Topological implications of negative curvature for biological and social networks

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Joint work with

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- Nasim Mobasher (UIC)
Outline of talk

1. **Introduction**

2. **Basic definitions and notations**

3. **Computing hyperbolicity for real networks**

4. **Implications of hyperbolicity of networks**
   - Hyperbolicity and crosstalk in regulatory networks
   - Geodesic triangles and crosstalk paths
   - Identifying essential edges and nodes in regulatory networks
   - A social network application
Introduction
Various network measures

Graph-theoretical analysis leads to useful insights for many complex systems, such as

- World-Wide Web
- social network of jazz musicians
- metabolic networks
- protein-protein interaction networks

Examples of useful network measures for such analyses

- **degree based**, *e.g.*
  - maximum/minimum/average degree, degree distribution, …
- **connectivity based**, *e.g.*
  - clustering coefficient, largest cliques or densest sub-graphs, …
- **geodesic based**, *e.g.*
  - diameter, betweenness centrality, …
- **other more complex measures**
network curvature as a network measure

network measure for this talk

network curvature via (Gromov) hyperbolicity measure

- originally proposed by Gromov in 1987 in the context of group theory
  - observed that many results concerning the fundamental group of a Riemann surface hold true in a more general context
  - defined for infinite continuous metric space with bounded local geometry via properties of geodesics
  - can be related to standard scalar curvature of Hyperbolic manifold

- adopted to finite graphs using a so-called 4-node condition
Outline of talk

1. Introduction

2. Basic definitions and notations

3. Computing hyperbolicity for real networks

4. Implications of hyperbolicity of networks
   - Hyperbolicity and crosstalk in regulatory networks
   - Geodesic triangles and crosstalk paths
   - Identifying essential edges and nodes in regulatory networks
   - A social network application
Basic definitions and notations
Graphs, geodesics and related notations

\[ G = (V, E) \] connected undirected graph of \( n \geq 4 \) nodes
\[ P \ni u \leftrightarrow v \] path \( P = (u_0, u_1, \ldots, u_{k-1}, u_k) \) between nodes \( u \) and \( v \)
\( \ell(P) \) length (number of edges) of the path \( u \leftrightarrow v \)
\[ P \ni u_i \leftrightarrow u_j \] sub-path \( (u_i, u_{i+1}, \ldots, u_j) \) of \( P \) between nodes \( u_i \) and \( u_j \)
\[ u \leftrightarrow v \] a shortest path between nodes \( u \) and \( v \)
\( d_{u,v} \) length of a shortest path between nodes \( u \) and \( v \)

\[ u_2 \leftrightarrow u_6 \] is the path \( P = (u_2, u_4, u_5, u_6) \)
\( \ell(P) = 3 \)
\( d_{u_2,u_6} = 2 \)
Consider four nodes $u_1, u_2, u_3, u_4$ and the six shortest paths among pairs of these nodes.
Basic definitions and notations

4 node condition (Gromov, 1987)

Consider four nodes $u_1, u_2, u_3, u_4$ and the six shortest paths among pairs of these nodes.

Assume, without loss of generality, that

\[
\begin{align*}
    d_{u_1,u_4} + d_{u_2,u_3} &\geq d_{u_1,u_3} + d_{u_2,u_4} \\
    &= L \\
    &\geq d_{u_1,u_2} + d_{u_3,u_4} \\
    &= M \\
    &\geq S
\end{align*}
\]
Consider four nodes $u_1, u_2, u_3, u_4$ and the six shortest paths among pairs of these nodes.

Assume, without loss of generality, that

$$d_{u_1,u_4} + d_{u_2,u_3} \geq d_{u_1,u_3} + d_{u_2,u_4} \geq d_{u_1,u_2} + d_{u_3,u_4} = L$$

Let $\delta_{u_1,u_2,u_3,u_4} = \frac{L-M}{2}$
Consider four nodes $u_1, u_2, u_3, u_4$ and the six shortest paths among pairs of these nodes.

Assume, without loss of generality, that

\[
\begin{align*}
\delta_{u_1, u_4} + \delta_{u_2, u_3} &\geq \delta_{u_1, u_3} + \delta_{u_2, u_4} \\
&\geq \delta_{u_1, u_2} + \delta_{u_3, u_4}
\end{align*}
\]

Let $\delta_{u_1, u_2, u_3, u_4} = \frac{L - M}{2}$

Definition (hyperbolicity of G)

\[
\delta(G) = \max_{u_1, u_2, u_3, u_4} \left\{ \delta_{u_1, u_2, u_3, u_4} \right\}
\]
Basic definitions and notations
Hyperbolic graphs (graphs of negative curvature)

Definition (Δ-hyperbolic graphs)

$G$ is $\Delta$-hyperbolic provided $\delta(G) \leq \Delta$

Definition (Hyperbolic graphs)

If $\Delta$ is a constant independent of graph parameters, then a $\Delta$-hyperbolic graph is simply called a hyperbolic graph
Basic definitions and notations
Hyperbolic graphs (graphs of negative curvature)

Definition ($\Delta$-hyperbolic graphs)
$G$ is $\Delta$-hyperbolic provided $\delta(G) \leq \Delta$

Definition (Hyperbolic graphs)
If $\Delta$ is a constant independent of graph parameters, then a $\Delta$-hyperbolic graph is simply called a hyperbolic graph.

Example (Hyperbolic and non-hyperbolic graphs)

- **Tree**: $\Delta(G) = 0$
  hyperbolic graph
- **Chordal (triangulated) graph**: $\Delta(G) = 1/2$
  hyperbolic graph
- **Simple cycle**: $\Delta(G) = \lceil n/4 \rceil$
  non-hyperbolic graph

$n = 10$
Are there real-world networks that are hyperbolic?

Yes, for example:

- Preferential attachment networks were shown to be scaled hyperbolic
  - [Jonckheere and Lohsoonthorn, 2004; Jonckheere, Lohsoonthorn and Bonahon, 2007]

- Networks of high power transceivers in a wireless sensor network were empirically observed to have a tendency to be hyperbolic
  - [Ariaei, Lou, Jonckheere, Krishnamachari and Zuniga, 2008]

- Communication networks at the IP layer and at other levels were empirically observed to be hyperbolic
  - [Narayan and Saniee, 2011]

- Extreme congestion at a very limited number of nodes in a very large traffic network was shown to be caused due to hyperbolicity of the network together with minimum length routing
  - [Jonckheere, Loua, Bonahona and Baryshnikova, 2011]

- Topology of Internet can be effectively mapped to a hyperbolic space
  - [Bogun, Papadopoulos and Krioukov, 2010]
Basic definitions and notations

Average hyperbolicity measure, computational issues

Definition (average hyperbolicity)

\[ \delta_{\text{ave}}(G) = \frac{1}{\binom{n}{4}} \sum_{u_1, u_2, u_3, u_4} \delta_{u_1, u_2, u_3, u_4} \]

expected value of \( \delta_{u_1, u_2, u_3, u_4} \) if \( u_1, u_2, u_3, u_4 \) are picked uniformly at random

Computation of \( \delta(G) \) and \( \delta_{\text{ave}}(G) \)

- Trivially in \( O(n^4) \) time
  - Compute all-pairs shortest paths
    - Floyd–Warshall algorithm \( O(n^3) \) time
  - For each combination \( u_1, u_2, u_3, u_4 \), compute \( \delta_{u_1, u_2, u_3, u_4} \) \( O(n^4) \) time
- Open problem: can we compute in \( O(n^{4-\epsilon}) \) time?
Outline of talk

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2. Basic definitions and notations

3. Computing hyperbolicity for real networks

4. Implications of hyperbolicity of networks
   - Hyperbolicity and crosstalk in regulatory networks
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Computing hyperbolicity for real networks

Direct calculation

Real networks used for empirical validation

20 well-known biological and social networks

- 11 biological networks that include 3 transcriptional regulatory, 5 signalling, 1 metabolic, 1 immune response and 1 oriented protein-protein interaction networks
- 9 social networks range from interactions in dolphin communities to the social network of jazz musicians
- hyperbolicity of the biological and directed social networks was computed by ignoring the direction of edges
- hyperbolicity values were calculated by writing codes in C using standard algorithmic procedures

Next slide: List of 20 networks
## 11 biological networks

<table>
<thead>
<tr>
<th>Network Type</th>
<th># Nodes</th>
<th># Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. <em>E. coli</em> transcriptional</td>
<td>311</td>
<td>451</td>
</tr>
<tr>
<td>2. Mammalian signaling</td>
<td>512</td>
<td>1047</td>
</tr>
<tr>
<td>3. <em>E. coli</em> transcriptional</td>
<td>418</td>
<td>544</td>
</tr>
<tr>
<td>4. T-LGL signaling</td>
<td>58</td>
<td>135</td>
</tr>
<tr>
<td>5. <em>S. cerevisiae</em> transcriptional</td>
<td>690</td>
<td>1082</td>
</tr>
<tr>
<td>6. <em>C. elegans</em> metabolic</td>
<td>453</td>
<td>2040</td>
</tr>
<tr>
<td>7. Drosophila segment polarity</td>
<td>78</td>
<td>132</td>
</tr>
<tr>
<td>(6 cells)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. ABA signaling</td>
<td>55</td>
<td>88</td>
</tr>
<tr>
<td>9. Immune response network</td>
<td>18</td>
<td>42</td>
</tr>
<tr>
<td>10. T cell receptor signaling</td>
<td>94</td>
<td>138</td>
</tr>
<tr>
<td>11. Oriented yeast PPI</td>
<td>786</td>
<td>2445</td>
</tr>
</tbody>
</table>

## 9 social networks

<table>
<thead>
<tr>
<th>Network Type</th>
<th># Nodes</th>
<th># Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Dolphin social network</td>
<td>62</td>
<td>160</td>
</tr>
<tr>
<td>2. American College Football</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Zachary Karate Club</td>
<td>34</td>
<td>78</td>
</tr>
<tr>
<td>4. Books about US politics</td>
<td>105</td>
<td>442</td>
</tr>
<tr>
<td>5. Sawmill communication network</td>
<td>36</td>
<td>62</td>
</tr>
<tr>
<td>6. Jazz Musician network</td>
<td>198</td>
<td>2742</td>
</tr>
<tr>
<td>7. Visiting ties in San Juan</td>
<td>75</td>
<td>144</td>
</tr>
<tr>
<td>8. World Soccer Data, Paris 1998</td>
<td>35</td>
<td>118</td>
</tr>
<tr>
<td>9. Les Miserables characters</td>
<td>77</td>
<td>251</td>
</tr>
</tbody>
</table>
Computing hyperbolicity for real networks

Direct calculation

<table>
<thead>
<tr>
<th>Biological networks</th>
<th>Average degree</th>
<th>δ_{ave}</th>
<th>δ</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. E. coli transcriptional</td>
<td>1.45</td>
<td>0.132</td>
<td>2</td>
</tr>
<tr>
<td>2. Mammalian Signaling</td>
<td>2.04</td>
<td>0.013</td>
<td>3</td>
</tr>
<tr>
<td>3. E. Coli transcriptional</td>
<td>1.30</td>
<td>0.043</td>
<td>2</td>
</tr>
<tr>
<td>4. T LGL signaling</td>
<td>2.32</td>
<td>0.297</td>
<td>2</td>
</tr>
<tr>
<td>5. S. cerevisiae transcriptional</td>
<td>1.56</td>
<td>0.004</td>
<td>3</td>
</tr>
<tr>
<td>6. C. elegans Metabolic</td>
<td>4.50</td>
<td>0.010</td>
<td>1.5</td>
</tr>
<tr>
<td>7. Drosophila segment polarity</td>
<td>1.69</td>
<td>0.676</td>
<td>4</td>
</tr>
<tr>
<td>8. ABA signaling</td>
<td>1.60</td>
<td>0.302</td>
<td>2</td>
</tr>
<tr>
<td>9. Immune Response Network</td>
<td>2.33</td>
<td>0.286</td>
<td>1.5</td>
</tr>
<tr>
<td>10. T Cell Receptor Signalling</td>
<td>1.46</td>
<td>0.323</td>
<td>3</td>
</tr>
<tr>
<td>11. Oriented yeast PPI</td>
<td>3.11</td>
<td>0.001</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>social networks</th>
<th>Average degree</th>
<th>δ_{ave}</th>
<th>δ</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Dolphins social network</td>
<td>5.16</td>
<td>0.262</td>
<td>2</td>
</tr>
<tr>
<td>2. American College Football</td>
<td>10.64</td>
<td>0.312</td>
<td>2</td>
</tr>
<tr>
<td>3. Zachary Karate Club</td>
<td>4.58</td>
<td>0.170</td>
<td>1</td>
</tr>
<tr>
<td>4. Books about US Politics</td>
<td>8.41</td>
<td>0.247</td>
<td>2</td>
</tr>
<tr>
<td>5. Sawmill communication</td>
<td>3.44</td>
<td>0.162</td>
<td>1</td>
</tr>
<tr>
<td>6. Jazz musician</td>
<td>27.69</td>
<td>0.140</td>
<td>1.5</td>
</tr>
<tr>
<td>7. Visiting ties in San Juan</td>
<td>3.84</td>
<td>0.422</td>
<td>3</td>
</tr>
<tr>
<td>8. World Soccer data, 1998</td>
<td>3.37</td>
<td>0.270</td>
<td>2.5</td>
</tr>
<tr>
<td>9. Les Miserable</td>
<td>6.51</td>
<td>0.278</td>
<td>2</td>
</tr>
</tbody>
</table>

- Hyperbolicity values of almost all networks are small
- For all networks δ_{ave} is one or two orders of magnitude smaller than δ
  - Intuitively, this suggests that value of δ may be a rare deviation from typical values of δ_{u_1,u_2,u_3,u_4} for most combinations of nodes \{u_1, u_2, u_3, u_4\}
- No systematic dependence of δ on number of nodes/edges or average degree
Computing hyperbolicity for real networks

Direct calculation

<table>
<thead>
<tr>
<th>Definition (Diameter of a graph)</th>
<th>Fact</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{D} = \max_{u,v} {d_{u,v}}$</td>
<td>$\delta \leq \mathcal{D}/2$</td>
</tr>
</tbody>
</table>

longest shortest path

small diameter implies small hyperbolicity

We found no systematic dependence of $\delta$ on $\mathcal{D}$
Computing hyperbolicity for real networks

**Direct calculation**

**Definition (Diameter of a graph)**

\[ D = \max_{u,v} \{d_{u,v}\} \]

longest shortest path

**Fact**

\[ \delta \leq D/2 \]

small diameter implies small hyperbolicity

We found no systematic dependence of \( \delta \) on \( D \)

For more rigorous checks of hyperbolicity of finite graphs and for evaluation of statistical significance of the hyperbolicity measure see our paper

R. Albert, B. DasGupta and N. Mobasheri,

Topological implications of negative curvature for biological and social networks.

Physical Review E 89(3), 032811 (2014)
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Implications of hyperbolicity

We discuss topological implications of hyperbolicity somewhat informally.

Precise Theorems and their proofs are available in our paper:
R. Albert, B. DasGupta and N. Mobasher, Topological implications of negative curvature for biological and social networks.
Physical Review E 89(3), 032811 (2014)
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Implications of hyperbolicity

Hyperbolicity and crosstalk in regulatory networks

Definition (Path chord and chord)

Path chord

\[ u_4 - u_5 - u_0 - u_1 \]

Chord

\[ u_4 - u_5 - u_0 - u_1 \]

Bhaskar DasGupta (UIC)
Implications of hyperbolicity
Hyperbolicity and crosstalk in regulatory networks

Definition (Path chord and chord)

\begin{align*}
\text{Path chord: } & \quad u_4 \ldots u_v \ldots u_1 \\
\text{Chord: } & \quad u_4 \ldots u_0 \ldots u_1 \\
\end{align*}

Theorem (large cycle without path-chord imply large hyperbolicity)

$G$ has a cycle of $k$ nodes which has no path-chord $\implies \delta \geq \lceil k/4 \rceil$

Corollary

Any cycle containing more than $4\delta$ nodes must have a path-chord

Example

$\delta < 1 \implies G$ is chordal graph
Implications of hyperbolicity
Hyperbolicity and crosstalk in regulatory networks

An example of a regulatory network

Network associated to the Drosophila segment polarity

Implications of hyperbolicity

Hyperbolicity and crosstalk in regulatory networks

Hyperbolicity and crosstalk in regulatory networks

short-cuts in long feedback loops

node regulates itself through a long feedback loop

⇒ this loop must have a path-chord

⇒ a shorter feedback cycle through the same node

interpreting chord or short path-chord as crosstalk

“source” regulates “target” through two long paths

⇒ must exist a crosstalk path between these two paths

number of crosstalk paths increases at least linearly with total length of two paths

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Implications of hyperbolicity
Geodesic triangles and crosstalk paths

Geodesic triangles and crosstalk paths

$u_0$ $u_1$ $u_2$

Implications of hyperbolicity
Geodesic triangles and crosstalk paths

Geodesic triangles and crosstalk paths

 shortest path

Implications of hyperbolicity

Geodesic triangles and crosstalk paths

\[ d_{u_0,u_1} = \left\lfloor \frac{d_{u_0,u_1} + d_{u_0,u_2} - d_{u_1,u_2}}{2} \right\rfloor \]

\[ d_{u_1,u_0} = \left\lceil \frac{d_{u_1,u_2} + d_{u_1,u_0} - d_{u_2,u_0}}{2} \right\rceil \]

\[ d_{u_1,u_0} = d_{u_1,u_2} \quad d_{u_0,u_1} = d_{u_0,u_2} \]

\[ d_{u_2,u_0} = d_{u_2,u_1} \]
Implications of hyperbolicity
Geodesic triangles and crosstalk paths

∀ \nu \text{ in one path} \quad \exists v' \text{ in the other path} \quad \text{such that} \quad d_{\nu,v'} \leq \max \{6\delta, 2\}

\begin{align*}
d_{u_0,u_0,1} &= \left[ \frac{d_{u_0,u_1} + d_{u_0,u_2} - d_{u_1,u_2}}{2} \right] \\
d_{u_1,u_0,1} &= \left[ \frac{d_{u_1,u_2} + d_{u_1,u_0} - d_{u_2,u_0}}{2} \right] \\
d_{u_1,u_0,1} &= d_{u_1,u_1,2} \\
d_{u_0,u_0,1} &= d_{u_0,u_0,2} \\
d_{u_2,u_0,2} &= d_{u_2,u_1,2}
\end{align*}
Implications of geodesic triangles for regulatory networks

Consider feedback or feed-forward loop formed by the shortest paths among three nodes

We can expect short cross-talk paths between these shortest paths

Feedback/feed-forward loop is nested with additional feedback/feed-forward loops
Implications of geodesic triangles for regulatory networks

Consider feedback or feed-forward loop formed by the shortest paths among three nodes

We can expect short cross-talk paths between these shortest paths

Feedback/feed-forward loop is nested with additional feedback/feed-forward loops


Network motifs\(^a\) are often nested

Two generations of nested assembly for a common *E. coli* motif
[DeDeo and Krakauer, 2012]

\(^a\) e.g., feed-forward or feedback loops of small number of nodes
Implications of hyperbolicity

Hausdorff distance between shortest paths

Definition (Hausdorff distance between two paths $P_1$ and $P_2$)

$$d_H(P_1, P_2) \overset{\text{def}}{=} \max \left\{ \max_{v_1 \in P_1} \min_{v_2 \in P_2} \{ d_{v_1, v_2} \}, \, \max_{v_2 \in P_2} \min_{v_1 \in P_1} \{ d_{v_1, v_2} \} \right\}$$

small Hausdorff distance implies every node of either path is close to some node of the other path
Implications of hyperbolicity
Hausdorff distance between shortest paths

Definition (Hausdorff distance between two paths $P_1$ and $P_2$)

$$d_H(P_1, P_2) \overset{\text{def}}{=} \max \left\{ \max_{v_1 \in P_1} \min_{v_2 \in P_2} \{ d_{v_1, v_2} \}, \max_{v_2 \in P_2} \min_{v_1 \in P_1} \{ d_{v_1, v_2} \} \right\}$$

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Implications of hyperbolicity

Hausdorff distance between shortest paths

**Definition (Hausdorff distance between two paths $P_1$ and $P_2$)**

$$d_H(P_1, P_2) \overset{\text{def}}{=} \max \left\{ \max_{v_1 \in P_1} \min_{v_2 \in P_2} \left\{ d_{v_1, v_2} \right\}, \max_{v_2 \in P_2} \min_{v_1 \in P_1} \left\{ d_{v_1, v_2} \right\} \right\}$$

small Hausdorff distance implies every node of either path is close to some node of the other path
Implications of hyperbolicity
Hausdorff distance between shortest paths

Definition (Hausdorff distance between two paths \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \))

\[
d_H(\mathcal{P}_1, \mathcal{P}_2) \defeq \max \left\{ \max_{v_1 \in \mathcal{P}_1} \min_{v_2 \in \mathcal{P}_2} \{ d_{v_1,v_2} \}, \max_{v_2 \in \mathcal{P}_2} \min_{v_1 \in \mathcal{P}_1} \{ d_{v_1,v_2} \} \right\}
\]

small Hausdorff distance implies every node of either path is close to some node of the other path

\[
d_H(\mathcal{P}_1, \mathcal{P}_2) \leq \max \{ 6 \delta, 2 \}
\]
Implications of hyperbolicity
Hausdorff distance between shortest paths

this result versus our previous path-chord result

path-chord result

long cycle ⇒ there is a path chord

this result

\[ d_H(P_1, P_2) \leq \max\{ 6\delta, 2 \} \]

Which result is more general in nature?

Implications of hyperbolicity
Hausdorff distance between shortest paths

this result versus our previous path-chord result

path-chord result

long cycle ⇒ there is a path chord

this result

\[ d_H(\mathcal{P}_1, \mathcal{P}_2) \leq \max\{6\delta, 2\} \]

Which result is more general in nature?

---

Implications of hyperbolicity

A notational simplification

unless $G$ is a tree or a complete graph ($K_n$), $\delta > 0$

$\delta > 0 \equiv \delta \geq \frac{1}{2}$

$\delta \geq \frac{1}{2} \Rightarrow \max\{6\delta, 2\} = 6\delta$

Hence, we will simply write $6\delta$ instead of $\max\{6\delta, 2\}$
Implications of hyperbolicity
Distance between geodesic and arbitrary path

Distance from a shortest path $u_0 \leftrightarrow^5 u_1$ to another arbitrary path $u_0 \rightleftharpoons^P u_1$

$n$ is the number of nodes in the graph

$\ell(P)$ is length of path $P$

shortest path $u_0 \leftrightarrow^5 u_1$

$\mathcal{P} \equiv u_0 \rightleftharpoons^P u_1$

arbitrary node
Implications of hyperbolicity
Distance between geodesic and arbitrary path

Distance from a shortest path $u_0 \leftrightarrow^5 u_1$ to another arbitrary path $u_0 \leftrightarrow \mathcal{P} u_1$

- $n$ is the number of nodes in the graph
- $\ell(\mathcal{P})$ is length of path $\mathcal{P}$

$\exists v' \ d_{v,v'} \leq 6\delta \log_2 \ell(\mathcal{P})$

$< 6\delta \log_2 n$

$O(\log n)$ if $\delta$ is constant

Implications of hyperbolicity
Distance between geodesic and arbitrary path

An interesting implication of this bound

$$\exists v' \quad d_{v,v'} \leq 6\delta \log_2 \ell(P)$$

An interesting implication of this bound

\[ \forall v' \in \mathcal{P} \quad d_{v,v'} \geq \gamma \]
An interesting implication of this bound

\[ \gamma \]

\[ \forall v' \in P, \quad d_{v,v'} \geq \gamma \]

\[ \Rightarrow \quad \ell(P) \geq 2^{\frac{\gamma}{6\delta}} = \Omega\left(2^{\Omega(\gamma)}\right) \]

if \( \delta \) is constant

Next: better bounds for approximately short paths
Implications of hyperbolicity

Approximately short path

Why consider approximately short paths?

Regulatory networks

Up/down-regulation of a target node is mediated by two or more “close to shortest” paths starting from the same regulator node.

Additional “very long” paths between the same regulator and target node do not contribute significantly to the target node’s regulation.

Definition: $\varepsilon$-additive-approximate short path $\mathcal{P}$

$$\ell(\mathcal{P}) \leq \text{length of shortest path} + \varepsilon$$
Implications of hyperbolicity

Approximately short path

Why consider approximately short paths?

Algorithmic efficiency reasons
Approximate short path may be faster to compute as opposed to exact shortest path

Routing and navigation problems (traffic networks)
Routing via approximate short path

Definition $\mu$-approximate short path $u_0 \overset{P}{\rightarrow} u_k = (u_0, u_1, \ldots, u_k)$

- $\ell(u_i \overset{P}{\rightarrow} u_j) \leq \mu$
  - length of sub-path from $u_i$ to $u_j$
- $d_{u_i, u_j}$
  - distance between $u_i$ and $u_j$

for all $0 \leq i < j \leq k$

2-approximate path $u_0 \overset{P}{\rightarrow} u_7 = (u_0, u_1, \ldots, u_7)$

shortest path $u_0 \overset{s}{\rightarrow} u_7$
Implications of hyperbolicity

Distance between geodesic and approximately short path

Distance from shortest path to an approximately short path $u_0 \leftrightarrow u_1$

$\epsilon$-additive approximate or, $\mu$-approximate

Implications of hyperbolicity
Distance between geodesic and approximately short path

Distance from shortest path to an approximately short path $u_0 \overset{\mathcal{P}}{\rightarrow} u_1$

$\epsilon$-additive approximate
or, $\mu$-approximate

$\forall \nu \exists \nu' \quad d_{\nu,\nu'} \leq \left(6\delta + 2\right) \log_2 \left(8 \left(6\delta + 2\right) \log_2 \left[ (6\delta + 2) (4 + 2\epsilon) \right] + 1 + \frac{\epsilon}{2} \right)$

$O(\delta \log(\epsilon + \delta \log \epsilon))$

depends only on $\delta$ and $\epsilon$

short crosstalk path for small $\epsilon$ and $\delta$

Implications of hyperbolicity

Distance between geodesic and approximately short path

Distance from shortest path to an \( \text{approximately short} \) path \( u_0 \leftrightarrow u_1 \)

\( \varepsilon \)-additive approximate

or, \( \mu \)-approximate

\[ u_0 \leftrightarrow u_1 \]

\( \mathcal{P} \equiv u_0 \leftrightarrow u_1 \)

arbitrary node

\( d_{v,v'} \)

shortest path \( u_0 \leftrightarrow u_1 \)

\( u_0 \leftrightarrow u_1 \) is \( \mu \)-approximate

\[ \forall v \exists v' \quad d_{v,v'} \leq (6\delta + 2) \log_2 \left( (6\mu + 2) (6\delta + 2) \log_2 \left[ (6\delta + 2) (3\mu + 1) \mu \right] + \mu \right) \]

\( O(\delta \log(\mu\delta)) \)

depends only on \( \delta \) and \( \mu \)

short crosstalk path for small \( \mu \) and \( \delta \)

Implications of hyperbolicity
Distance between geodesic and approximately short path

Contrast the new bounds with the old bound of $d_{v,v'} = O\left( \delta \log \ell(\mathcal{P}) \right)$

$d_{u_0,u_1}$ is the length of a shortest path between $u_0$ and $u_1$

<table>
<thead>
<tr>
<th>Old bound</th>
<th>New bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_0 \leftrightarrow u_1$ is $\epsilon$-additive approximate</td>
<td>$u_0 \leftrightarrow u_1$ is $\mu$-approximate</td>
</tr>
<tr>
<td>$\ell(\mathcal{P}) \leq d_{u_0,u_1} + \epsilon$</td>
<td>$\ell(\mathcal{P}) \leq \mu d_{u_0,u_1}$</td>
</tr>
<tr>
<td>$O\left( \delta \log (\epsilon + d_{u_0,u_1}) \right)$</td>
<td>$O\left( \delta \log (\mu d_{u_0,u_1}) \right)$</td>
</tr>
<tr>
<td>no dependency on $d_{u_0,u_1}$</td>
<td>no dependency on $d_{u_0,u_1}$</td>
</tr>
</tbody>
</table>

Implications of hyperbolicity
Distance between geodesic and approximately short path

Distance from an approximately short path $u_0 \leftrightarrow u_1$ to a shortest path

$\varepsilon$-additive approximate
or, $\mu$-approximate

for simplified exposition, we show bounds only in asymptotic $O(\cdot)$ notation
please refer to our paper for more precise bounds

Implications of hyperbolicity

Distance between geodesic and approximately short path

**Distance from an approximately short path** $u_0 \xrightarrow{\mathcal{P}} u_1$ to a shortest path

ε-additive approximate
or, μ-approximate

for simplified exposition, we show bounds only in asymptotic $O(\cdot)$ notation
please refer to our paper for more precise bounds

$\forall v' \exists v \quad d_{v',v} \leq O(\varepsilon + \delta \log(\varepsilon + \delta \log \varepsilon))$
depends only on δ and ε

Implications of hyperbolicity

Distance between geodesic and approximately short path

Distance from an approximately short path \( u_0 \leftrightarrow u_1 \) to a shortest path

\[ \epsilon \text{-additive approximate or, } \mu \text{-approximate} \]

for simplified exposition, we show bounds only in asymptotic \( O(\cdot) \) notation
please refer to our paper for more precise bounds

\[ \forall v' \exists v \quad d_{v',v} \leq O(\mu \delta \log(\mu \delta)) \]

depends only on \( \delta \) and \( \mu \)

Implications of hyperbolicity
Distance between geodesic and approximately short path

Distance from approximate short path $\mathcal{P}_1$ to approximate short path $\mathcal{P}_2$

arbitrary node $v$
nearest node $v'$

Implications of hyperbolicity
Distance between geodesic and approximately short path

\[ \text{Distance from approximate short path } \mathcal{P}_1 \text{ to approximate short path } \mathcal{P}_2 \]

arbitrary node \( v \)

nearest node \( v' \)

go to any shortest path

\[ u_0 \rightarrow v'' \rightarrow u_1 \]

shortest path

arbitrary node

Implications of hyperbolicity

Distance between geodesic and approximately short path

Distance from approximate short path $\mathcal{P}_1$ to approximate short path $\mathcal{P}_2$

- arbitrary node $v$
- nearest node $v'$

$\mathcal{P}_2$

$v''$

$\mathcal{P}_1$

$u_0$

$u_1$

arbitrary node

shortest path

continue to the other path

Implications of hyperbolicity

Distance between geodesic and approximately short path

Distance from approximate short path $P_1$ to approximate short path $P_2$

arbitrary node $v$  

nearest node $v'$

we sometimes overestimate quantities to simplify expression

$P_1$ is $\varepsilon_1$-additive approximate
$P_2$ is $\varepsilon_2$-additive approximate

$O(\varepsilon_1 + \delta \log(\varepsilon_1 \varepsilon_2) + \delta \log \delta)$

$P_1$ is $\varepsilon$-additive approximate
$P_2$ is $\mu$-approximate

$O(\varepsilon + \delta \log(\varepsilon \mu) + \delta^2 \log \log \varepsilon)$

$P_1$ is $\mu$-approximate
$P_2$ is $\varepsilon$-additive approximate

$O(\mu \delta \log(\mu \delta) + \varepsilon + \delta \log \varepsilon)$

$P_1$ is $\mu_1$-approximate
$P_2$ is $\mu_2$-approximate

$O(\mu_1 \delta \log(\mu_1 \delta) + \delta \log \mu_2)$

Implications of hyperbolicity
Distance between geodesic and approximately short path

Interesting implications of these improved bounds

Implications of hyperbolicity

Distance between geodesic and approximately short path

Interesting implications of these improved bounds

Assume $\forall v' \in P$ $d_{v,v'} \geq \gamma$

Implications of hyperbolicity
Distance between geodesic and approximately short path

Interesting implications of these improved bounds

If $\mathcal{P}$ is $\epsilon$-additive-approximate short then

$$\epsilon = \Omega \left( \frac{2 \gamma / \delta}{\delta} - \log \delta \right)$$

assume $\forall \, v' \in \mathcal{P} \, d_{v,v'} \geq \gamma$
Implications of hyperbolicity
Distance between geodesic and approximately short path

Interesting implications of these improved bounds

assume $\forall v' \in P \ d_{v,v'} \geq \gamma \ \Rightarrow \ 
\mu = \Omega\left(\frac{2\gamma/\delta}{\gamma}\right)$

if $P$ is $\mu$-approximate short then

approximately short path $P$
To wrap it up, approximate shortest paths look like the following cartoon.
Implications of hyperbolicity
Distance between geodesic and approximately short path

To wrap it up, approximate shortest paths look like the following cartoon

Interpretation for regulatory networks

- It is reasonable to assume that, when up- or down-regulation of a target node is mediated by two or more approximate short paths starting from the same regulator node, additional very long paths between the same regulator and target node do not contribute significantly to the target node’s regulation.

- We refer to the short paths as relevant, and to the long paths as irrelevant.

- Then, our finding can be summarized by saying that almost all relevant paths between two nodes have crosstalk paths between each other.

Outline of talk

1. Introduction
2. Basic definitions and notations
3. Computing hyperbolicity for real networks
4. Implications of hyperbolicity of networks
   - Hyperbolicity and crosstalk in regulatory networks
   - Geodesic triangles and crosstalk paths
   - Identifying essential edges and nodes in regulatory networks
   - A social network application
Implications of hyperbolicity
Identifying essential edges and nodes in regulatory networks

**Influence of a node on the geodesics between other pair of nodes**

**integer parameters used in this result**

\[ \kappa \geq 4 \quad \alpha > 0 \quad r > 3(\kappa - 2)\delta \]

Example: 5 1 9\(\delta + 1\)

---

Implications of hyperbolicity
Identifying essential edges and nodes in regulatory networks

Influence of a node on the geodesics between other pair of nodes

Integer parameters used in this result

\[ \kappa \geq 4 \quad \alpha > 0 \quad r > 3(\kappa - 2)\delta \]

Example:

\[ 5 \quad 1 \quad 9\delta + 1 \]

Implications of hyperbolicity
Identifying essential edges and nodes in regulatory networks

Influence of a node on the geodesics between other pair of nodes

Integer parameters used in this result

\[ \kappa \geq 4 \quad \alpha > 0 \quad r > 3(\kappa - 2) \delta \]

Example: \(5, 1, 9\delta + 1\)

---

Implications of hyperbolicity
Identifying essential edges and nodes in regulatory networks

Influence of a node on the geodesics between other pair of nodes

integer parameters used in this result

\[ \kappa \geq 4 \quad \alpha > 0 \quad r > 3(\kappa - 2)\delta \]

Example:

\[ 5 \quad 1 \quad 9\delta + 1 \]

consider any shortest path \( \mathcal{P} \) between \( u_3 \) and \( u_4 \)

\( \mathcal{P} \) must look like this

Influence of a node on the geodesics between other pair of nodes

**integer parameters used in this result**

\[ \kappa \geq 4 \quad \alpha > 0 \quad r > 3(\kappa - 2)\delta \]

Example: \[ 5 \quad 1 \quad 9\delta + 1 \]

Consider any shortest path \( \mathcal{P} \) between \( u_3 \) and \( u_4 \)

\( \mathcal{P} \) must look like this

\[ \gamma = d_{u_0,v} \leq r - \left( \frac{3}{2}\kappa - 1 \right)\delta \]

\[ r - \Theta(\kappa \delta) \]

Implications of hyperbolicity

Identifying essential edges and nodes in regulatory networks

Influence of a node on the geodesics between other pair of nodes

**integer parameters used in this result**

\[ \kappa \geq 4 \quad \alpha > 0 \quad r > 3(\kappa - 2)\delta \]

Example: \[ 5 \quad 1 \quad 9\delta + 1 \]

Consider any shortest path \( P \) between \( u_3 \) and \( u_4 \)

\( P \) must look like this

\[ \gamma = d_{u_0,v} \leq r - \left( \frac{3}{2}\kappa - 1 \right)\delta \]

\[ r - \Theta(\kappa\delta) \]

\[ \ell(P) \geq (3\kappa - 2)\delta + 2\alpha \]

\[ \Omega(\kappa\delta + \alpha) \]

---

Corollary (of previous results)

Consider any path $\mathcal{P}$ between $u_3$ and $u_4$

Suppose that $\mathcal{P}$ does not intersect the shaded region

$$r \geq 3 \kappa \delta$$
Implications of hyperbolicity
Identifying essential edges and nodes in regulatory networks

Influence of a node on the geodesics between other pair of nodes

**Corollary (of previous results)**

Consider any path $\mathcal{P}$ between $u_3$ and $u_4$

Suppose that $\mathcal{P}$ does not intersect the shaded region

\[
\ell(\mathcal{P}) \geq \frac{\alpha}{2^6 \delta} + \frac{\kappa}{4}
\]

\[
2 \Omega \left( \frac{\alpha}{\delta} + \kappa \right)
\]

\[
\geq 3 \kappa \delta
\]
Implications of hyperbolicity
Identifying essential edges and nodes in regulatory networks

Influence of a node on the geodesics between other pair of nodes

Corollary (of previous results)

Consider any path $\mathcal{P}$ between $u_3$ and $u_4$

Suppose that $\mathcal{P}$ does not intersect the shaded region

$$\ell(\mathcal{P}) \geq \frac{\alpha}{2^6 \delta} + \frac{\kappa}{4} \geq 2 \Omega \left(\frac{\alpha}{\delta} + \kappa\right)$$

$\mathcal{P}$ $\epsilon$-additive-approximate $\Rightarrow$

$$\epsilon > \frac{2 \frac{\alpha}{6 \delta} + \frac{\kappa}{4}}{48 \delta} - \log_2 (48 \delta) \Omega \left(2^{\Theta(\alpha + \kappa)}\right)$$

if $\delta$ is constant

Implications of hyperbolicity

Identifying essential edges and nodes in regulatory networks

Influence of a node on the geodesics between other pair of nodes

Corollary (of previous results)

Consider any path $\mathcal{P}$ between $u_3$ and $u_4$

Suppose that $\mathcal{P}$ does not intersect the shaded region.

$$\ell(\mathcal{P}) \geq \frac{\alpha}{2} + \frac{\kappa}{4}$$

$$\geq 2 \Omega \left( \frac{\alpha}{\delta} + \kappa \right)$$

$\mathcal{P}$ $\varepsilon$-additive-approximate $\Rightarrow$

$$\varepsilon > \frac{2 \alpha + \kappa}{4 \delta} + \log_2 (48 \delta)$$

$$\Omega \left( 2^{\Theta (\alpha + \kappa)} \right)$$

if $\delta$ is constant

$\mathcal{P}$ $\mu$-approximate $\Rightarrow$

$$\mu \geq \frac{\alpha}{12 \alpha + 6 \delta (3 \kappa - 26)}$$

$$\Omega \left( \frac{2^{\Theta (\frac{\alpha}{\delta} + \kappa)}}{\alpha + \kappa \delta} \right)$$


Bhaskar DasGupta (UIC)

Negative curvature for networks

November 29, 2014 41 / 52
Interesting implications of these bounds for regulatory networks

Interesting implications of these bounds for regulatory networks

Interesting implications of these bounds for regulatory networks

\[ \xi = O(\delta) \]
Implications of hyperbolicity
Identifying essential edges and nodes in regulatory networks

Interesting implications of these bounds for regulatory networks

All shortest paths between $u_{\text{source}}$ and $u_{\text{target}}$ must intersect the $\xi$-neighborhood.

Therefore, “knocking out” nodes in $\xi$-neighborhood cuts off all shortest regulatory paths between $u_{\text{source}}$ and $u_{\text{target}}$.

$u_{\text{source}}$ $u_{\text{target}}$ $u_{\text{middle}}$

$\xi = O(\delta)$

Implications of hyperbolicity
Identifying essential edges and nodes in regulatory networks

Interesting implications of these bounds for regulatory networks

But, it gets even more interesting!

\[ \xi = O(\delta) \]

Interesting implications of these bounds for regulatory networks

But, it gets even more interesting!

shifting the $\xi$-neighborhood does not change claim

Implications of hyperbolicity
Identifying essential edges and nodes in regulatory networks

Interesting implications of these bounds for regulatory networks

how about enlarging the $\xi$-neighborhood?

$\xi = O(\delta)$

Interesting implications of these bounds for regulatory networks

how about enlarging the $\xi$-neighborhood?

approximately short paths start intersecting the neighborhood

Consider a ball (neighborhood) of radius $\xi \log n$ \( (n \text{ is the number of nodes}) \)
Interesting implications of these bounds for regulatory networks

Consider a ball (neighborhood) of radius $\xi \log n$  \((n \text{ is the number of nodes})\)

All paths intersect the neighborhood.
Implications of hyperbolicity
Identifying essential edges and nodes in regulatory networks

Empirical estimation of neighborhoods and number of essential nodes

We empirically investigated these claims on relevant paths passing through a neighborhood of a central node for the following two biological networks:

- *E. coli* transcriptional
- T-LGL signaling

by selecting a few biologically relevant source-target pairs

Our results show much better bounds for real networks compared to the worst-case pessimistic bounds in the mathematical theorems

see our paper for further details

The following cartoon informally depicts some of the preceding discussions.
The following cartoon informally depicts some of the preceding discussions.
The following cartoon informally depicts some of the preceding discussions.

As we move further from the central node, the more a shortest path bends inward towards the central node.
Implications of hyperbolicity

Identifying essential edges and nodes in regulatory networks

eavesdropper may succeed with limited sensor range

eavesdropper need not be a hub
Implications of hyperbolicity
Identifying essential edges and nodes in regulatory networks

Traffic network
need not be a hub
Outline of talk

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Implications of hyperbolicity

Effect of hyperbolicity on structural holes in social networks

Visual illustration of a well-known social network

Zachary’s Karate Club (http://networkdata.ics.uci.edu/data.php?id=105)
Implications of hyperbolicity
Effect of hyperbolicity on structural holes in social networks

**Structural hole in a social network** [Burt, 1995; Borgatti, 1997]

**Definition** (Adjacency matrix of an undirected unweighted graph)

\[
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

\[
a_{u,v} = \begin{cases} 
1, & \text{if } \{u, v\} \text{ is an edge} \\
0, & \text{otherwise}
\end{cases}
\]

**Definition** (measure of structural hole at node \(u\) [Burt, 1995; Borgatti, 1997])

(assume \(u\) has degree at least 2)

\[
M_u \overset{\text{def}}{=} \sum_{v \in V} \left( \frac{a_{u,v} + a_{v,u}}{\max_{x \neq u} \{a_{u,x} + a_{x,u}\}} \right) \left[ 1 - \sum_{y \in V} \left( \frac{a_{u,y} + a_{y,u}}{\sum_{x \neq u} (a_{u,x} + a_{x,u})} \right) \left( \frac{a_{v,y} + a_{y,v}}{\max_{z \neq y} \{a_{v,z} + a_{z,v}\}} \right) \right]
\]

too complicated 😞
Implications of hyperbolicity

Effect of hyperbolicity on structural holes in social networks

**Structural hole in a social network** [Burt, 1995; Borgatti, 1997]

**Definition** (Adjacency matrix of an undirected unweighted graph)

\[
\begin{pmatrix}
  u & \cdots & a_{u,v} & \cdots & \cdots \\
  \cdots & \ddots & \cdots & \ddots & \cdots \\
  \cdots & \cdots & a_{u,v} & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \ddots & \cdots \\
  \cdots & \cdots & \cdots & \cdots & u \\
\end{pmatrix}
\]

\[
a_{u,v} = \begin{cases} 
1, & \text{if } \{u, v\} \text{ is an edge} \\
0, & \text{otherwise}
\end{cases}
\]

**Definition** (Measure of structural hole at node \(u\) [Burt, 1995; Borgatti, 1997])

(assume \(u\) has degree at least 2)

Let \(\text{Nbr}(u)\) be set of nodes adjacent to \(u\)

\[
M_u = |\text{Nbr}(u)| - \frac{\sum_{v,y \in \text{Nbr}(u)} a_{v,y}}{|\text{Nbr}(u)|}
\]

Next: An intuitive interpretation of \(M_u\)
Implications of hyperbolicity

Effect of hyperbolicity on structural holes in social networks

An intuitive interpretation of $M_u$

**Definition (weak dominance $<^{\rho, \lambda}_{weak}$)**

Nodes $v, y$ are weakly $(\rho, \lambda)$-dominated by node $u$ provided

1. $\rho < d_{u,v}, d_{u,y} \leq \rho + \lambda$, and
2. for at least one shortest path $\mathcal{P}$ between $v$ and $y$, $\mathcal{P}$ contains a node $z$ such that $d_{u,z} \leq \rho$

**Definition (strong dominance $<^{\rho, \lambda}_{strong}$)**

Nodes $v, y$ are strongly $(\rho, \lambda)$-dominated by node $u$ provided

1. $\rho < d_{u,v}, d_{u,y} \leq \rho + \lambda$, and
2. for every shortest path $\mathcal{P}$ between $v$ and $y$, $\mathcal{P}$ contains a node $z$ such that $d_{u,z} \leq \rho$
Implications of hyperbolicity
Effect of hyperbolicity on structural holes in social networks

An intuitive interpretation of $M_u$

Notation (boundary of the $\xi$-neighborhood of node $u$)

$B_\xi (u) = \{ v \mid d_{u,v} = \xi \}$

the set of all nodes at a distance of precisely $\xi$ from $u$

Observation

$M_u = \mathbb{E} \left[ \begin{array}{c}
\text{number of pairs of nodes } v, y \text{ such that } \\
v, y \text{ is weakly } (0,1)-\text{dominated by } u
\end{array} \right]$

$\geq \mathbb{E} \left[ \begin{array}{c}
\text{number of pairs of nodes } v, y \text{ such that } \\
v, y \text{ is strongly } (0,1)-\text{dominated by } u
\end{array} \right]$

$v$ is selected uniformly randomly from $\bigcup_{0<j \leq 1} B_j(u)$

always true

equality does not hold in general

Implications of hyperbolicity

Effect of hyperbolicity on structural holes in social networks

Generalize $M_u$ to $M_{u,\rho,\lambda}$ for larger ball of influence of a node
replace (0, 1) by ($\rho, \lambda$)

$$M_u = \mathbb{E} \left[ \text{number of pairs of nodes } v, y \text{ such that } v, y \text{ is weakly (0, 1)-dominated by } u \right]$$

$$M_{u,\rho,\lambda} = \mathbb{E} \left[ \text{number of pairs of nodes } v, y \text{ such that } v, y \text{ is weakly (\rho, \lambda)-dominated by } u \right]$$

$v$ is selected uniformly randomly from $\bigcup_{\rho < j \leq \lambda} B_j(u)$
Implications of hyperbolicity
Effect of hyperbolicity on structural holes in social networks

Generalize $M_u$ to $M_{u,ρ,λ}$ for larger ball of influence of a node
replace $(0, 1)$ by $(ρ, λ)$

Lemma (equivalence of strong and weak domination)

If $λ \geq 6δ\log_2 n$ then

$$M_{u,ρ,λ} \overset{\text{def}}{=} \mathbb{E} \left[ \begin{array}{c} \text{number of pairs of nodes } v, y \text{ such that } \\
\text{ } v, y \text{ is weakly } (ρ, λ)-\text{dominated by } u \\
\end{array} \right]$$

$$= \mathbb{E} \left[ \begin{array}{c} \text{number of pairs of nodes } v, y \text{ such that } \\
\text{ } v, y \text{ is strongly } (ρ, λ)-\text{dominated by } u \\
\end{array} \right]$$

$v$ is selected uniformly randomly from $\bigcup_{ρ < j \leq λ} B_j(u)$

equality holds now

Implications of hyperbolicity

Effect of hyperbolicity on structural holes in social networks

Lemma (equivalence of strong and weak domination)

If $\lambda \geq 6 \delta \log_2 n$ then

$$M_{u,\rho,\lambda} \overset{\text{def}}{=} \mathbb{E} \left[ \begin{array}{c}
\text{number of pairs of nodes } v, y \text{ such that } \\
v, y \text{ is weakly } (\rho, \lambda)-\text{dominated by } u
\end{array} \right]$$

$$= \mathbb{E} \left[ \begin{array}{c}
\text{number of pairs of nodes } v, y \text{ such that } \\
v, y \text{ is strongly } (\rho, \lambda)-\text{dominated by } u
\end{array} \right]$$

$v$ is selected uniformly randomly from $\bigcup_{\rho < j \leq \lambda} B_j(u)$

What does this lemma mean intuitively?
Implications of hyperbolicity
Effect of hyperbolicity on structural holes in social networks

What does this lemma mean intuitively?

\[ B_{\rho}(u) \]

\[ \rho \]

\[ \lambda \geq 6\delta \log_2 n \]

\[ B_{\rho+\lambda}(u) \]
Implications of hyperbolicity
Effect of hyperbolicity on structural holes in social networks

What does this lemma mean intuitively?

\[ B_\rho(u) \]

either all the shortest paths are completely inside \( B_{\rho+\lambda}(u) \)

\[ \lambda \geq 6\delta \log_2 n \]
Implications of hyperbolicity

Effect of hyperbolicity on structural holes in social networks

What does this lemma mean intuitively?

or all the shortest paths are completely outside of $\mathcal{B}_{\rho+\lambda}(u)$

$\mathcal{B}_{\rho}(u)$

$\mathcal{B}_{\rho+\lambda}(u)$

$\rho$

$\lambda \geq 6\delta \log_2 n$

Plato

Socrates
Implications of hyperbolicity

Effect of hyperbolicity on structural holes in social networks

What does this lemma mean intuitively?

But not both!

\[ B_\rho(u) \]

\[ B_\rho + \lambda(u) \]

\[ \lambda \geq 6\delta \log_2 n \]
Thank you for your attention

"But before we move on, allow me to belabor the point even further..."

Questions??