CS 594 – Advanced Crypto (Spring 2026)

Alex Block

Lecture 9

February 16, 2026

Verifiable Secret Sharing

The Story So Far

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Dealer D

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P_5

P_1

P_2

P_4

P_3

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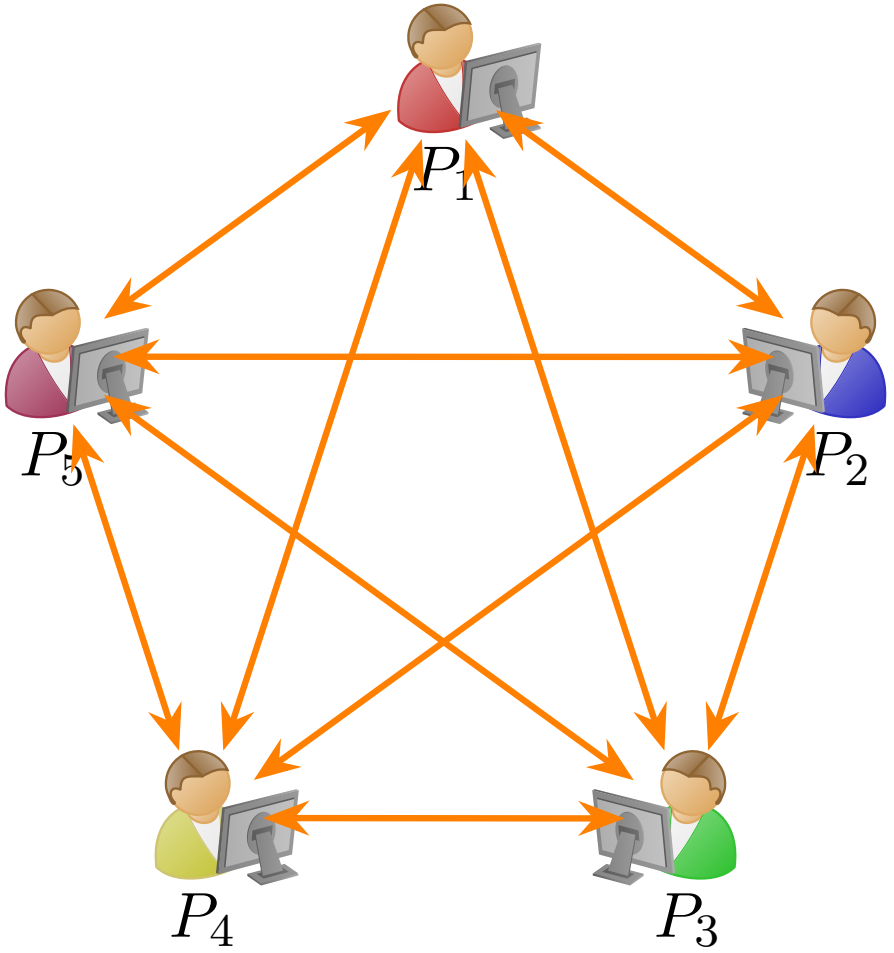

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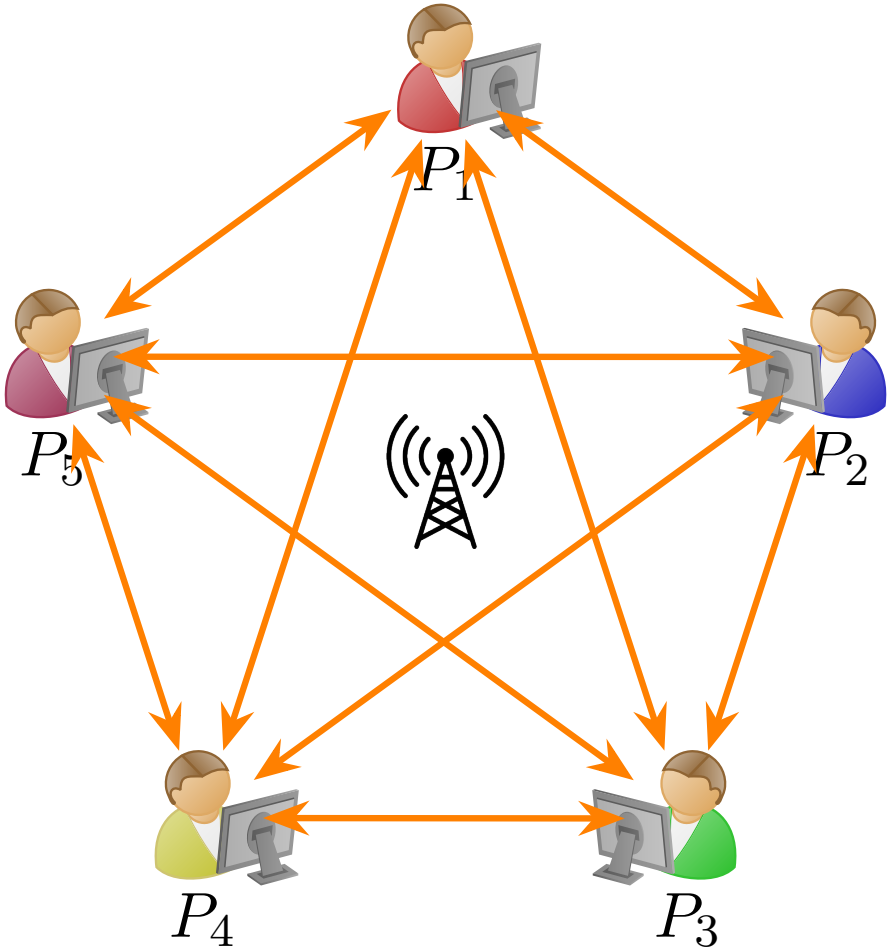

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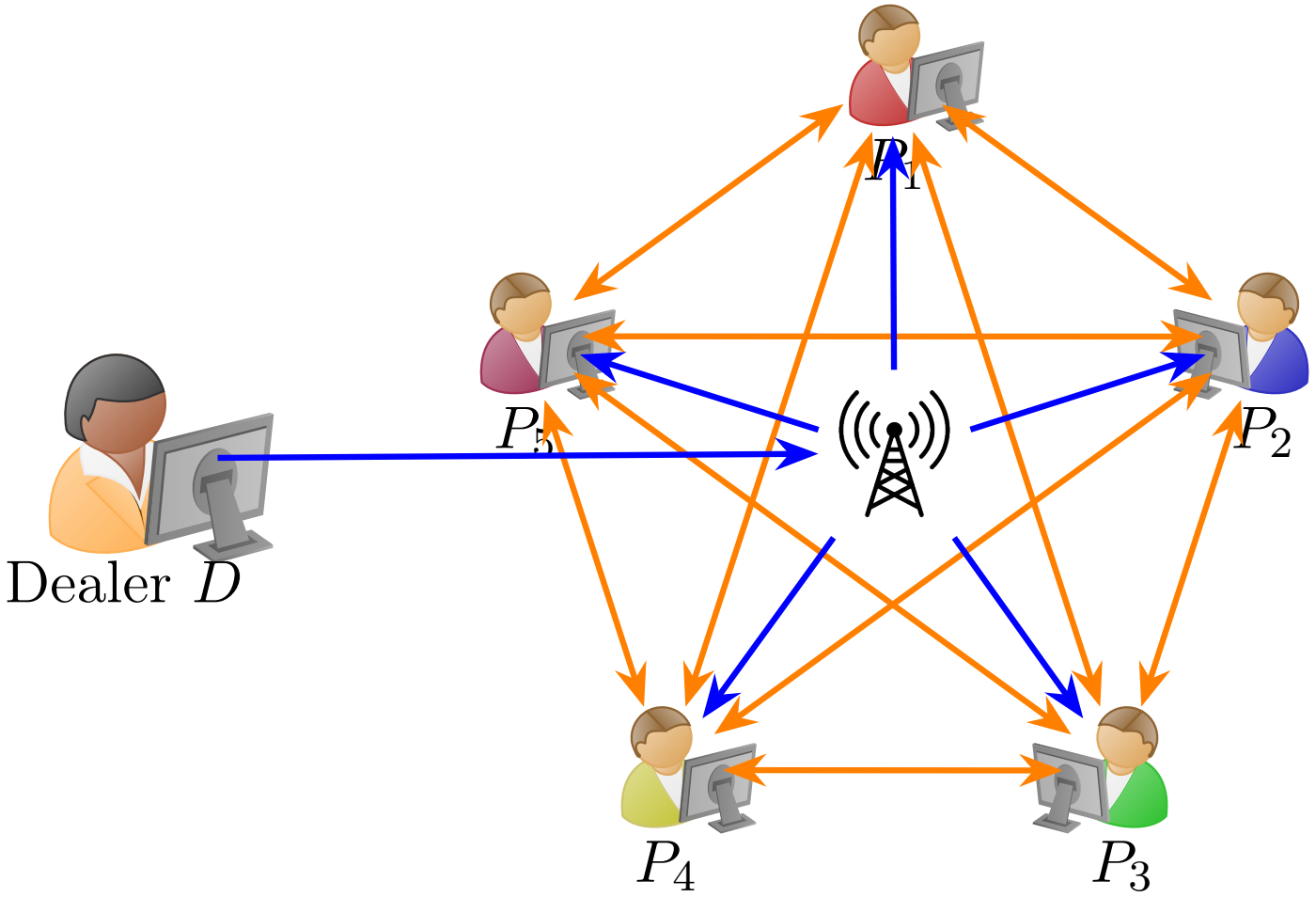

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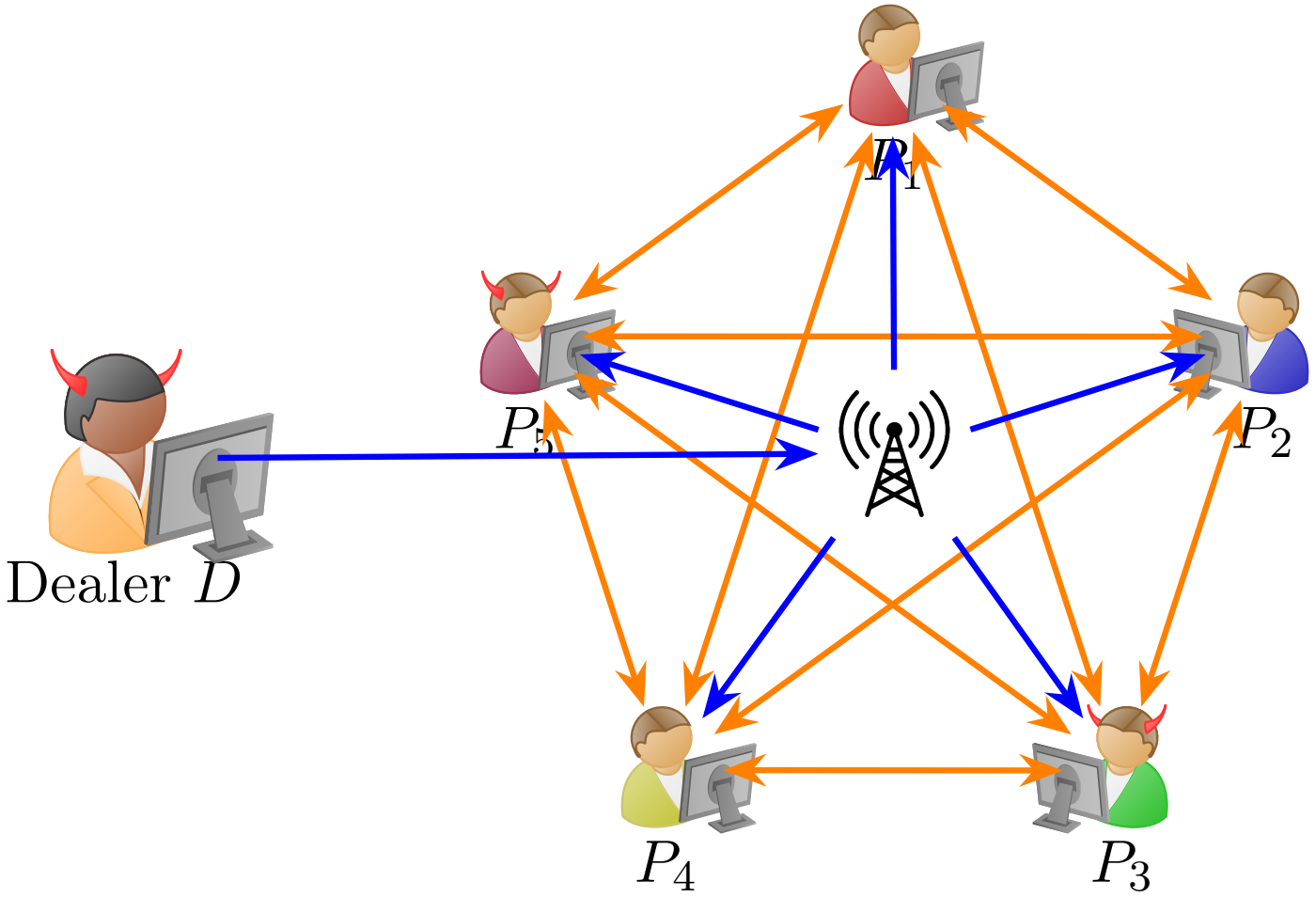

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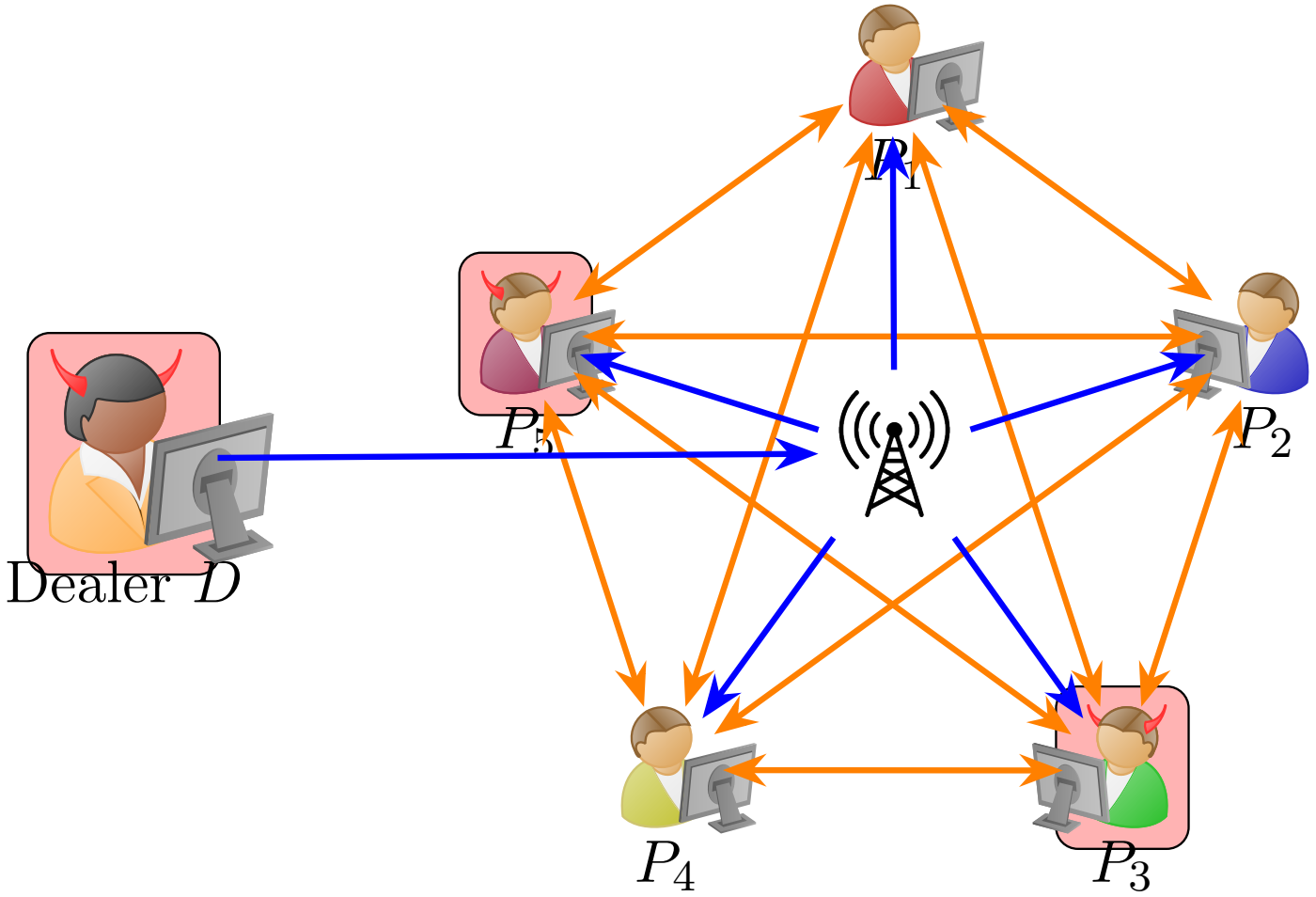

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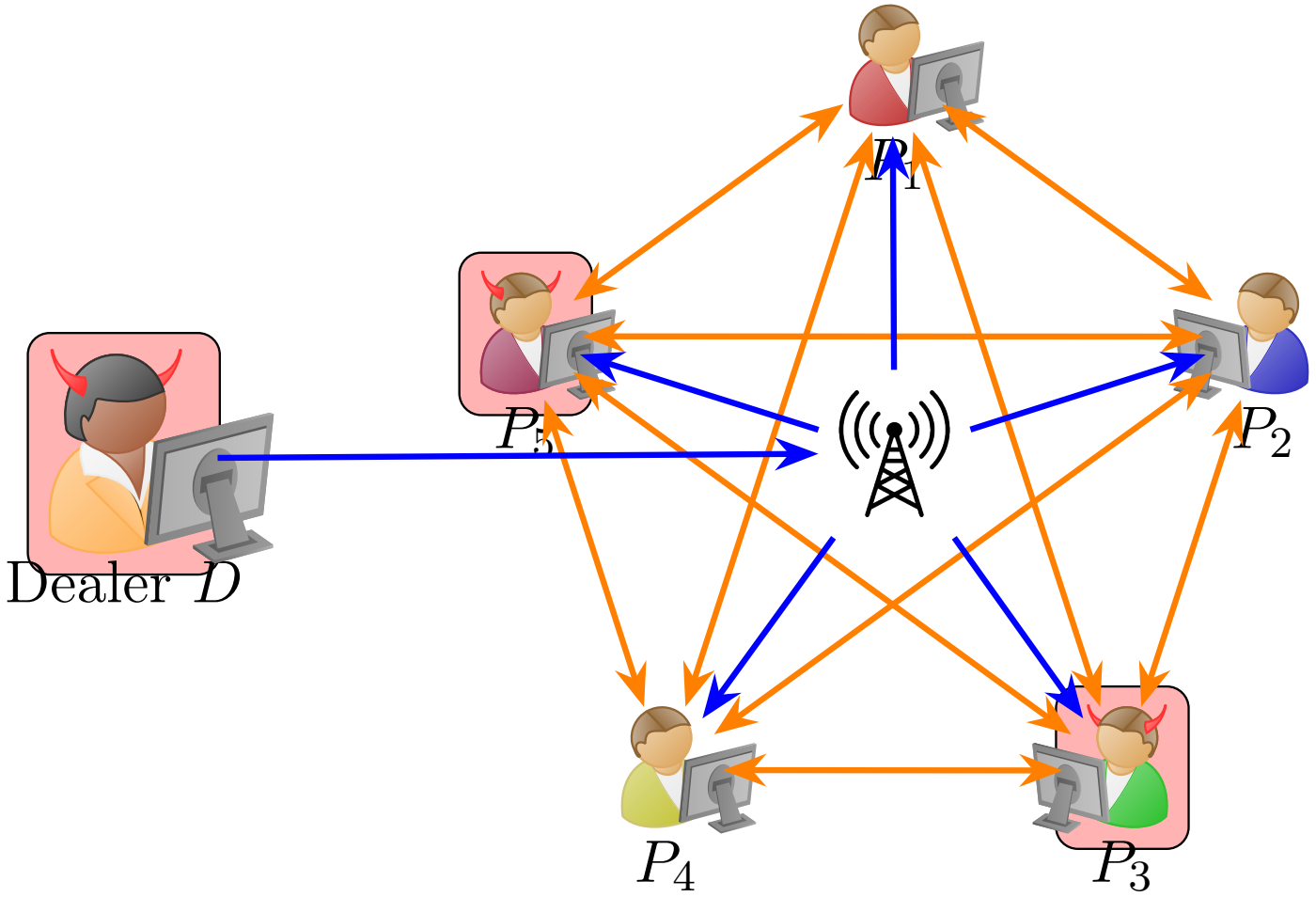

How do we know the dealer has given honest parties *valid* shares?

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- VSS also extends Sharing and Reconstruction to *protocols*.

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- **Privacy**: if D is honest, corrupt parties learn *nothing* about s during Π_{S}.
- **Strong Commitment**: if D is *corrupt*, the execution of Π_{S} with honest parties defines some secret value $\widetilde{s}$ such that (1) $\widetilde{s}$ is shared among $\mathcal{P}$ according to Share, and (2) all honest parties output $\widetilde{s}$ at the end of Π_{R}.

(2) ✗

Weak (Type-I) : doesn't hold w.r.t SS

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  - Security with abort or guaranteed output delivery.

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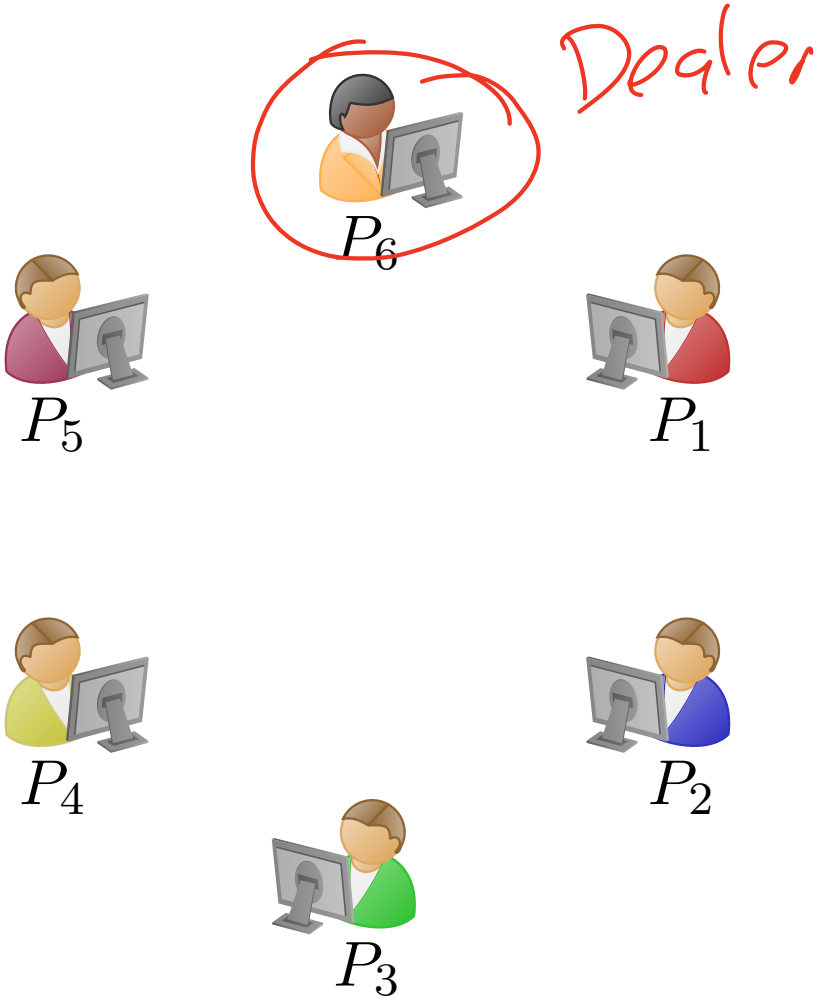
- Synchronous Communication.
- Secure Broadcast and Secure Point-to-point Channels.
- Adversary: static, malicious, computationally (un)bounded.

VSS Adversary and Communication Model



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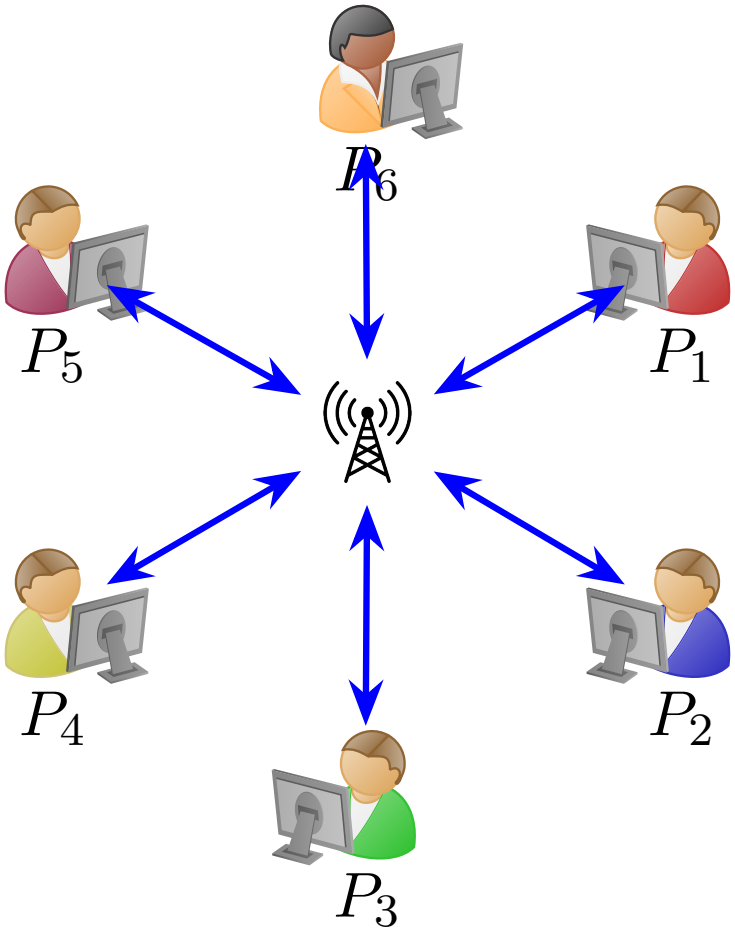

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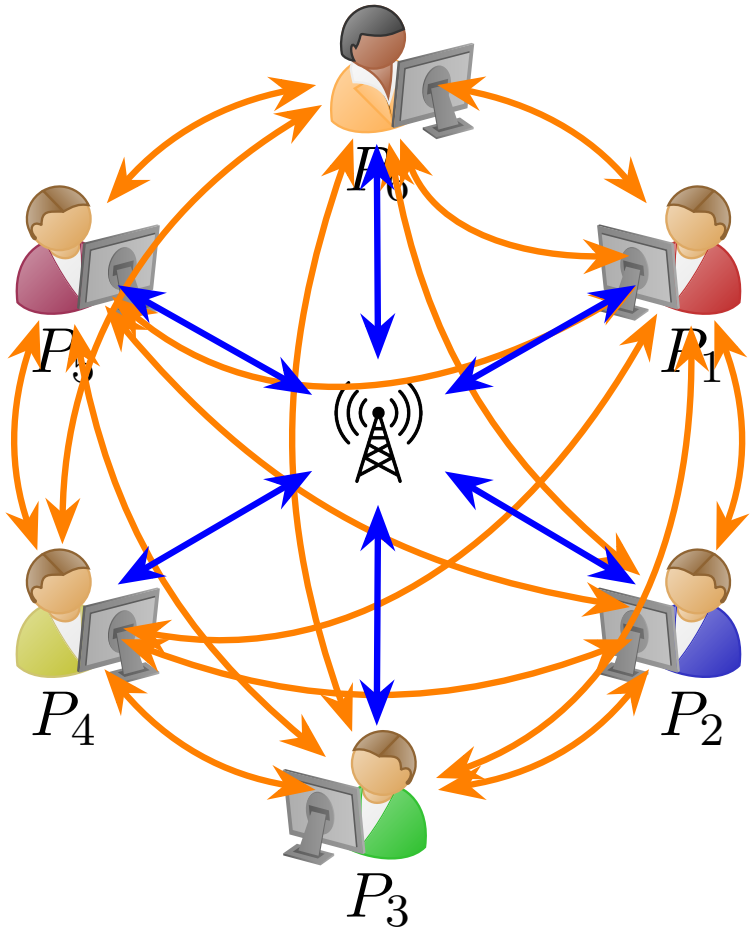

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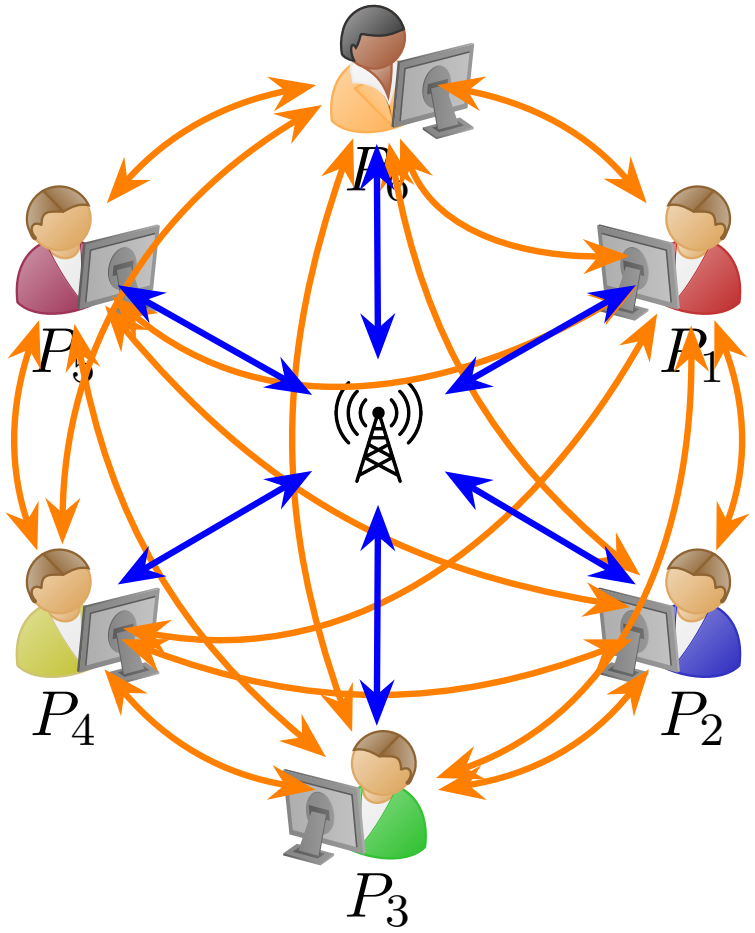

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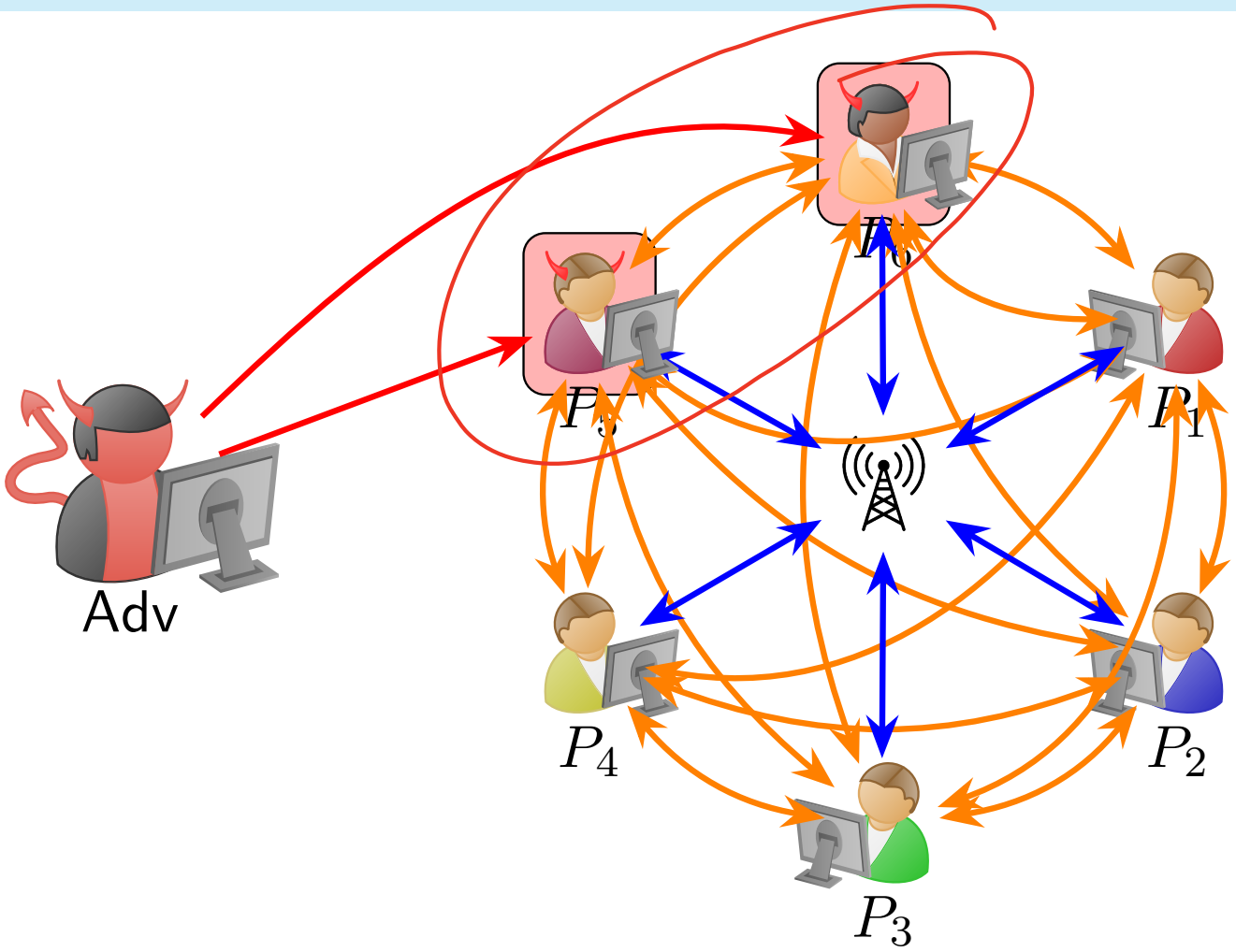

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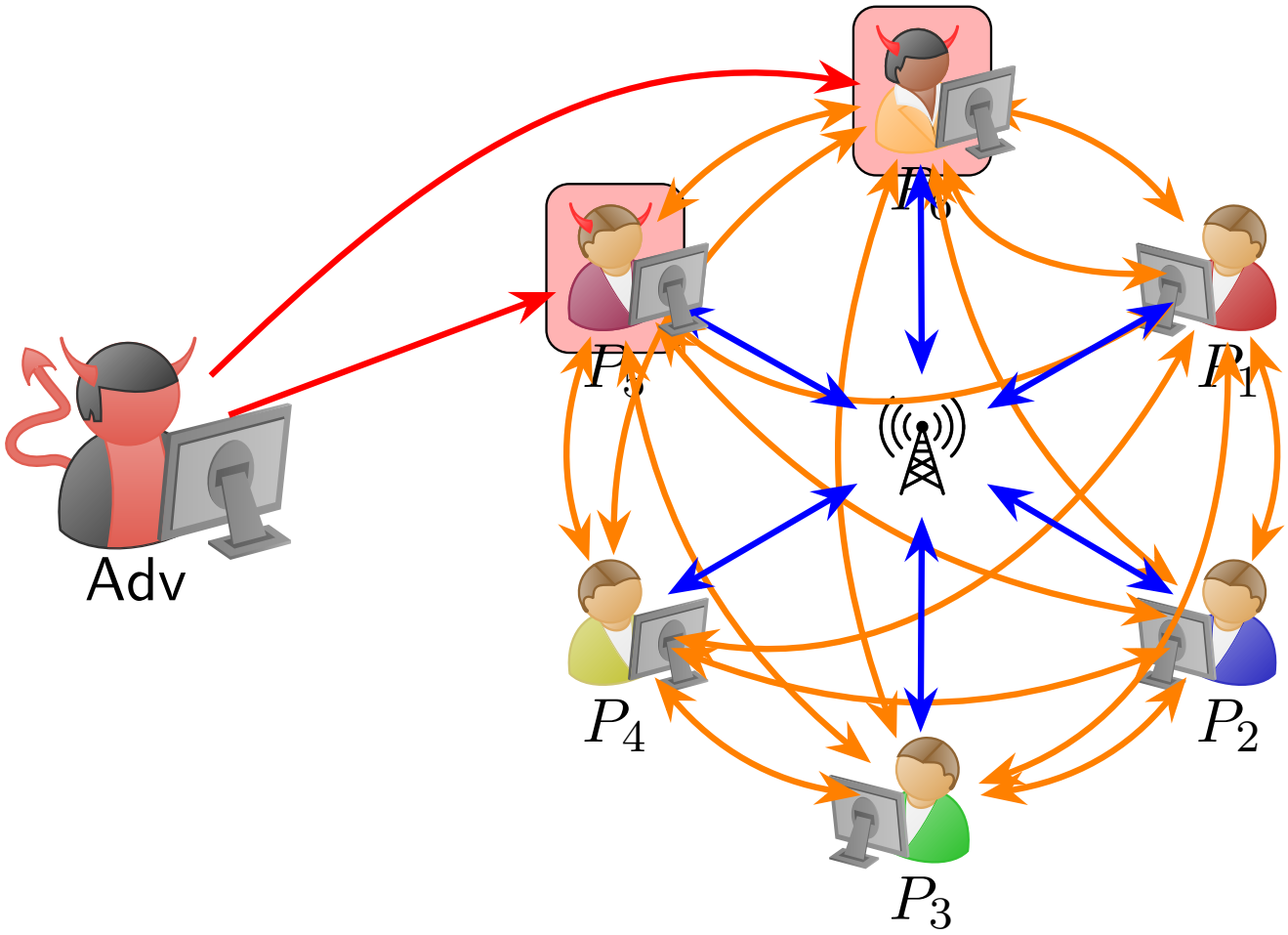

(R, R') VSS

Both Π_{S} and Π_{R} proceed for at most R rounds, where the broadcast channel is used for at most R' rounds.

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$(g, g \cdot a)$ for rand. a, then $\approx_c (g, h)$ $h \xleftarrow{\$} \mathbb{G}$

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g·p in additive group

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- Idea: Perform Shamir over $\mathbb{Z}_p$, commit to the coefficients using g.

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$p > n$

(n, t) fixed, $\mathcal{M} = \mathbb{Z}_p$, $\mathcal{S} = \mathbb{Z}_p$, $g \in \mathbb{G}$ of order p

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  - Broadcast $(C_0, \ldots, C_{t-1})$ to all parties.

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(1,1) VSS

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  - Broadcast $(C_0, \ldots, C_{t-1})$ to all parties.
  - Compute $s_i = f(i)$ for $i \in [n]$.
  - Output secret shares $(s_1, \ldots, s_n)$.

<- P2P channel

$(s_i, (C_0, \ldots, C_{t-1})) \longrightarrow P_i$

each party i checks $g^{s_i} = \prod_{k=0}^{t-1} C_k^{i^k}$

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- Return $\mathsf{Shamir.Reconstruct}(s_{i_1}, \ldots, s_{i_t})$.

$s_i = f(i) = \sum_{k=0}^{t-1} f_k \cdot i^k$

$g \cdot s_i = \sum_{k=0}^{t-1} g \cdot f_k \cdot i^k$ → C_k if D_i is honest

Feldman's NIVSS: Correctness

VSS Correctness

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(Shamir, done!

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- For (2), if one party is dishonest, then checking consistency with $C_0, \ldots, C_{t-1}$ will fail (unless Adv can break the discrete-log assumption in $\mathbb{G}$). $\tilde{s}_i$ s.t. $g \cdot \tilde{s}_i = \sum_{k=0}^{t-1} C_k \cdot i^k$ but $\tilde{s}_i \neq f(i)$

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  - Collect the first t shares which authenticate, output $\mathsf{Shamir.Reconstruct}$ of those values.

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- For (1), if D is honest, it honestly shares s as $f(i)$ for $i \in [n]$, where $f(X) = \sum_{j=0}^{t-1} f_j X^j$, $f_0 = s$, and $f_j \xleftarrow{\$} \mathbb{Z}_p$ for $j \in [t-1]$.
- For (2), if one party is dishonest, then checking consistency with $C_0, \ldots, C_{t-1}$ will fail (unless Adv can break the discrete-log assumption in $\mathbb{G}$).
  - If a share is not authenticated, kick that party out of the protocol.
  - Collect the first t shares which authenticate, output $\mathsf{Shamir.Reconstruct}$ of those values.
- Honesty of D implies the honest parties output s. □

Feldman's VSS: Privacy

VSS Privacy

If D is honest, then Adv (i.e., all corrupt parties) learn nothing about s during Π_{S}.

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$C_0 = g \cdot s$

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  - E.g., if $s \equiv 0 \bmod p$, then $g \cdot s = 1$.

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- If s is *not random*, then Adv learns some information about s.
  - E.g., if $s \equiv 0 \bmod p$, then $g \cdot s = 1$.
- In the context of many uses of VSS (and Feldman's), this is just fine, so we ignore it for now. □

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Proof.

- Without loss of generality, we can examine Π_{R} from the view of all honest parties P_i for $i \in H := [n] \setminus T$, where $T \subseteq [n]$ of size $|T| \leqslant t-1$ is the set of corrupt parties.

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- During Π_{R}, each P_i for $i \in H$ authenticates the shares s_i.
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    1. shares some secret $\widetilde{s}$ according to Shamir.Share and correctly constructs C_k for $k \in \{0, \ldots, t-1\}$; or
    2. can break the discrete-log assumption in $\mathbb{G}$.
- The statement follows. □

Adv. D:

Broadcast $(\tilde{c}_0, \ldots, \tilde{c}_{t-1}) \in G^t$

- P_i gets $\tilde{s}_i$

Honest parties $h \in H$:

$$\tilde{s}_h \cdot g = \sum_{k=0}^{t-1} \tilde{c}_k \cdot h^k \in G_q \qquad = g \cdot \sum_{k=0}^{t-1} \tilde{\alpha}_k \cdot h^k$$

$$g \cdot \tilde{\alpha}_k \to g(\tilde{\alpha}_k \cdot h^k)$$

Pedersen's NIVSS

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- A relatively straightforward modification of Feldman's VSS.
- Hides the secret s from even a *computationally unbounded* adversary Adv.
- Adversary Adv still must be computationally bounded when the dealer is corrupt, as ~~security~~ relies on hardness of discrete-log.

Strong Comm.

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  - Broadcast $(C_0, \ldots, C_{t-1})$ to all parties.
  - Compute $S_i = (s_i, s'_i) = (f(i), f'(i))$ for $i \in [n]$.
  - Output secret shares $(s_1, \ldots, s_n)$.

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- Let $s_{i_1}, \ldots, s_{i_t}$ be the (first) t shares that correctly authenticated.
- Return $\mathsf{Shamir.Reconstruct}(s_{i_1}, \ldots, s_{i_t})$.

Pedersen's NIVSS: Correctness and Strong Commitment

VSS Correctness

If D is honest, then (1) $s \in \mathbb{Z}_p$ is shared according to Shamir.Share and (2) all honest parties output s at the end of Π_{R}.

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If D is *corrupt*, then the execution of Π_{S} uniquely defines some secret value $\widetilde{s}$ such that $\widetilde{s}$ is shared among $\mathcal{P}$ according to Shamir.Share and all honest parties output $\widetilde{s}$ at the end of Π_{R}.

By Feldman's scheme

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Proof.

Follows directly from Feldman's VSS. Crucially, for Strong Commitment, Adv must be computationally bounded so they cannot break the discrete-log assumption in $\mathbb{G}$. □

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- We argue that even if Adv is computationally *unbounded*, Adv learns nothing about secret s shared by honest dealer D.
- The values broadcast to all parties: $C_k = g \cdot f_k + h \cdot f'_k$ for $k \in \{0, 1, \ldots, t-1\}$, where $f_0 = s$ and all other f_i, f'_j sampled uniformly at random.

$k > 0 \quad C_k \equiv U_G$

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- Each $C_k \equiv U_{\mathbb{G}}$ by construction. C_0 : fixed + random = random

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- Each $C_k \equiv U_{\mathbb{G}}$ by construction.
- Moreover, since Adv corrupts at most $t-1$ parties, the shares (s_i, s'_i) information-theoretically hide the polynomial interpolating s. □

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Upper and Lower Bounds for VSS

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- The field is much larger and very deeply related to other parts of crypto, especially
  - MPC; and
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- I'll give you a brief overview of known upper and lower bounds, and some open questions related to VSS.

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Theorem 2

Perfectly secure VSS implies the existence of perfectly-secure reliable broadcast over P2P channels.

Theorem 3 (Pease, Shostak, Lamport)

Perfectly secure reliable broadcast over P2P channels requires $n > 3t$.

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Let $R \geqslant 1$ and $R' \leqslant R$ be integers. Then:

- *If $R = 1$, there does not exist any perfectly secure VSS scheme with (R, R') rounds in the sharing phase if either (a) $t > 1$ (for arbitrary n), or (b) $t = 1$ and $n \leqslant 4$*

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- *If $R = 2$, then perfectly secure VSS with (R, R') rounds in the sharing phase exists only if $n > 4t$.*
- *If $R \geqslant 3$, then perfectly secure VSS with (R, R') rounds in the sharing phase exists only of $n > 3t$.*

Upper Bounds for VSS

Scheme	n	Round Complexity	Type	Sharing Semantic	Algebraic Structure	Communication Complexity
7BGW-VSS [12]	$n > 3t$	$(7, 5)$	Type-II	Shamir	$\mathbb{F}$	$\mathcal{O}(n^2 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^2 \log \lvert\mathbb{F}\rvert))$
5BGW-VSS [33]	$n > 3t$	$(5, 3)$	Type-II	Shamir	$\mathbb{F}$	$\mathcal{O}(n^2 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^2 \log \lvert\mathbb{F}\rvert))$
4GIKR-VSS [33]	$n > 3t$	$(4, 3)$	Type-II	Shamir	$\mathbb{F}$	$\mathcal{O}(n^2 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^2 \log \lvert\mathbb{F}\rvert))$
3GIKR-VSS [33]	$n > 3t$	$(3, 2)$	Type-II	RSS	$\mathbb{G}$	$\mathcal{O}(n \cdot \binom{n}{t} \log \lvert\mathbb{G}\rvert + \mathcal{BC}(n \cdot \binom{n}{t} \log \lvert\mathbb{G}\rvert))$
3FGGRS-VSS [32]	$n > 3t$	$(3, 2)$	Type-I	Not Applicable	$\mathbb{F}$	$\mathcal{O}(n^3 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^3 \log \lvert\mathbb{F}\rvert))$
3KKK-VSS [39]	$n > 3t$	$(3, 1)$	Type-II	Shamir	$\mathbb{F}$	$\mathcal{O}(n^3 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^3 \log \lvert\mathbb{F}\rvert))$
3AKP-VSS [3]	$n > 3t$	$(3, 2)$	Type-II	Shamir	$\mathbb{F}$	$\mathcal{O}(n^3 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^3 \log \lvert\mathbb{F}\rvert))$
2GIKR-VSS [33]	$n > 4t$	$(2, 1)$	Type-II	Shamir	$\mathbb{F}$	$\mathcal{O}(n^2 \log \lvert\mathbb{F}\rvert + \mathcal{BC}(n^2 \log \lvert\mathbb{F}\rvert))$
1GIKR-VSS [33]	$n = 5, t = 1$	$(1, 0)$	Type-I	Not Applicable	$\mathbb{F}$	$\mathcal{O}(n \log \lvert\mathbb{F}\rvert)$

Table 1: Summary of the sharing phase of the perfectly-secure VSS schemes, with $\mathcal{BC}$ denoting communication over the broadcast channel.

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- Taken from https://eprint.iacr.org/2021/445.pdf

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+ Comp. unbounded adversaries

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- Can replace Secure Broadcast with *reliable broadcast protocols* over P2P channels.

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- Can also model the network as *hybrid*: async phases and sync phases.

(Some) Open Problems in Perfectly Secure VSS

- *Communication Complexity* of VSS (i.e., number of bits sent during the protocols).
  - 3-round *efficient* VSS schemes have $n\times$ more communication than $\geqslant 4$ round schemes. Can this gap be closed?
  - Better than trivial lower bound on communication complexity of perfectly secure VSS?

- *Broadcast Complexity*: number of bits sent via broadcast.
  - All schemes with $n > 3t$ have broadcast complexity $O(L \cdot n^3 \log |\mathbb{F}|)$, where L is the number of secrets shared by the scheme.
  - For asynchronous VSS, there is a scheme with $O(n^2 \cdot \log(n))$ broadcast complexity (independent of L).
  - Can we get this same bound for the synchronous setting?

Next Time: Function Secret Sharing