CS 594 – Advanced Crypto (Spring 2026)

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Lecture 8

February 11, 2026

Linear Secret Sharing (Continued)

Blakley Secret Sharing

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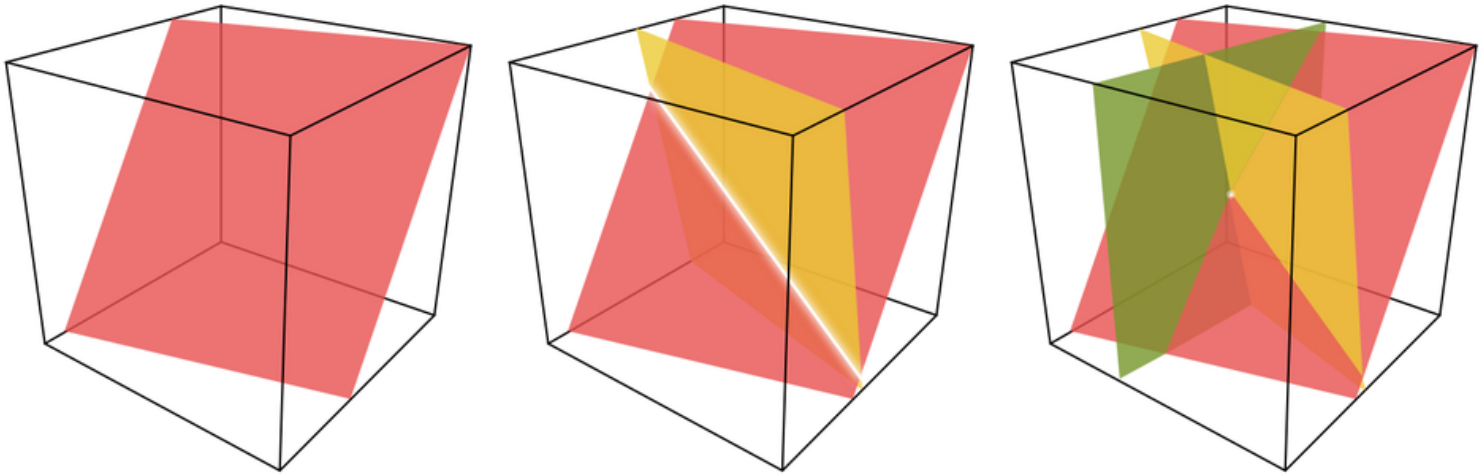

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  - Output $(\mathbf{s}_1, \ldots, \mathbf{s}_n)$.

$z \equiv a_{i,1} x_1 + \ldots + a_{i,t-1} x_{t-1} + c_i \pmod{\mathbb{F}}$

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  - Solve the linear system and output $m = x_1$.

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- Need to argue: $\forall m \in \mathbb{F}$ and any $T \subset [n]$ such that $|T| \geqslant t$, we have

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Issue!

What if the matrix is not full rank?

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- "Easy fix" ensure that the system is full rank when running Share

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$s_i := (a_{i,1}, \ldots, a_{i,t-1}, c_i)$

$|T| = t-1$

$\$ \quad \$ \quad \$$

$\Sigma \$$

non-parallel

Intersection of $(t-1)$ hyperplanes of dim $= (t-1)$? $\longrightarrow$ a line in dim $= t$

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  - Sample $t-1$ random points in $\mathbb{F}$;
  - For each party $i \in [n]$, sample $t-1$ additional random points in $\mathbb{F}$, and compute an inner product of size t.

$(m, x_2, \ldots, x_t)$

$c_i = \langle x, (-a_{i,1}, \ldots, -a_{i,t-1}, -1) \rangle$ $O(t)$

$\$\$$

$O(t)$ random samples per party

$O(t)$ $\mathbb{F}$-ops per party

$\rightarrow O(nt)$

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  - Worst case: $O(t^3)$, or a fancier algorithm which does $O(t^3/\log_q(t))$, where $q = |\mathbb{F}|$.

$$A X^T = C^T$$

$$\rightarrow X^T = A^{-1} C^T$$

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- Share Size
  - Each party stores t elements of $\mathbb{F}$

Linear Secret Sharing from Linear Codes

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↑ encoding algorithm / encoder of C

Linear Codes: Distance and Dual Codes

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Hamming wt : # non-zeros of x

Definition 2 (Distance of a Code)

Let C be a $[n, k]_q$ code. We say that C is a $[n, k, d]_q$ code if for all $\mathbf{x} \in C \setminus \{0^n\}$, we have $\mathbf{wt}(\mathbf{x}) \geqslant d$. We say that d is the *distance* of the code C.

$d = \max_{d'} \{ wt(x) \geq d' \ \forall x \in C \setminus \{0^n\} \}$

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- The generator matrix $\mathbf{H} \in \mathbb{F}^{(n-k)\times n}$ is called the *parity check matrix*

$G \cdot H^T = 0$, $H \cdot G^T = 0$

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Useful Facts About Linear Codes

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Theorem 1 (Singleton Bound)

For any $[n, k, d]_q$ code, we have $k + d \leqslant n + 1$.

non-linear:

$A(n, d, q) \leq q^{n+1-d}$

$\frac{}{\#\text{ of vectors of length } n \text{ s.t. } \mathrm{HM}(x, x') \geq d}$

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- Note that if C is a MDS code, then $C^{\perp}$ is also a MDS code.

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Lemma 1

Let C be a $[n, k]_q$ code and let $\mathbf{H}$ be the parity check matrix. Then C has distance d if and only if $\mathbf{H}$ satisfies the following properties:

1. *There exists a set of d linearly dependent columns; and*
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- Additionally, if C is a MDS code, (1) above is replaced with "all sets of at least d columns are linearly dependent."

Linear Secret Sharing from MDS Codes

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$\mathbf{u} = (m, \mathbf{v})$, where $\mathbf{v} \xleftarrow{\$} \mathbb{F}^{k-1}$

return $\mathbf{s} = \mathbf{uG} \in \mathbb{F}^n$

$s_i = \langle u, g_i \rangle \in \mathbb{F}$

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$\underline{\mathsf{Reconstruct}(s_{i_1}, \ldots, s_{i_m}):}$

Parse $\mathbf{G} = [\mathbf{g}_1, \ldots, \mathbf{g}_n]$

Solve for $\mathbf{x}$:

$\mathbf{e}_1 = \sum_{j=1}^{m} x_j \cdot \mathbf{g}_{i_j}$

return $m = \sum_{j=1}^{m} x_j \cdot s_{i_j}$

$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{F}^k$

$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (x_1 \cdots x_m)(g_{i_1}, \ldots, g_{i_m})$

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    - where $\mathbf{z} = (\alpha m + \beta m', \alpha\mathbf{v} + \beta\mathbf{v}')$.

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- This implies that every set of k columns of $\mathbf{G}$ is linearly dependent.

$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow e_1 = x_1 g_1 + x_2 g_2 + \cdots + x_k g_k$

can always solve

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- This implies that every set of k columns of $\mathbf{G}$ is linearly dependent.
- So reconstruction with at least k parties will always succeed.

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- More formally, any set of at most $k-1$ shares are uniformly random.
- Follows since any set of $\leqslant k-1$ columns of $\mathbf{G}$ are linearly independent, and by construction of the secret sharing scheme.

$T \subseteq [n]$, $|T| \leq k-1$ $\quad T = \{1, \ldots, k-1\}$

$= k-1$

$u = (m, \$, \ldots, \$)$

$(\text{Share}(m)_i)_{i \in T} = (s_1, \ldots, s_{k-1})$

$s_i = (u_i \; g_i) = m \cdot g_{i,1} + \$$

$= \$$

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$$\mathsf{RS}[\mathbb{F}, L, k] := \{(f(\ell))_{\ell \in L} \colon f(X) \in \mathbb{F}[X], \mathsf{deg}(f) \leqslant k\}$$

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  1. To share secret $m \in \mathbb{F}$, sample $\mathbf{c} \xleftarrow{\$} \mathsf{RS}[\mathbb{F}, L, t]$ such that $m = \mathsf{Interpolate}(\{(\ell_i, \mathbf{c}_i)\}_{i \in [n]})(0)$

ℓ_i is ith element of L

$(f(\ell_1), \ldots, f(\ell_n)) \in \mathbb{F}^n$

$1, \ldots, n$

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$$f(X) = m + \sum_{i=1}^{t-1} x_i X^i$$

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$L = [n]$

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The Vandermonde Matrix! (with respect to L)

$$\begin{matrix} 1 \\ \ell_0 \\ \ell_0^2 \\ \vdots \\ \ell_0^{t-1} \end{matrix} \begin{bmatrix} 1 & \cdots & 1 \\ \ell_1 & \cdots & \ell_{n-1} \\ \ell_1^2 & \cdots & \ell_{n-1}^2 \\ \vdots & \ddots & \vdots \\ \ell_1^{t-1} & \cdots & \ell_{n-1}^{t-1} \end{bmatrix} \begin{matrix} n \\ n^2 \\ \\ n^{t-1} \end{matrix}$$

$\mathbb{F}^{t \times n} \ni$ ℓ_i are all distinct

$\omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$

- If I have t valid shares

$S_{i_1}, \ldots, S_{i_t}$, then

$$\sum_{k=1}^{t} \lambda_{i_k} \cdot S_{i_k} = 0$$

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An *access structure* is a monotone collection $\Gamma \subseteq 2^{[n]}$ of non-empty subsets of $[n]$. $\{1,2,3,4\}$

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- (Security) For all $m \neq m' \in \mathcal{M}$ and any $T \notin \Gamma$, the following two distributions are identical:

$$(\mathsf{Share}(m)_i)_{i \in T} \equiv \left(\mathsf{Share}(m')_i\right)_{i \in T}.$$

Next Time: Verifiable Secret Sharing