Existence and Construction of a Universal Space of Infinite Hierarchies of Beliefs

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Outline

The Problem

In an interactive decision-making setting,
is it possible to represent and reason with
an agent's beliefs about other agents, it's beliefs about other agents' beliefs,
and so on, ad infinitum ?

Introductory remarks

- Game theory involves the study of interactive decision-making environments where multiple agents (with possibly conflicting objectives) seek to maximize their own utility
- Both *descriptive* (social mores, conventions, rules-of-thumb etc.) and *prescriptive* (elegant solution concepts, for e.g. Nash equilibria, Bayesian equilibria, Shapley value, stable sets etc.)
- Applications in artificial intelligence, economics, political science, operations research and other areas
- In settings *without* uncertainty, the pertinent aspects of the game are *commonly known*, leading to a relatively simpler *game-theoretic* analysis
- **Problem:** Most real-world scenarios involve uncertainty!

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Solutions for Now-or-Never Game (Blind version)

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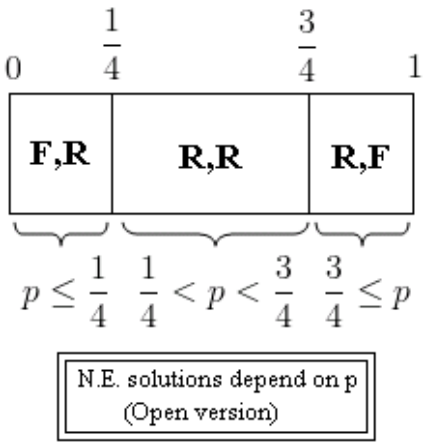

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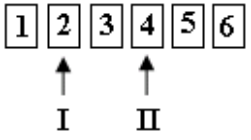

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- $B_I(p \leq \frac{1}{4})$; this implies that *I* should *Fold*; but wait....
- What if $B_I(B_{II}(p \geq \frac{3}{4}))$? Then $B_I($*II will Fold*$)$; in this case *I* should *Raise*!!
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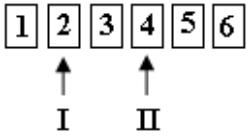

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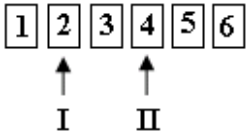

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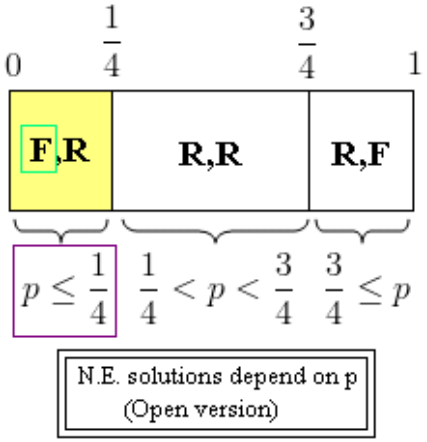

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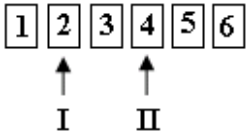

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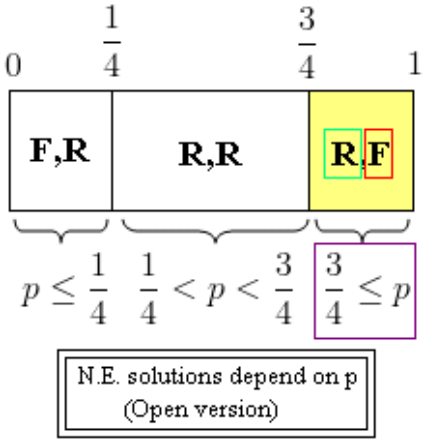

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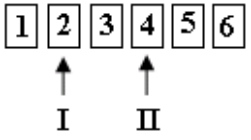

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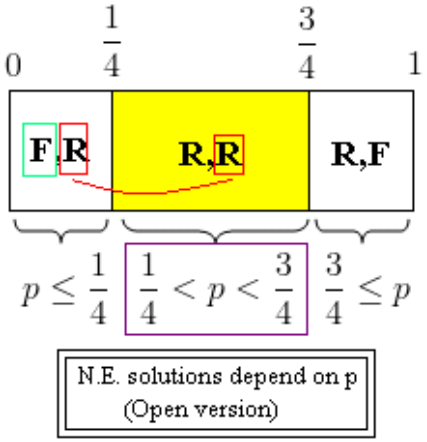

Now-or-Never : Epistemic Analysis

- If the players' relative strength, or uncertainty thereof, is commonly known, the *Nash Equilibrium* solution is easily obtained
- The *open* version does not belong to the above two cases, therefore, it becomes necessary for the agents to represent and reason with deeper levels of mutual beliefs
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  - But, this may involve dealing with infinite hierarchies of beliefs!

Problem: Explicitly modeling an agent's epistemic situation may necessitate the handling of infinite hierarchies of beliefs

Outline

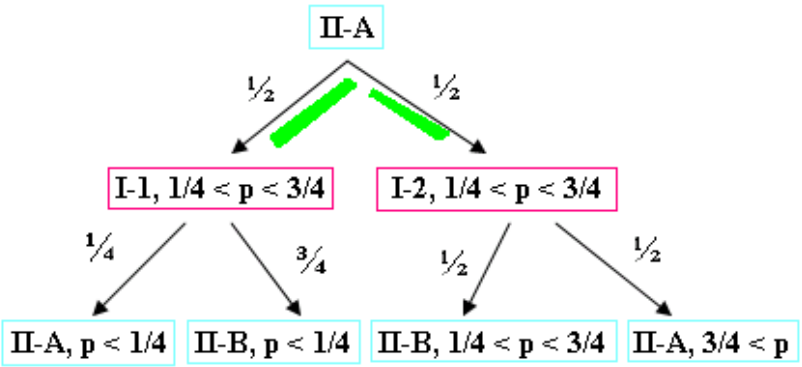
Harsanyi's Solution : Type Spaces

- Design a space where all levels of mutual beliefs are *implicity* specified
- Example
  - There are two types of player *I* (1 and 2) and two types of player *II* (*A* and *B*)
  - Each type is characterized by a probability distribution over *p* (the ground space of uncertainty) and the opponents' types

	$p \leq \frac{1}{4}$		$\frac{1}{4} < p < \frac{3}{4}$		$\frac{3}{4} \leq p$	
	A	B	A	B	A	B
1	¼	¾	0	0	0	0
2	0	0	0	½	½	0

	$p \leq \frac{1}{4}$		$\frac{1}{4} < p < \frac{3}{4}$		$\frac{3}{4} \leq p$	
	1	2	1	2	1	2
A	0	0	½	½	0	0
B	0	0	0	0	¼	¾

Unfolding a type space



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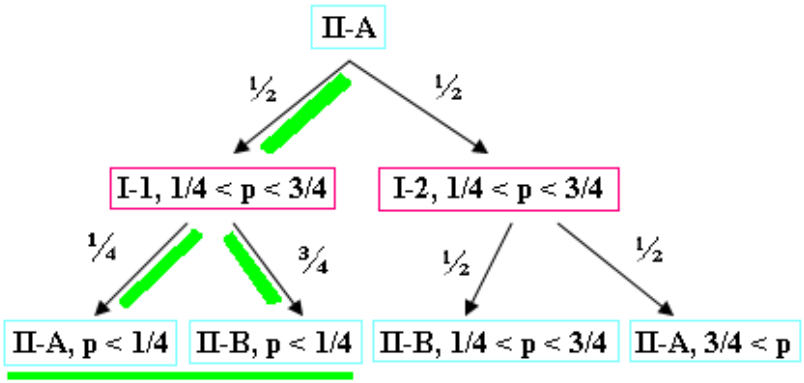

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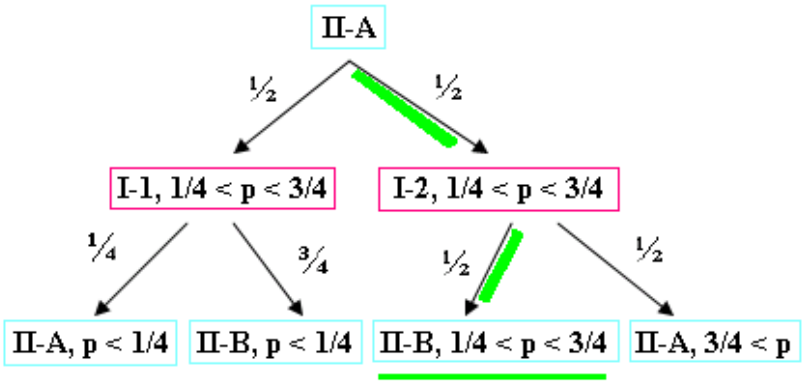

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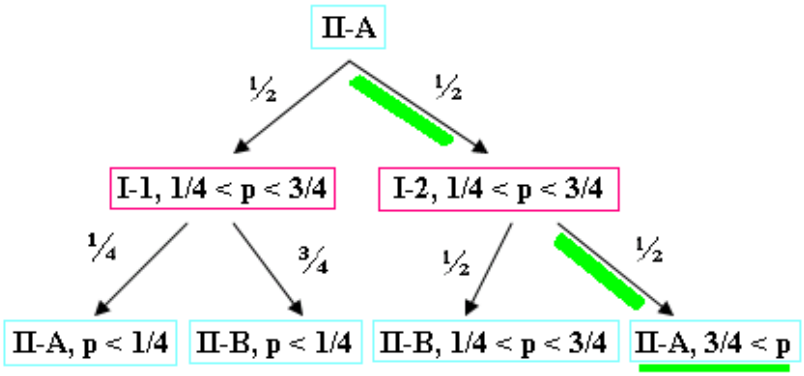

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Universal Type Spaces

- Implicit descriptions are not expressed in terms of space of uncertainty – instead, on one particular type space
- What if some types are missed in this space, and can be found only in a larger type space that contains the former?
- If this is true for any type space, then the concept of type space is necessarily retrictive

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Problem: Can we find a *universal* type space, i.e. one that contains *all type spaces*?

The Problem (restated)

Does the universal type space exist?

The Problem (restated)

Does the universal type space exist?
The Answer : **YES !**

Outline

Universal Beliefs Space and Universal Type Space

Given a basic space of uncertainty, S, of all possible values of the parameters of the game, there exists

- $\mathcal{Y}$, the universal beliefs space generated by S, and
- $\mathcal{T}$, the universal type space (of all possible types of a player in the game), such that,
  - $\mathcal{Y} = S \times [\mathcal{T}]^n$
  - $\mathcal{T} = \Pi(S \times [\mathcal{T}]^{n-1})$

S-based abstract beliefs space (BL-space)

Definition

An S-based abstract beliefs space (BL-space) is an (n+3) tuple $(C, S, f, (t^i)_{i=1}^n)$ where C is a compact set, S is some compact space, f is a continuous mapping $f : C \to S$ and $t^i, i = 1, 2, ..., n$, are continuous mappings $t^i : C \to \Pi(C)$ satisfying

$$\tilde{c} \in Sup(t^i(c)) \Rightarrow t^i(\tilde{c}) = t^i(c)$$

Remarks:

- Our objective is to obtain a space of states C, where each point uniquely determines a set of parameters $s \in S$ and the type t^i of each player
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Spaces of hierarchies of beliefs

Definition

Define level-k belief hierarchy spaces,

$$X_0 = S$$
$$T_k = \Pi(X_{k-1})$$
$$X_k = X_{k-1} \times [T_k]^n = S \times \mathop{\times}_{l=1}^{k} [T_l]^n; k = 1, 2, ...$$

Define the space of infinite hierarchies of beliefs generated by S

$$X(S) := X = S \times \mathop{\times}_{l=1}^{\infty} [T_l]^n$$

Coherent beliefs hierarchies of level K $(K = 1, 2, ...)$

Definition

A *coherent beliefs hierarchy* of level $K(K = 1, 2, ...)$ is a sequence $(C_0, C_1, ..., C_K)$ where:

1. $C_0 \subseteq S$ and for $k = 1, ..., K$, $C_k \subseteq C_{k-1} \times [\Pi(C_{k-1})]^n$

for e.g. $C_2 \subseteq C_0 \times \Pi(C_0) \times \Pi(C_0)$

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Denote by ρ_{k-1} and t^i the projections of C_k on C_{k-1} and the i-th copy of $\Pi(C_{k-1})$ respectively and $\forall c_k \in C_k$ let $c_{k-1} = \rho_{k-1}(c_k)$, then $\forall i$:

H1: the marginal distribution of $t^i(c_k)$ on C_{k-2} is $t^i(c_{k-1})$; and,

H2: the marginal distribution of $t^i(c_k)$ on the i-th copy of $\Pi(C_{k-2})$ is the unit mass at $t^i(c_{k-1})$

2. $\rho_{k-1}(C_k) = C_{k-1}; k = 1, ..., K$

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Coherent belief hierarchies for two players

$$C_0 = S$$

$$C_1 = S \times \Pi(S) \times \Pi(S)$$

$$C_2 \subseteq S \times \Pi(S) \times \Pi(S)$$

$$\times [\Pi(S \times \Pi(S) \times \Pi(S))]$$

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Player 1
Player 2

Outline

Theorem

For any compact S and positive integer n, there are spaces $\mathcal{Y}$ and $\mathcal{T}$ such that:

1. $\mathcal{Y} = S \times [\mathcal{T}]^n$

2. $\mathcal{T} = \Pi(S \times [\mathcal{T}]^{n-1})$

3. *There are compact spaces $\{Y_k\}_{k=0}^{\infty}$ s.t. $\forall k, (Y_0, Y_1, ..., Y_k)$ is a coherent beliefs hierarchy and $\mathcal{Y}$ is the projective limit $\{Y_k\}_{k=0}^{\infty}$ (with respect to the natural projection $\rho_{k-1} : Y_k \to Y_{k-1}$. We denote by ρ_k also the projection of $\mathcal{Y}$ on Y_k).*

4. *$\mathcal{Y}$ is an S-based BL-space (with the projections $f : \mathcal{Y} \to S$ and $t^i : \mathcal{Y} \to \mathcal{T}^i$*

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Proof Idea

Idea: The sequence $\{Y_k\}_{k=0}^{\infty}$ in [3] will be constructed, and $\mathcal{Y}$ will be defined as its projective limit and $\mathcal{T}^i$ as the projection of $\mathcal{Y}$ on player i's coordinates.

Construction of $\mathcal{Y}$

Define the sequence of spaces $\{Y_k\}_{k=0}^{\infty}$ as follows:

Definition

$Y_0 = S$ and for $k = 1, 2, ...$
$Y_k = \{y_k \in Y_{k-1} \times [\Pi(Y_{k-1})]^n$
such that,
Condition 1,H1 and 1,H2 of a *coherent beliefs hierarchy* are satisfied, i.e. $\forall i$ the marginal distribution of $t^i(y_k)$ on $Y_{k-2} \times [\Pi(Y_{k-2})]_i$ is $t^i(y_{k-1}) \times \delta_{t^i(y_{k-1})}\}$

Construction of $\mathcal{Y}$

Remarks:

- By definition, $\forall k, (Y_0, ..., Y_k)$ satisfies Condition 1 (H1 and H2) of a *coherent beliefs hierarchy*
- If, in addition, Condition 2 is also satisfied, i.e. if $\rho_k(Y_{k+1}) = Y_k$, we then have that:

$\forall k, (Y_0, ..., Y_k)$ is a coherent beliefs hierarchy, and,
$\forall k, \rho_k(\mathcal{Y}) = Y_k$, in particular $\mathcal{Y} \neq \emptyset$

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We will prove that Condition 2 is indeed satisfied

Proof of $\forall k, \rho_k(Y_{k+1}) = Y_k$

The proof proceeds by induction on k

- Base case: It holds for $k = 0$ that $\rho_0(Y_1) = Y_0$ since $Y_0 = S$ and $Y_1 = S \times [\Pi(S)]^n$ and all conditions are satisfied vacuously
- Assume that $\rho_{k-1}(Y_k) = Y_{k-1}$. Then, we have to show that $\rho_k(Y_{k+1}) = Y_k$, i.e. that any point $y \in Y_k$ can be extended to a point $(y, t^1_{k+1}, ..., t^n_{k+1}) \in Y_{k+1}$
  - i.e. we have to establish the existence of an n-tuple $t^1_{k+1}, ..., t^n_{k+1}$ of probability distributions $t^i_{k+1} \in \Pi(Y_k)$ satisfying Conditions 1,H1 and 1,H2, namely that the marginal distribution of t^i_{k+1} on $Y_{k-1} \times [\Pi(Y_{k-1})]_i$ is $t^i_k \times \delta_{t^i(y)}$
  - we have to show that each of these marginals can be extended to a probability distribution $t^i k + 1$ on $Y_{k-1} \times [\Pi(Y_{k-1})]_1 \times ... \times [\Pi(Y_{k-1})]_n$ supported by its subset Y_k

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  - we have to show that each of these marginals can be extended to a probability distribution $t^i k + 1$ on $Y_{k-1} \times [\Pi(Y_{k-1})]_1 \times ... \times [\Pi(Y_{k-1})]_n$ supported by its subset Y_k

Proof of $\forall k, \rho_k(Y_{k+1}) = Y_k$

The proof proceeds by induction on k

- Base case: It holds for $k = 0$ that $\rho_0(Y_1) = Y_0$ since $Y_0 = S$ and $Y_1 = S \times [\Pi(S)]^n$ and all conditions are satisfied vacuously
- Assume that $\rho_{k-1}(Y_k) = Y_{k-1}$. Then, we have to show that $\rho_k(Y_{k+1}) = Y_k$, i.e. that any point $y \in Y_k$ can be extended to a point $(y, t^1_{k+1}, ..., t^n_{k+1}) \in Y_{k+1}$
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Marginal Extension Lemma on Compact Spaces

Lemma

Let A and B be compact sets, D a compact subset of $A \times B$ and $q \in \Pi(A)$. A necessary and sufficient condition for the existence of $p \in \Pi(D)$ whose marginal distribution on A is q, is that $q(D_A) = 1$, where D_A is the projection of D on A.

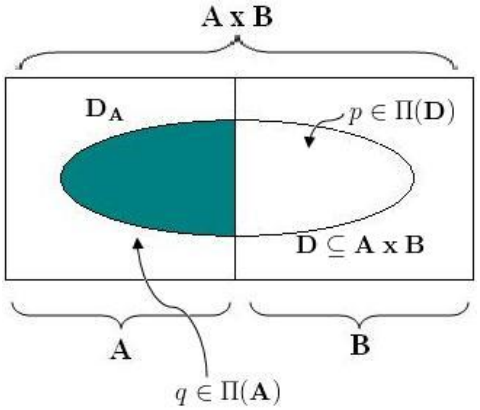

Figure: Marginal Extension Lemma

Proof of $\forall k, \rho_k(Y_{k+1}) = Y_k$

- Using Marginal Extension Lemma, it remains to prove that

$$\text{Support } (t^i(y) \times \delta_{t^i(y)}) \subseteq \text{projection of } Y_k \text{ on } Y_{k-1} \times [\Pi(Y_{k-1})]_i$$

$$\Leftrightarrow$$

$$\text{Support } (t^i(y)) \times \{t^i(y)\} \subseteq \text{projection of } Y_k \text{ on } Y_{k-1} \times [\Pi(Y_{k-1})]_i$$

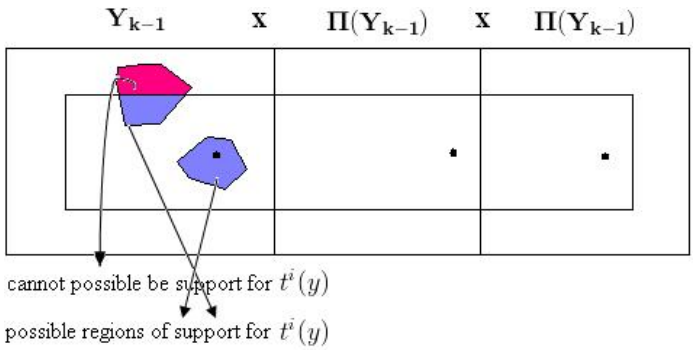

Figure: Extending a point $y \in Y_k$ to a point in Y_{k+1}

Proof of $\forall k, \rho_k(Y_{k+1}) = Y_k$

- Let $(\tilde{(y)}_{k-1}, t^i(y)) \in Support(t^i(y)) \times \{t^i(y)\}$
  i.e $\tilde{y}_{k-1} \in Support(t^i(y)) \subseteq Y_{k-1}$
- Since by induction hypothesis, $\rho_{k-1}(Y_k) = Y_{k-1}$, there is an extension $(\tilde{y}_{k-1}, \tilde{t}_k^1, ..., \tilde{t}_k^n) \in Y_k$.
  - <u>Claim</u>: If in this point, we replace $\tilde{t}_k^i$ by $t^i(y)$ we obtain a point which is also in Y_k, proving that $(\tilde{y}_{k-1}, t^i(y))$ is in the projection of Y_k on $Y_{k-1} \times [\Pi(Y_{k-1})]_i$, thus completing the proof.

Proof of claim : $(\tilde{y}_{k-1}, \tilde{t}_k^1, ..., t^i(y), ...\tilde{t}_k^n) \in Y_k$

- Claim: If at the point $(\tilde{y}_{k-1}, \tilde{t}_k^1, ..., \tilde{t}_k^n) \in Y_k$, we replace $\tilde{t}_k^i$ by $t^i(y)$, we obtain a point which is also in Y_k
- Note that all conditions concerning $\tilde{t}_k^j, j \neq i$ are satisfied since $\tilde{t}_k^1, ..., \tilde{t}_k^n) \in Y_k$
- Since $t^i(y)$ assigns probability 1 to $t^i(\rho_{k-1}(y))$ it follows that $t^i(\tilde{y}_{k-1}) = t^i(\rho_{k-1}(y))$. Therefore, the conditions concerning $t^i(y)$ are satisfied

Proof of claim : $(\tilde{y}_{k-1}, \tilde{t}_k^1, ..., t^i(y), ...\tilde{t}_k^n) \in Y_k$

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This ends the proof

Type Space for the *Now-or-Never* game

	$p \leq \frac{1}{4}$			$\frac{1}{4} < p < \frac{3}{4}$			$\frac{3}{4} \leq p$		
	A	**B**	**C**	**A**	**B**	**C**	**A**	**B**	**C**
A	0	**1**	0	0	0	0	0	0	0
B	0	0	0	0	**1**	0	0	0	0
C	0	0	0	0	0	0	0	**1**	0

Thank You ! Any Questions ?