Removing partisan bias in redistricting: computational complexity meets the science of gerrymandering

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**Gerrymandering**

Creation of district plans with highly asymmetric electoral outcomes to disenfranchise voters

- **Long history starting from as early as 1812**
  - 1812: shape of South Essex district (Massachusetts) resembling a salamander created to favor selected candidates

- **Extensive legal history too!**
  - **1986**: US Supreme Court: gerrymandering is justiciable
  - **2006**: US Supreme Court: some measure of partisan symmetry may be used to remedy gerrymandering
    - Which measure? Court did not say. Depends case by case.
  - **2019**: US Supreme Court: best settled at the legislative and political level (ALAS!)

- **Major impediment to removing gerrymandering**
  - How to formulate an effective and precise measure for partisan bias that will be acceptable in courts?
Some tools politicians use for partisan gerrymandering in 2-party system

- **Packing** → concentrate voters of opposition party in a single district
- **Cracking** → spread voters of opposition party across many districts

Other methods include
- Hijacking
- Kidnapping etc.
“Efficiency Gap” measure for partisan gerrymandering

- Introduced by Stephanopoulos and McGhee in 2014 for a 2-party system (such as USA)

- Minimizes absolute difference of total “wasted votes” between the parties

- Very promising in several aspects, e.g.,
  - provides a “mathematically precise” measure of gerrymandering with desirable properties
  - was found legally convincing in a US appeals court case
    - ALAS, Supreme Court overturned the ruling in 2019
"Wasted votes" for a district

- Total votes 100 (need 51 to win)
  - Party A vote 59
  - Party B vote 41

- Wasted votes for Party A 59-51=8
- Wasted votes for Party B 41
“Efficiency gap” measure for the whole map

Efficiency gap

= \left| \frac{\text{sum of Party A wasted votes over all districts} - \text{sum of Party B wasted votes over all districts}}{\text{Total votes over all districts}} \right|
Formalization of the efficiency gap calculation problem

Basic assumption: only two parties: Party A and Party B
(3rd party votes are negligible, like in USA)

Topological part of an input: a “map” $\mathcal{P}$
▷ partitioned into atomic elements or cells
    e.g., subdivisions of counties

Two possible types of maps:

**Rectilinear polygon $\mathcal{P}$ without holes**
▷ $\mathcal{P}$ placed on a unit grid of size $m \times n$
▷ atomic elements (cells) $\Rightarrow$ unit squares of grid inside $\mathcal{P}$
▷ $v_{i,j}$ : cell on $i^{th}$ row and $j^{th}$ column

**Arbitrary polygon $\mathcal{P}$ without holes:**
▷ atomic elements (cells) $\Rightarrow$ sub-polygons (without holes) inside $\mathcal{P}$
▷ *Alternate* way of looking: **planar graph** $G(\mathcal{P})$
    • nodes are cells
    • edge connects two cells if they share boundary
Formalization of the efficiency gap calculation problem

*only two* parties: Party A and Party B

**Parameters of our gerrymandering problem**

- **Map $\mathcal{P}$:**
  - *size* $|\mathcal{P}|$: number of cells or nodes in $\mathcal{P}$

- **Cell or node $y$ of $\mathcal{P}$:**
  - $\text{PartyA}(y)$: total number of voters for Party A
  - $\text{PartyB}(y)$: total number of voters for Party B
  - $\text{Pop}(y) = \text{PartyA}(y) + \text{PartyB}(y)$: total number of voters

- **Global:**
  - $\kappa$: *required* (legally mandated) number of districts ($1 < \kappa < |\mathcal{P}|$)
    - Hard constraint: solution with different value of $\kappa$ would be *illegal*
    - precludes designing approximation algorithm in which the value of $\kappa$ changes even by just $\pm 1$
    - computational hardness for a value of $\kappa$ may *not* necessarily imply hardness for another value of $\kappa
Formalization of the efficiency gap calculation problem

*only two* parties: Party A and Party B

**Granularities of numeric parameters**

- **Course granularity:**
  - Pop($y$)'s are numbers of arbitrary size
  - total number of bits contributes to input size
  - data at the “county” level or “census block group” level

- **Fine granularity:**
  - $\forall$ cell or node $y$: $0 < \text{Pop}(y) \leq c$ for some fixed constant $c$
  - data at the “Voting Tabulation District” (VTD) level or “census block” level

- **Ultra-fine granularity:**
  - $\forall$ cell or node $y$: $\text{Pop}(y) = c$ for some fixed constant $c$
  - theoretically interesting case, but practically a bit unrealistic
Formalization of the efficiency gap calculation problem

*only two* parties: Party A and Party B

\[ \kappa \] number of districts

\[ S \] set of all cells in given polygonal map \( \mathcal{P} \)

or, set of all nodes in given planar graph \( G(\mathcal{P}) \)

districting scheme: partition of \( S \) into \( \kappa \) subsets \( S_1, \ldots, S_\kappa \)

Notations for each \( S_j \):

**Party affiliations in** \( S_j \)

\[
\text{PartyA}(S_j) = \sum_{y \in S_j} \text{PartyA}(y)
\]

\[
\text{PartyB}(S_j) = \sum_{y \in S_j} \text{PartyB}(y)
\]

**Population of** \( S_j \)

\[
\text{Pop}(S_j) = \text{PartyA}(S_j) + \text{PartyB}(S_j)
\]

Legal requirements for valid re-districting plans:

- Every \( S_j \) must be a connected polygon
- Populations of different \( S_j \)’s must be as equal as possible
Formalization of the efficiency gap calculation problem

*only two* parties: Party A and Party B

**Legal requirements for valid re-districting plans**

- Every $S_j$ must be a connected polygon
- Populations of different $S_j$’s must be as equal as possible
  - **Strict partitioning criteria**
    \[ \{S_1, \ldots, S_\kappa\} \text{ is an exact } \kappa\text{-equipartition of } S, \ i.e., \ \forall j : \text{Pop}(S_j) \in \left\lfloor \text{Pop}(S)/\kappa \right\rfloor, \left\lceil \text{Pop}(S)/\kappa \right\rceil \]
  - (Multiplicatively) approximate partitioning criteria
    \[ \{S_1, \ldots, S_\kappa\} \text{ is a } \varepsilon\text{-approximate } \kappa\text{-equipartition of } S, \ i.e., \ \frac{\max\{\text{Pop}(S_j)\}}{\min\{\text{Pop}(S_j)\}} \leq 1 + \varepsilon \]
    
    courts may allow a maximum value of $\varepsilon$ in the range of 0.05 to 0.1
    
    e.g., , (US Supreme Court ruling in Karcher v. Daggett, 1983)
  - **Additively approximate partitioning criteria**
    \[ \{S_1, \ldots, S_\kappa\} \text{ is an additive } \varepsilon\text{-approximate } \kappa\text{-equipartition of } S, \ i.e., \ \max\{\text{Pop}(S_j)\} \leq \min\{\text{Pop}(S_j)\} + \varepsilon \]
Formalization of the efficiency gap calculation problem

only two parties: Party A and Party B

“Wasted votes” for a district

- Total votes 100 (Party A needs 50 to win)
  - Party A vote 59
  - Party B vote 41
- Wasted votes for Party A 59 – 50 = 9
- Wasted votes for Party B 41
- Efficiency gap for $S_j$ 9 – 41 = -32

\[
\text{Effgap}(S_j) = \begin{cases} 
(\text{PartyA}(S_j) - \frac{1}{2}\text{Pop}(S_j)) - \text{PartyB}(S_j) & \text{if } \text{PartyA}(S_j) \geq \frac{1}{2}\text{Pop}(S_j) \\
2\text{PartyA}(S_j) - \frac{3}{2}\text{Pop}(S_j) & \text{otherwise}
\end{cases}
\]

from the point of view of Party A (the victim party of gerrymandering)
Formalization of the efficiency gap calculation problem

*only two* parties: Party A and Party B

“Wasted votes” for a district

- Total votes **100** (Party A needs 50 to win)
  - Party A vote      **41**
  - Party B vote      **59**
- Wasted votes for Party A                   **41**
- Wasted votes for Party B  **59 - 50 = 9**
- Efficiency gap for  **$S_j$**     **41 - 9 = -32**

Effgap($S_j$) = \[
\begin{cases}
  \text{PartyA}(S_j) - \left( \text{PartyB}(S_j) - \frac{1}{2} \text{Pop}(S_j) \right) & \text{if } \text{PartyA}(S_j) < \frac{1}{2} \text{Pop}(S_j) \\
  2\text{PartyA}(S_j) - \frac{1}{2} \text{Pop}(S_j) & \text{otherwise}
\end{cases}
\]

from the point of view of Party A (the victim party of gerrymandering)
Formalization of the efficiency gap calculation problem

*only two parties: Party A and Party B*

\[ \kappa \quad \text{number of districts} \]

\[ S \quad \text{set of all cells in given polygonal map } \mathcal{P} \]

\[ \text{or, set of all nodes in given planar graph } G(\mathcal{P}) \]

\[ \text{districting scheme} \quad \text{partition of } S \text{ into } \kappa \text{ subsets } S_1, \ldots, S_\kappa \]

\[
\text{Effgap}_{\kappa}(\mathcal{P}, S_1, \ldots, S_\kappa) = \left| \sum_{j=1}^{\kappa} \text{Effgap}(S_j) \right|
\]

(to be minimized)

from the point of view of Party A (the victim party of gerrymandering)
Formalization of the efficiency gap calculation problem

*only two* parties: Party A and Party B

\[ \kappa\text{-district Minimum Wasted Vote Problem (MIN-WVP}_\kappa) \]

**Input**
- map \( \mathcal{P} \) with \( \text{Pop}(y), \text{PartyA}(y), \text{PartyB}(y) \) for every cell \( y \in \mathcal{P} \)
- integer \( 1 < \kappa \leq |\mathcal{P}| \)

**Assumption**
\( \mathcal{P} \) has at least one \( \kappa \)-equipartition

**Valid solution**
Any \( \kappa \)-equipartition \( S_1, \ldots, S_\kappa \) of \( \mathcal{P} \)

**Objective**
\[ \text{minimize} \ \text{Effgap}_\kappa(\mathcal{P}, S_1, \ldots, S_\kappa) = | \sum_{j=1}^{\kappa} \text{Effgap}(S_j) | \]

**Notation**
\[ \text{OPT}_\kappa(\mathcal{P}) \overset{\text{def}}{=} \min \{ \text{Effgap}_\kappa(\mathcal{P}, S_1, \ldots, S_\kappa) \mid S_1, \ldots, S_\kappa \text{ is a } \kappa \text{-equipartition of } \mathcal{P} \} \]

\(\dagger\) in exact or approximate sense
A numerical example to illustrate efficiency gap calculation problem

\[
\begin{array}{c}
\text{PartyA}_{2,3} \\
\text{Pop}_{2,3} = 11 + 39 \quad \text{PartyB}_{2,3} \\
\text{Pop}_{2,3} = 50 \\
\kappa = 3 \\
|\mathcal{P}| = 15 \\
\sum_{i,j} \text{Pop}_{i,j} = 1200
\end{array}
\]

Two possible district maps

<table>
<thead>
<tr>
<th>PartyA(Q₁)</th>
<th>PartyB(Q₁)</th>
<th>Effgap(Q₁)</th>
</tr>
</thead>
<tbody>
<tr>
<td>208</td>
<td>192</td>
<td>-184</td>
</tr>
<tr>
<td>170</td>
<td>230</td>
<td>140</td>
</tr>
<tr>
<td>88</td>
<td>312</td>
<td>-24</td>
</tr>
</tbody>
</table>

\[
\text{Effgap}(\mathcal{P}, Q₁, Q₂, Q₃) = |− 184 + 140 − 24| = 68
\]

<table>
<thead>
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<td>192</td>
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</tr>
<tr>
<td>134</td>
<td>266</td>
<td>58</td>
</tr>
<tr>
<td>124</td>
<td>276</td>
<td>48</td>
</tr>
</tbody>
</table>

\[
\text{Effgap}(\mathcal{P}, Q₁, Q₂, Q₃) = |− 184 + 58 + 48| = 78
\]
Mathematical properties of $\text{Effgap}_κ(\mathcal{P}, S_1, \ldots, S_κ)$: set of attainable values
assume strict $κ$-equipartition, i.e., $\text{Pop}(S_1) = \cdots = \text{Pop}(S_κ)$

Lemma 1

$\triangleright$ $\text{Effgap}_κ(\mathcal{P}, S_1, \ldots, S_κ)$ assumes one of the $κ + 1$ values:

$$2 \times \text{PartyA}(\mathcal{P}) - \left( z + \frac{κ}{2} \right) \frac{\text{Pop}(\mathcal{P})}{κ} \quad \text{for } z = 0, 1, \ldots, κ$$

$\triangleright$ If $\text{Effgap}_κ(\mathcal{P}, S_1, \ldots, S_κ) = 2 \text{PartyA}(\mathcal{P}) - \left( z + \frac{κ}{2} \right) \frac{\text{Pop}(\mathcal{P})}{κ}$ then

$$\frac{\text{Pop}(\mathcal{P})}{2κ} z \leq \text{PartyA}(\mathcal{P}) \leq \frac{\text{Pop}(\mathcal{P})}{2κ} z + \frac{1}{2} \text{Pop}(\mathcal{P})$$

Illustrative example: $κ = 2$

only 3 possible values of $\text{Effgap}_2(\mathcal{P}, S_1, S_2)$

$$2 \times \text{PartyA}(\mathcal{P}) - \frac{1}{2} \text{Pop}(\mathcal{P}) \quad \text{or} \quad 2 \times \text{PartyA}(\mathcal{P}) - \text{Pop}(\mathcal{P}) \quad \text{or} \quad 2 \times \text{PartyA}(\mathcal{P}) - \frac{3}{2} \text{Pop}(\mathcal{P})$$
First a “somewhat” bad news
(worst-case computational complexity meets gerrymandering)

Theorem (informal description)

Not only calculation of efficiency gap is NP-complete, but

assuming $P \neq NP$, no non-trivial approximation is possible in polynomial time

But, have no fear!

We have only shown hardness in theoretical worst-case
Worst-case computational complexity meets gerrymandering

Assumptions

- Map $\mathcal{P}$: rectilinear polygon without holes
- Strict partitioning criteria: $\{S_1, \ldots, S_\kappa\}$ is exact $\kappa$-equipartition of $S$
- Course granularity: $\text{Pop}(y)$’s are numbers of arbitrary size
- $P \neq NP$

Theorem 1

For any rational constant $\varepsilon \in (0, 1)$, for any $\rho$ and all $2 \leq \kappa \leq \varepsilon |\mathcal{P}|$, the $\text{MIN-WVP}_\kappa$ problem for rectilinear polygon $\mathcal{P}$ does not admit a $\rho$-approximation algorithm.

Reduction: from PARTITION problem
Worst-case computational complexity meets gerrymandering

Assumptions

- Map $\mathcal{P}$: planar graph $G = (V, E)$
- (Multiplicatively) approximate partitioning criteria:
  $$\{S_1, \ldots, S_\kappa\} \text{ is a } \epsilon\text{-approximate } \kappa\text{-equipartition of } S, \text{ i.e., } \frac{\max\{\text{Pop}(S_j)\}}{\min\{\text{Pop}(S_j)\}} \leq 1 + \epsilon$$
- Fine granularity: $\forall$ node $y$: $0 < \text{Pop}(y) \leq c$ for some fixed constant $c$

Theorem 2

For any constant $0 < \epsilon < 1/2$,
- computing an exact solution of the $\text{MIN-WVP}_\kappa$ problem is NP-complete

Proof does not provide any non-trivial inapproximability ratio

Reduction: from maximum independent set for planar cubic graphs
However, even in theory, we can efficiently compute efficiency gap under “reasonable” assumptions

e.g., with these assumptions:

- Input map: a rectilinear polygon $\mathcal{P}$ (without holes)
- Every district must have a “nice” shape ($y$-convex shape)
- $\kappa$ (number of districts) is constant
- Total population $\text{Pop}(\mathcal{P})$ is polynomial in number of cells $|\mathcal{P}|$
We developed and implemented a simple heuristic algorithm based on “local search” method

• Start with some existing or random valid solution

• Search for nearby valid solutions by randomly “swapping” local regions among various districts

  o Pitfall: can get stuck with far-away local optima but, does not seem to often occur for real maps

➢ Next few slides: results for real maps
Local search algorithm

START

i-th iteration $i<100$?

F

Return solution with min EG. If EG>7%, re-execute the algorithm

T

Select n random counties from the original dataset. $n \in (1,K)$ with $K<\text{tot_counties}$

j-th county $j<\text{n}$?

T

County already analyzed?

T

j-th county on district boundary?

F

Pop dev < 10 % AND new EG < old EG?

T

Calculate new districts and compute new EG

F: Go to (B)

Go to (B)

Next input: new EG and new map

Go to (B)

j-th county on district boundary?

F

Disconnected map?

T

Shift jth county to kth district.

F: go to (A)

k-th neighb $k<\text{tot_n}$?

(2)

County already analyzed?

T

k-th neighb $k<\text{tot_n}$?

F

Go to (B)

(2)

(2): Go to (B)

STOP
Wisconsin

Total votes: 2,841,407
Dem votes: 1,441,804 ~ 51%
Rep votes: 1,399,603 ~ 49%

Current EG: 14.8%
Dem #seats: 3
Rep #seats: 5

New EG: 3.8%
Dem #seats: 3
Rep #seats: 5
Virginia

Total votes: 3,569,498
Dem votes: 1,736,164 ~ 49%
Rep votes: 1,833,334 ~ 51%

Current EG: 22%
Dem #seats: 3
Rep #seats: 8

New EG: 3.6%
Dem #seats: 5
Rep #seats: 6
New EG: 3.3%
Dem #seats: 12
Rep #seats: 24

Current EG: 4.01%
Dem #seats: 12
Rep #seats: 24

Total votes: 7,379,170
Dem votes: 2,949,900 ~ 40%
Rep votes: 4,429,270 ~ 60%
Pennsylvania

2012 House Elections

Total votes: 5,374,461
Dem votes: 2,722,560 ~ 51%
Rep votes: 2,651,901 ~ 49%

Current EG: 23.8%
Dem #seats: 5
Rep #seats: 13

New EG: 8.64%
Dem #seats: 6
Rep #seats: 12
2016 Presidential Elections

Total votes: 5,896,628
Dem votes: 2,925,776 ~ 50%
Rep votes: 2,970,852 ~ 50%

New EG: 8.05%
Dem #seats: 7
Rep #seats: 11

Current EG: 14.34%
Dem #seats: 6
Rep #seats: 12

New EG: 3%
Dem #seats: 8
Rep #seats: 10

Created on Feb. 2018 (by a local court in PA) and based on symmetry between seat share and vote share.
Some Interesting Insights based on simulation results

- **Seat gain vs. efficiency gap**
  - lower efficiency gap does not necessarily lead to seat gains for the loosing party

- **Compactness vs. efficiency gap**
  - Our new district map have fewer districts that are oddly shaped compared to the current gerrymandered maps

- **How natural are current gerrymandered districts?**
  - It seems that original gerrymandered districts are far from being a product of arbitrarily random decisions
Future research

Science of gerrymandering is a huge garden with so many unknown fruits for hungry theoretical computer scientists!

So many questions, so few answers

- Define and analyze new quantitative measures of gerrymandering
  - What about 3 or more party systems?

- Analyze computational complexities of existing measures of gerrymandering

- Join court cases as an expert witness and convince judges that computational complexity matters
Thank you for your attention!

Questions?