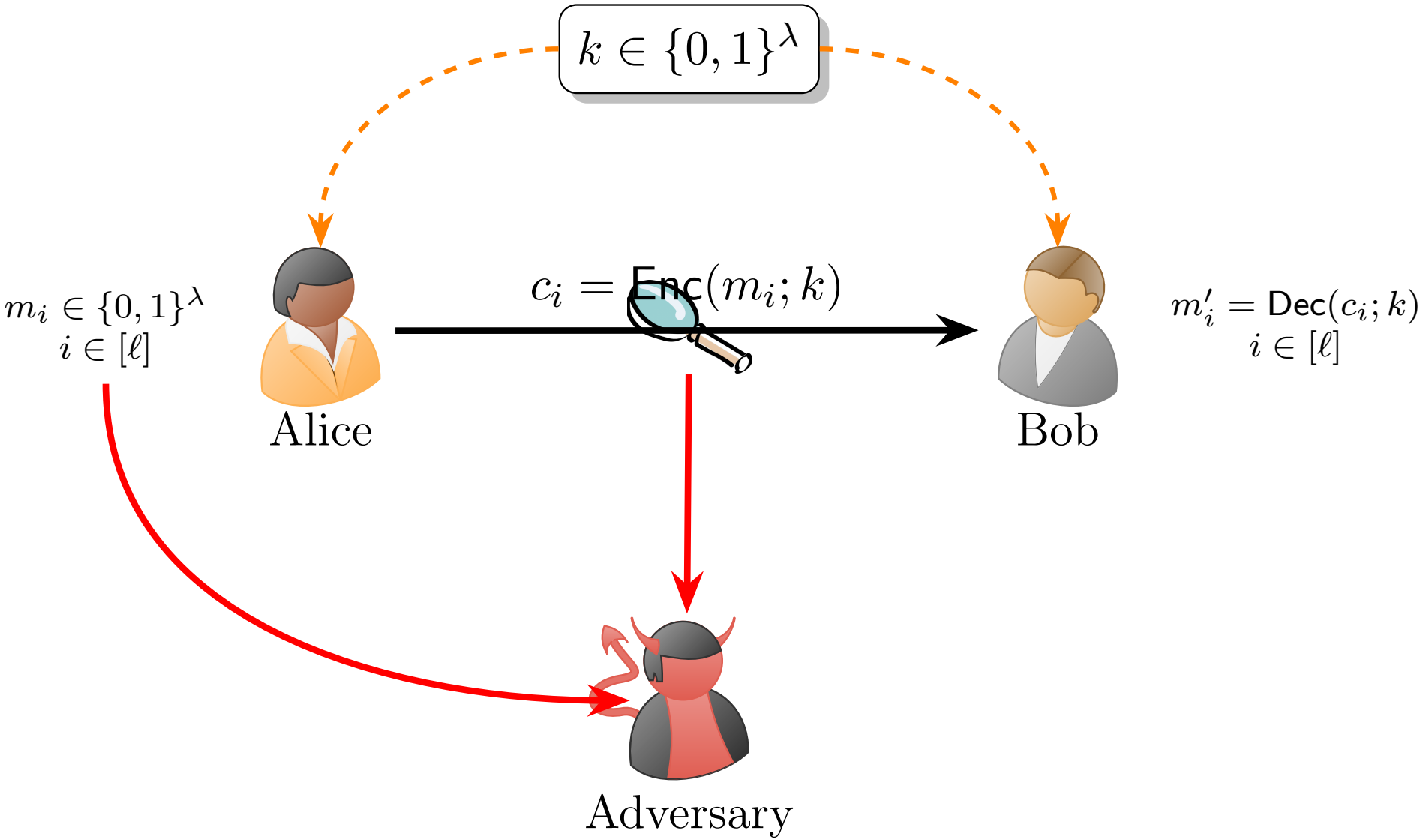

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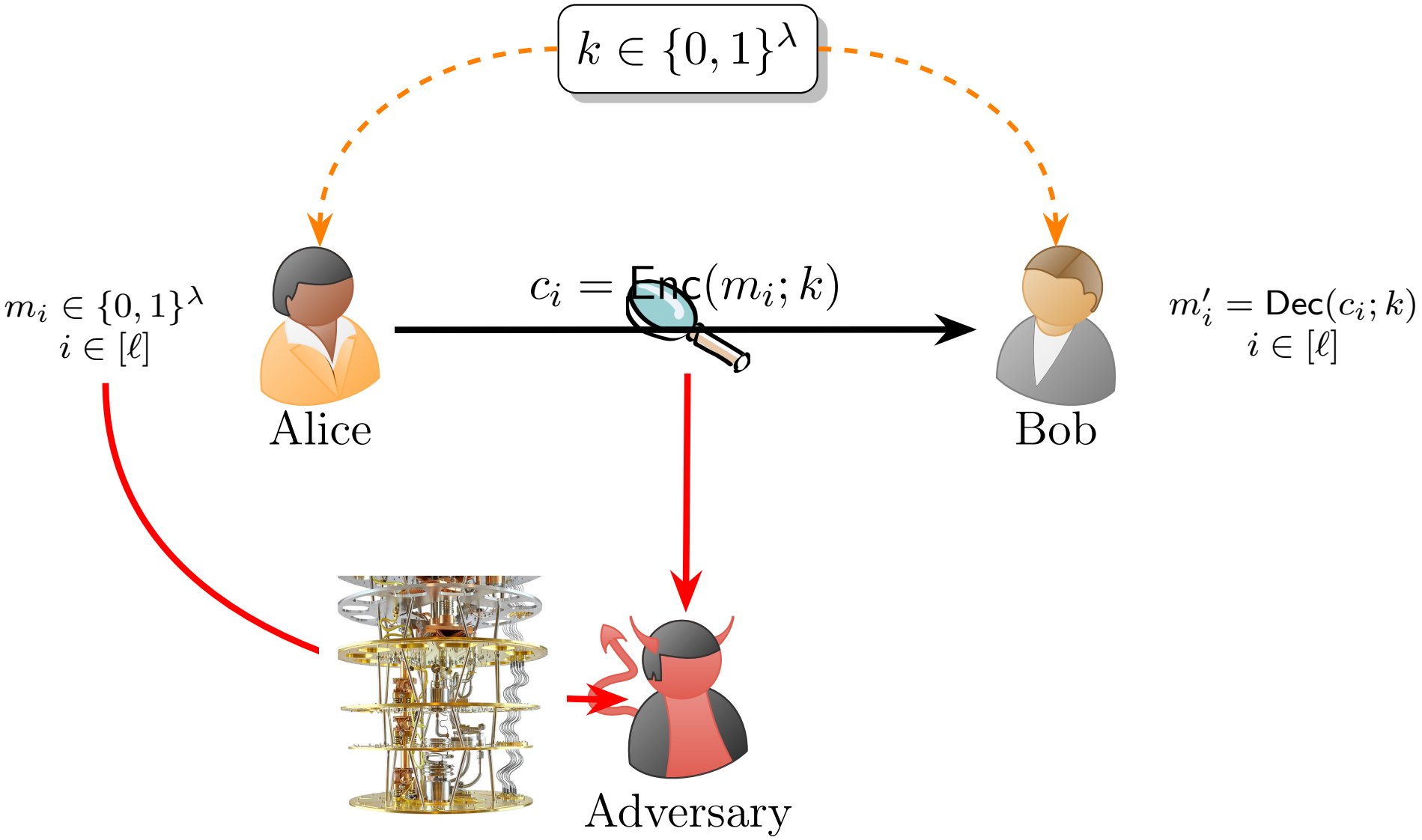

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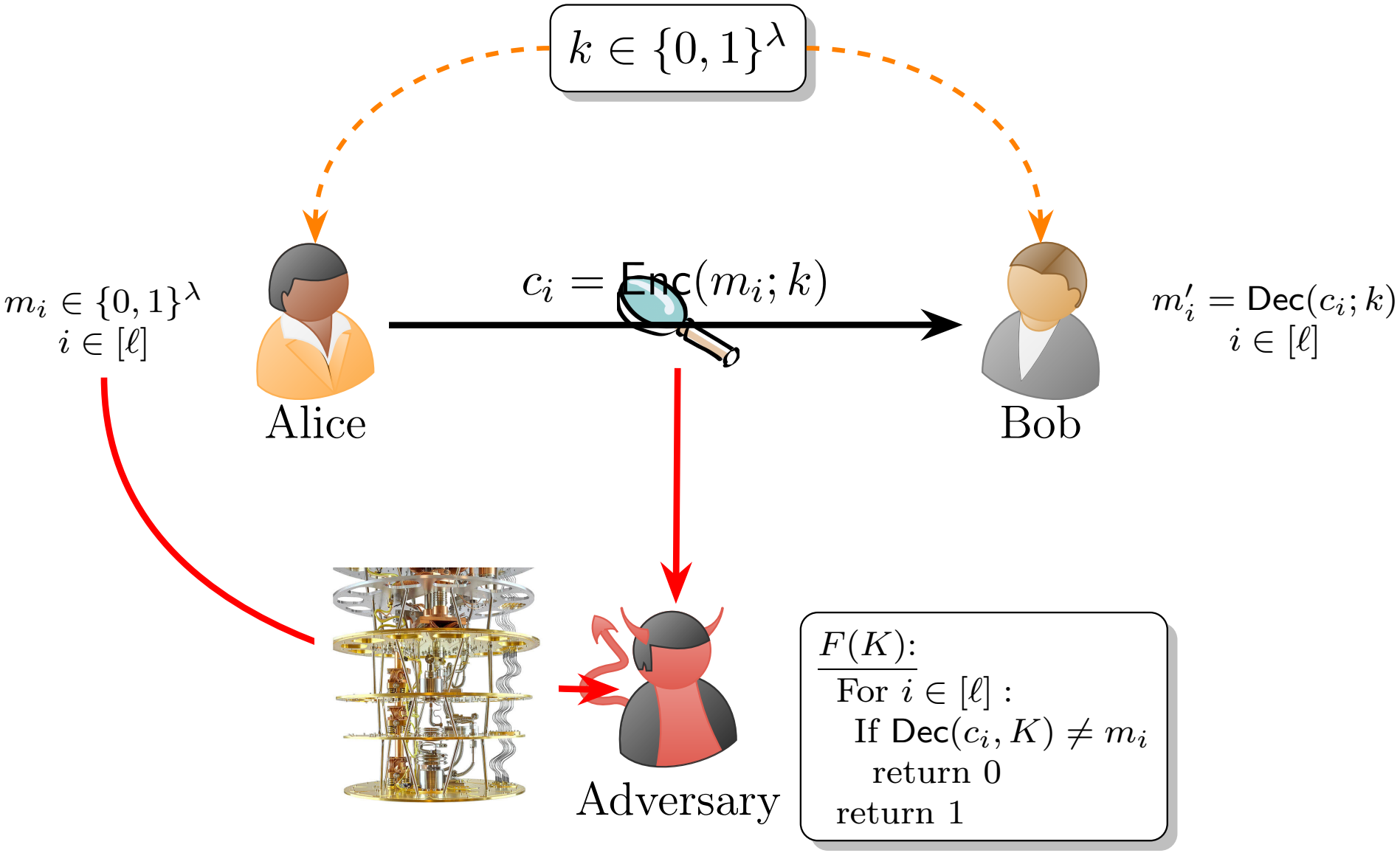

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- For symmetric-key encryption like AES, the previous attack brute-forces the key with Grover Search in $O(2^{\lambda/2})$ time on a quantum computer.

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- Another application of Grover Search: collisions in a hash function $H\colon \{0,1\}^n \to \{0,1\}^\lambda$.

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Question

Can we do better with Grover Search?

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- Run Grover Search on the function f above.

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- Setting $\ell = \Theta(2^{\lambda/3})$ gives us a constant probability of finding a collision using only $O(2^{\lambda/3})$ evaluations of H!

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     - We will see shortly that there are better algorithms for certain problems!

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- Implications: the following cryptosystems are broken by quantum computers (this list is non-exhaustive).

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  - **Factoring-based:** RSA encryption and signatures.

Quantum Computers vs. Asymmetric-key Crypto

- In 1994, Peter Shor gave a quantum algorithm that could break most (at the time) asymmetric-key cryptography in *polynomial time*.

Theorem 6 (Shor's Algorithm)

There exists a quantum algorithm with $\text{poly}(\lambda)$ *run-time for solving the following problems:*

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- Implications: the following cryptosystems are broken by quantum computers (this list is non-exhaustive).
  - **Factoring-based:** RSA encryption and signatures.
  - **DLog-based:** DDH and El-Gamal encryption, Schnorr signatures.

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  - Extended/generalized to other groups by subsequent work.

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- Use Shor's Algorithm to find this period δ, can be used to factor N in *classical* polynomial time since $x^\delta = 1 \bmod N$.

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- Can use Shor's Algorithm to find period $(\delta, 1)$, can be used to compute $\log_g(h)$ in *classical* polynomial-time since $f_{g,h}(\delta, 1) = 1_{\mathbb{G}}$.

Next Time: Post-quantum Crypto Assumptions and Constructions