Existence and Construction of a Universal Space of Infinite Hierarchies of Beliefs

Paul Varkey

Artificial Intelligence Laboratory, Department of Computer Science, UIC

3rd Graduate Student Conference in Probability, May 1st, 2009
Introduction

The Problem
Introductory Remarks
An Example

The Harsanyi Program
Universal Type Spaces
The Problem (restated)

Definitions and Preliminary Results
Universal Beliefs Space and Universal Type Space
The $S$-based abstract beliefs space (BL-space)
Coherent beliefs hierarchies

Construction of Universal Beliefs and Type Spaces
Main Theorem
Proof
Conclusion: A Type Space for the Now-or-Never game

Paul Varkey
Universal Space of Infinite Hierarchies of Beliefs
In an interactive decision-making setting, is it possible to represent and reason with an agent’s beliefs about other agents, it’s beliefs about other agents’ beliefs, and so on, ad infinitum?
Game theory involves the study of interactive decision-making environments where multiple agents (with possibly conflicting objectives) seek to maximize their own utility. Both descriptive (social mores, conventions, rules-of-thumb etc.) and prescriptive (elegant solution concepts, for e.g. Nash equilibria, Bayesian equilibria, Shapley value, stable sets etc.) applications in artificial intelligence, economics, political science, operations research and other areas. In settings without uncertainty, the pertinent aspects of the game are commonly known, leading to a relatively simpler game-theoretic analysis. Problem: Most real-world scenarios involve uncertainty!
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**Problem:** Most real-world scenarios involve uncertainty!
Now-or-Never

- A simplified one-stage gambling game
- Two players: I and II
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- Two versions:
  - Blind: Players cannot look at their card
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Payoff matrices for Now-or-Never Game:

<table>
<thead>
<tr>
<th></th>
<th>Player II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raise</td>
<td>10, -10</td>
</tr>
<tr>
<td></td>
<td>5, -5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Player I</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raise</td>
<td>10, 10</td>
</tr>
<tr>
<td></td>
<td>-5, -5</td>
</tr>
</tbody>
</table>

<table>
<thead>
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</thead>
<tbody>
<tr>
<td>Raise</td>
<td>-10, 10</td>
</tr>
<tr>
<td></td>
<td>-5, -5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Player I</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raise</td>
<td>-5, 5</td>
</tr>
<tr>
<td></td>
<td>-5, -5</td>
</tr>
</tbody>
</table>

$\text{Player I} \succeq \text{Player II}$

$\text{Player II} \succeq \text{Player I}$

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Universal Space of Infinite Hierarchies of Beliefs
Now-or-Never

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Player I $\succeq$ Player II

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<tr>
<td>Raise</td>
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Player II $\succeq$ Player I

Payoff matrices for Now-or-Never Game
**Now-or-Never**

- A simplified one-stage gambling game
- Two players: I and II
- There are \( n \) cards (numbered from 1 through \( n \))
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The table below shows the possible outcomes for the Now-or-Never Game:

<table>
<thead>
<tr>
<th></th>
<th>Player II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raise</td>
<td>10, -10, 5, -5</td>
</tr>
<tr>
<td>Fold</td>
<td>-5, -5, -5, -5</td>
</tr>
</tbody>
</table>

\[ \text{Player I} \geq \text{Player II} \]

And:

<table>
<thead>
<tr>
<th></th>
<th>Player II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raise</td>
<td>-10, 10, -5, -5</td>
</tr>
<tr>
<td>Fold</td>
<td>-5, 5, -5, -5</td>
</tr>
</tbody>
</table>

\[ \text{Player II} \geq \text{Player I} \]

Payoff matrices for the Now-or-Never Game.
Now-or-Nevert

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<tr>
<td>Raise</td>
<td></td>
</tr>
<tr>
<td>[p]</td>
<td>10, -10</td>
</tr>
<tr>
<td>Fold</td>
<td>-5, -5</td>
</tr>
<tr>
<td>Player I ≥ Player II</td>
<td></td>
</tr>
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</table>

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</thead>
<tbody>
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<td>Raise</td>
<td></td>
</tr>
<tr>
<td>[1-p]</td>
<td>-10, 10</td>
</tr>
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**Payoff matrices for Now-or-Never Game**

![Payoff matrices for Now-or-Never Game](image)
Now-or-Never

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Player I $\succeq$ Player II

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Player II $\succeq$ Player I

Payoff matrices for Now-or-Never Game
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<table>
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<th>( P )</th>
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<th>Player I</th>
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<tbody>
<tr>
<td>Raise</td>
<td>10, -10</td>
<td>5, -5</td>
<td></td>
</tr>
<tr>
<td>Fold</td>
<td>-5, -5</td>
<td>-5, -5</td>
<td></td>
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<table>
<thead>
<tr>
<th>( 1 - P )</th>
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<th>Player I</th>
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<tr>
<td>Raise</td>
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Player I \( \succeq \) Player II
Player II \( \succeq \) Player I

Payoff matrices for Now-or-Never Game
A simplified one-stage gambling game
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\[
\begin{array}{cccc}
\text{Player I} & \text{Player II} & \text{Raise} & \text{Fold} \\
\text{Raise} & 10 & -10 & 5 & -5 \\
\text{Fold} & -5 & -5 & -5 & -5 \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{Player II} & \text{Player I} & \text{Raise} & \text{Fold} \\
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\end{array}
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Now-or-Ne ver: Solutions

- In the \textit{blind} version, both players \textit{Raise} in the Nash Equilibrium solution.
- In the \textit{open} version,
  - If I \textit{Raises}, II should \textit{Raise} if $-20 \times p + 10 > -5$, i.e. if $p \leq \frac{3}{4}$.
  - If I \textit{Folds}, II should \textit{Raise} if $-10 \times p + 5 > -5$, i.e. if $p \leq 1$.
  - If II \textit{Raises}, I should \textit{Raise} if $20 \times p - 10 > -5$, i.e. if $p > \frac{1}{4}$.
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\begin{itemize}
  \item \textbf{Player I} \hspace{2cm} \textbf{Player II}
  \begin{itemize}
    \item \text{Raise} \hspace{2cm} \text{Raise}
      \begin{itemize}
        \item $10 , -10$
      \end{itemize}
    \item \text{Fold} \hspace{2cm} \text{Fold}
      \begin{itemize}
        \item $-5 , -5$
      \end{itemize}
  \end{itemize}

\textit{Player I \succeq Player II}

\begin{itemize}
  \item \textbf{Player II} \hspace{2cm} \textbf{Player I}
  \begin{itemize}
    \item \text{Raise} \hspace{2cm} \text{Raise}
      \begin{itemize}
        \item $-10 , 10$
      \end{itemize}
    \item \text{Fold} \hspace{2cm} \text{Fold}
      \begin{itemize}
        \item $-5 , 5$
      \end{itemize}
  \end{itemize}

\textit{Player II \succeq Player I}

\end{itemize}
Now-or-Never: Solutions

- In the **blind** version, both players \textit{Raise} in the \textit{Nash Equilibrium} solution.

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![Game Matrix](image-url)
In the *blind* version, both players *Raise* in the *Nash Equilibrium* solution.

In the *open* version,

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\[
\begin{array}{c|cc}
 & \text{Raise} & \text{Fold} \\
\hline
\text{Raise} & 10, -10 & 5, -5 \\
\text{Fold} & -5, -5 & -5, -5 \\
\end{array}
\]

*Player I $\succeq$ Player II*

\[
\begin{array}{c|cc}
 & \text{Raise} & \text{Fold} \\
\hline
\text{Raise} & -10, 10 & -5, -5 \\
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\end{array}
\]

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\[
\begin{array}{c|cc}
\text{Player I} & \text{Raise} & \text{Fold} \\
\hline
\text{Raise} & 10, -10 & 5, -5 \\
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\end{array}
\]

*Player I ≥ Player II*

\[
\begin{array}{c|cc}
\text{Player II} & \text{Raise} & \text{Fold} \\
\hline
\text{Raise} & -10, 10 & -5, -5 \\
\text{Fold} & -5, 5 & -5, -5 \\
\end{array}
\]

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In the *blind* version, both players *Raise* in the *Nash Equilibrium* solution.

In the *open* version,
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- If I *Folds*, II should *Raise* if \(-10 \times p + 5 > -5\), i.e. if \(p \leq 1\)
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\[
\begin{array}{c|c|c}
\text{Player I} & \text{Player II} & \\
\hline
\text{Raise} & 10, -10 & 5, -5 \\
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\end{array}
\]

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\[
\begin{array}{c|c|c}
\text{Player I} & \text{Player II} & \\
\hline
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\end{array}
\]

Player II $\succeq$ Player I

Payoff matrices for Now-or-Never Game
Now-or-Never: Solutions

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Consider a game instance with 6 cards (1, 2, 3, 4, 5 and 6) where
I receives 2 and II receives 4

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
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<td>6</td>
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Notation: $B_n(\phi)$ will be used to denote “Player n believes that $\phi$”

- $B_I(p \leq \frac{1}{4})$; this implies that I should Fold; but wait....
- What if $B_I(B_{II}(p \geq \frac{3}{4}))$? Then $B_I(II will Fold); in this case I should Raise!!
- It turns out that $B_I(B_{II}(\frac{1}{4} < p < \frac{3}{4}))$; so I should indeed Fold
Now-or-Never: Example Instances

- Consider a game instance with 6 cards (1, 2, 3, 4, 5 and 6) where I receives 2 and II receives 4.

  \[
  \begin{array}{cccccc}
  1 & 2 & 3 & 4 & 5 & 6 \\
  \hline
  \uparrow & \uparrow \\
  I & II
  \end{array}
  \]

- **Notation**: \( B_n(\phi) \) will be used to denote "Player n believes that \( \phi \)"

- \( B_I(p \leq \frac{1}{4}) \); this implies that I should Fold; but wait....

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\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
& & & & \\
I & & & & II
\end{array}
\]

**Notation**: \( B_n(\phi) \) will be used to denote “Player n believes that \( \phi \)”

- \( B_I(p \leq \frac{1}{4}) \); this implies that I should Fold;
  but wait....
- What if \( B_I(B_{II}(p \geq \frac{3}{4})) \)?
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- It turns out that \( B_I(B_{II}(\frac{1}{4} < p < \frac{3}{4})) \);
  so I should indeed Fold
Consider a game instance with 6 cards (1, 2, 3, 4, 5 and 6) where I receives 2 and II receives 4:

\[ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \]

\[ \begin{array}{c}
  \uparrow \\
  \text{I} \\
  \text{II} \\
\end{array} \]

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\begin{array}{cccccc}
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\uparrow & \uparrow & & & & \\
I & II & & & &
\end{array}
\]

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Now-or-Never: Epistemic Analysis

- If the players’ relative strength, or uncertainty thereof, is commonly known, the *Nash Equilibrium* solution is easily obtained.
- The *open* version does not belong to the above two cases, therefore, it becomes necessary for the agents to represent and reason with deeper levels of mutual beliefs.
  - But, this may involve dealing with infinite hierarchies of beliefs!
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**Problem:** Explicitly modeling an agent’s epistemic situation may necessitate the handling of infinite hierarchies of beliefs.
Outline

1. Introduction
   - The Problem
   - Introductory Remarks
   - An Example

2. The Harsanyi Program
   - Universal Type Spaces
   - The Problem (restated)

3. Definitions and Preliminary Results
   - Universal Beliefs Space and Universal Type Space
   - The S-based abstract beliefs space (BL-space)
   - Coherent beliefs hierarchies

4. Construction of Universal Beliefs and Type Spaces
   - Main Theorem
   - Proof
   - Conclusion: A Type Space for the Now-or-Never game
Harsanyi’s Solution : Type Spaces

- Design a space where all levels of mutual beliefs are *implicitly* specified.

Example

- There are two types of player I (1 and 2) and two types of player II (A and B).
- Each type is characterized by a probability distribution over \( p \) (the ground space of uncertainty) and the opponents’ types.

<table>
<thead>
<tr>
<th></th>
<th>( p \leq \frac{1}{4} )</th>
<th>( \frac{1}{4} &lt; p &lt; \frac{3}{4} )</th>
<th>( \frac{3}{4} \leq p )</th>
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<tr>
<td>A</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{3}{4} )</td>
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<tr>
<td>B</td>
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Unfolding a type space

\[
\begin{align*}
\Pi - A, \ p < \frac{1}{4} & \\
\Pi - B, \ p < \frac{1}{4} & \\
\Pi - B, \ 1/4 < p < 3/4 & \\
\Pi - A, \ 3/4 < p & \\
I - 1, \ 1/4 < p < 3/4 & \\
I - 2, \ 1/4 < p < 3/4 & \\
\end{align*}
\]

\[
B_{II-A} \left( \frac{1}{4} < p < \frac{3}{4} \right)
\]

\[
B_{II-A} \left( B_I \left( \begin{array}{l}
\frac{1}{4} < p < \frac{3}{4} : \frac{1}{2} \\
\frac{3}{4} < p : \frac{1}{4} \\
p > \frac{3}{4} : \frac{1}{4}
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Unfolding a type space

$$B_{II-A} \left( \frac{1}{4} < p < \frac{3}{4} \right)$$

$$B_{II-A} \left( B_I \left( \frac{1}{4} < p < \frac{3}{4} ; \frac{1}{4}, \frac{3}{4} \right) \right)$$
Unfolding a type space

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Implicit descriptions are not expressed in terms of space of uncertainty – instead, on one particular type space.

What if some types are missed in this space, and can be found only in a larger type space that contains the former?

If this is true for any type space, then the concept of type space is necessarily restrictive.
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Universal Type Spaces

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Universal Type Spaces

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**Problem:** Can we find a *universal* type space, i.e. one that contains *all* type spaces?
The Problem (restated)

Does the universal type space exist?
The Problem (restated)

Does the universal type space exist?
The Answer: YES!
Given a basic space of uncertainty, $S$, of all possible values of the parameters of the game, there exists

- $\mathcal{Y}$, the universal beliefs space generated by $S$, and
- $\mathcal{T}$, the universal type space (of all possible types of a player in the game), such that,

$$\mathcal{Y} = S \times [\mathcal{T}]^n$$

$$\mathcal{T} = \prod(S \times [\mathcal{T}]^{n-1})$$
S-based abstract beliefs space (BL-space)

Definition

An $S$-based abstract beliefs space (BL-space) is an $(n+3)$ tuple $(C, S, f, (t^i)_{i=1}^n)$ where $C$ is a compact set, $S$ is some compact space, $f$ is a continuous mapping $f : C \rightarrow S$ and $t^i$, $i = 1, 2, ..., n$, are continuous mappings $t^i : C \rightarrow \Pi(C)$ satisfying

$$\tilde{c} \in \text{Sup}(t^i(c)) \Rightarrow t^i(\tilde{c}) = t^i(c)$$

Remarks:

- Our objective is to obtain a space of states $C$, where each point uniquely determines a set of parameters $s \in S$ and the type $t^i$ of each player.
- The type $t^i$ is a probability distribution on $C$ which is coherent in the sense that each player knows his own type (or, in other words, is certain of his own beliefs).
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Spaces of hierarchies of beliefs

Definition

Define level-k belief hierarchy spaces,

\[ X_0 = S \]

\[ T_k = \Pi(X_{k-1}) \]

\[ X_k = X_{k-1} \times [T_k]^n = S \times \prod_{l=1}^{k} [T_l]^n; \quad k = 1, 2, \ldots \]

Define the space of infinite hierarchies of beliefs generated by \( S \)

\[ X(S) := X = S \times \prod_{l=1}^{\infty} [T_l]^n \]
Coherent beliefs hierarchies of level $K$ ($K = 1, 2, ...$)

Definition

A *coherent beliefs hierarchy* of level $K$ ($K = 1, 2, ...$) is a sequence $(C_0, C_1, ..., C_K)$ where:

1. $C_0 \subseteq S$ and for $k = 1, ..., K$, $C_k \subseteq C_{k-1} \times [\prod(C_{k-1})]^n$

For e.g. $C_2 \subseteq C_0 \times \prod(C_0) \times \prod(C_0)$

$C_2 \subseteq C_0 \times \prod(C_0) \times \prod(C_0) \times \prod(C_0 \times \prod(C_0) \times \prod(C_0)) \times \prod(C_0 \times \prod(C_0) \times \prod(C_0))$

Denote by $\rho_{k-1}$ and $t^i$ the projections of $C_k$ on $C_{k-1}$ and the $i$-th copy of $\prod(C_{k-1})$ respectively and for all $c_k \in C_k$ let $c_{k-1} = \rho_{k-1}(c_k)$, then:

H1: the marginal distribution of $t^i(c_k)$ on $C_{k-2}$ is $t^i(c_{k-1})$; and,

H2: the marginal distribution of $t^i(c_k)$ on the $i$-th copy of $\prod(C_{k-2})$ is the unit mass at $t^i(c_{k-1})$

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2. $\rho_{k-1}(C_k) = C_{k-1}; \ k = 1, \ldots, K$
Coherent belief hierarchies for two players

\[ C_0 = S \]
\[ C_1 = S \times \Pi(S) \times \Pi(S) \]
\[ C_2 \subseteq S \times \Pi(S) \times \Pi(S) \]
\[ \times [\Pi(S \times \Pi(S) \times \Pi(S))] \]
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Player 1
Player 2
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   - Proof
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Theorem

For any compact $S$ and positive integer $n$, there are spaces $\mathcal{Y}$ and $\mathcal{T}$ such that:

1. $\mathcal{Y} = S \times [\mathcal{T}]^n$
2. $\mathcal{T} = \Pi(S \times [\mathcal{T}]^{n-1})$
3. There are compact spaces $\{Y_k\}_{k=0}^{\infty}$ s.t. $\forall k, (Y_0, Y_1, ..., Y_k)$ is a coherent beliefs hierarchy and $\mathcal{Y}$ is the projective limit $\{Y_k\}_{k=0}^{\infty}$ (with respect to the natural projection $\rho_{k-1} : Y_k \rightarrow Y_{k-1}$. We denote by $\rho_k$ also the projection of $\mathcal{Y}$ on $Y_k$).
4. $\mathcal{Y}$ is an $S$-based BL-space (with the projections $f : \mathcal{Y} \rightarrow S$ and $t^i : \mathcal{Y} \rightarrow \mathcal{T}^i$)

$\mathcal{Y}$ will be called the Universal Beliefs (or, BL-) Space generated by $S$ (and $n$), and, $\mathcal{T}$ will be called the Universal Type Space generated by $S$ (and $n$).
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Proof Idea

Idea: The sequence \( \{Y_k\}_{k=0}^{\infty} \) in [3] will be constructed, and \( \mathcal{Y} \) will be defined as its projective limit and \( T^i \) as the projection of \( \mathcal{Y} \) on player i’s coordinates.
Define the sequence of spaces \( \{Y_k\}_{k=0}^{\infty} \) as follows:

**Definition**

\[
Y_0 = S \quad \text{and for } k = 1, 2, \ldots \n
Y_k = \{y_k \in Y_{k-1} \times [\prod(Y_{k-1})]^n \text{ such that,} \}
\]

Condition 1,H1 and 1,H2 of a coherent beliefs hierarchy are satisfied, i.e. \( \forall i \) the marginal distribution of \( t^i(y_k) \) on \( Y_{k-2} \times [\prod(Y_{k-2})]_i \) is \( t^i(y_{k-1}) \times \delta_{t^i(y_{k-1})} \)
Remarks:

- By definition, ∀k, (Y_0, ..., Y_k) satisfies Condition 1 (H1 and H2) of a coherent beliefs hierarchy.
- If, in addition, Condition 2 is also satisfied, i.e. if \( \rho_k(Y_{k+1}) = Y_k \), we then have that:

\[ \forall k, (Y_0, ..., Y_k) \text{ is a coherent beliefs hierarchy, and, } \forall k, \rho_k(\mathcal{Y}) = Y_k, \text{ in particular } \mathcal{Y} \neq \emptyset \]
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We will prove that Condition 2 is indeed satisfied.
Proof of $\forall k, \rho_k(Y_{k+1}) = Y_k$

The proof proceeds by induction on $k$

- **Base case**: It holds for $k = 0$ that $\rho_0(Y_1) = Y_0$ since $Y_0 = S$ and $Y_1 = S \times [\prod(S)]^n$ and all conditions are satisfied vacuously.

- **Assume** that $\rho_{k-1}(Y_k) = Y_{k-1}$. Then, we have to show that $\rho_k(Y_{k+1}) = Y_k$, i.e. that any point $y \in Y_k$ can be extended to a point $(y, t_{k+1}^1, ..., t_{k+1}^n) \in Y_{k+1}$

  - i.e. we have to establish the existence of an $n$-tuple $t_{k+1}^1, ..., t_{k+1}^n$ of probability distributions $t_{k+1}^i \in \prod(Y_k)$ satisfying Conditions 1,H1 and 1,H2, namely that the marginal distribution of $t_{k+1}^i$ on $Y_{k-1} \times [\prod(Y_{k-1})]^i$ is $t_{k}^i \times \delta_{t_{k}^i}(y)$

  - we have to show that each of these marginals can be extended to a probability distribution $t_{k+1}^i$ on $Y_{k-1} \times [\prod(Y_{k-1})]^i \times ... \times [\prod(Y_{k-1})]^n$ supported by its subset $Y_k$
Proof of $\forall k, \rho_k(Y_{k+1}) = Y_k$

The proof proceeds by induction on $k$

- **Base case:** It holds for $k = 0$ that $\rho_0(Y_1) = Y_0$ since $Y_0 = S$ and $Y_1 = S \times [\Pi(S)]^n$ and all conditions are satisfied vacuously.

- Assume that $\rho_{k-1}(Y_k) = Y_{k-1}$. Then, we have to show that $\rho_k(Y_{k+1}) = Y_k$, i.e. that any point $y \in Y_k$ can be extended to a point $(y, t_{k+1}^1, ..., t_{k+1}^n) \in Y_{k+1}$.

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  - we have to show that each of these marginals can be extended to a probability distribution $t^i_{k+1}$ on $Y_{k-1} \times [\Pi(Y_{k-1})]^i \times ... \times [\Pi(Y_{k-1})]^n$ supported by its subset $Y_k$. 

Paul Varkey  
Universal Space of Infinite Hierarchies of Beliefs
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Proof of $\forall k, \rho_k(Y_{k+1}) = Y_k$

The proof proceeds by induction on $k$

- **Base case:** It holds for $k = 0$ that $\rho_0(Y_1) = Y_0$ since $Y_0 = S$ and $Y_1 = S \times [\prod(S)]^n$ and all conditions are satisfied vacuously.

- **Assume that $\rho_{k-1}(Y_k) = Y_{k-1}$.** Then, we have to show that $\rho_k(Y_{k+1}) = Y_k$, i.e. that any point $y \in Y_k$ can be extended to a point $(y, t^{1}_{k+1}, \ldots, t^{n}_{k+1}) \in Y_{k+1}$

  - i.e. we have to establish the existence of an n-tuple $t^{1}_{k+1}, \ldots, t^{n}_{k+1}$ of probability distributions $t^{i}_{k+1} \in \prod(Y_k)$ satisfying Conditions 1,H1 and 1,H2, namely that the marginal distribution of $t^{i}_{k+1}$ on $Y_{k-1} \times [\prod(Y_{k-1})]_i$ is $t^{i}_{k} \times \delta_{t^{i}(y)}$

  - we have to show that each of these marginals can be extended to a probability distribution $t^{i}k + 1$ on $Y_{k-1} \times [\prod(Y_{k-1})]_1 \times \ldots \times [\prod(Y_{k-1})]_n$ supported by its subset $Y_k$
Lemma

Let $A$ and $B$ be compact sets, $D$ a compact subset of $A \times B$ and $q \in \Pi(A)$. A necessary and sufficient condition for the existence of $p \in \Pi(D)$ whose marginal distribution on $A$ is $q$, is that $q(D_A) = 1$, where $D_A$ is the projection of $D$ on $A$.

Figure: Marginal Extension Lemma
Proof of $\forall k, \rho_k(Y_{k+1}) = Y_k$

- Using Marginal Extension Lemma, it remains to prove that
  \[
  \text{Support } (t^i(y) \times \delta_{t^i(y)}) \subseteq \text{projection of } Y_k \text{ on } Y_{k-1} \times [\Pi(Y_{k-1})]_i
  \]
  \[
  \iff
  \text{Support } (t^i(y)) \times \{t^i(y)\} \subseteq \text{projection of } Y_k \text{ on } Y_{k-1} \times [\Pi(Y_{k-1})]_i
  \]

**Figure:** Extending a point $y \in Y_k$ to a point in $Y_{k+1}$
Proof of $\forall k, \rho_k(Y_{k+1}) = Y_k$

- Let $(\tilde{y}_{k-1}, t^i(y)) \in \text{Support}(t^i(y)) \times \{t^i(y)\}$
  i.e $\tilde{y}_{k-1} \in \text{Support}(t^i(y)) \subseteq Y_{k-1}$
- Since by induction hypothesis, $\rho_{k-1}(Y_k) = Y_{k-1}$, there is an extension $(\tilde{y}_{k-1}, \tilde{t}^1_k, ..., \tilde{t}^n_k) \in Y_k$.
  - **Claim**: If in this point, we replace $\tilde{t}^i_k$ by $t^i(y)$ we obtain a point which is also in $Y_k$, proving that $(\tilde{y}_{k-1}, t^i(y))$ is in the projection of $Y_k$ on $Y_{k-1} \times [\Pi(Y_{k-1})]_i$, thus completing the proof.
Proof of claim: \((\tilde{y}_{k-1}, \tilde{t}^1_k, \ldots, t^i(y), \ldots \tilde{t}^n_k) \in Y_k\)

- **Claim:** If at the point \((\tilde{y}_{k-1}, \tilde{t}^1_k, \ldots, \tilde{t}^n_k) \in Y_k\), we replace \(\tilde{t}^i_k\) by \(t^i(y)\), we obtain a point which is also in \(Y_k\).

- Note that all conditions concerning \(\tilde{t}^j_k, j \neq i\) are satisfied since \(\tilde{t}^1_k, \ldots, \tilde{t}^n_k \in Y_k\).

- Since \(t^i(y)\) assigns probability 1 to \(t^i(\rho_{k-1}(y))\) it follows that \(t^i(\tilde{y}_{k-1}) = t^i(\rho_{k-1}(y))\). Therefore, the conditions concerning \(t^i(y)\) are satisfied.
Proof of claim: 
\[(\tilde{y}_{k-1}, \tilde{t}_k^1, \ldots, t^i(y), \ldots \tilde{t}_k^n) \in Y_k\]

- **Claim:** If at the point \((\tilde{y}_{k-1}, \tilde{t}_k^1, \ldots, \tilde{t}_k^n) \in Y_k\), we replace \(\tilde{t}_k^i\) by \(t^i(y)\), we obtain a point which is also in \(Y_k\).

- Note that all conditions concerning \(\tilde{t}_k^j, j \neq i\) are satisfied since \((\tilde{t}_k^1, \ldots, \tilde{t}_k^n) \in Y_k\).

- Since \(t^i(y)\) assigns probability 1 to \(t^i(\rho_{k-1}(y))\) it follows that \(t^i(\tilde{y}_{k-1}) = t^i(\rho_{k-1}(y))\). Therefore, the conditions concerning \(t^i(y)\) are satisfied.
Proof of claim: \((\tilde{y}_{k-1}, \tilde{t}_k^1, \ldots, t^i(y), \ldots \tilde{t}_k^n) \in Y_k\)

- **Claim:** If at the point \((\tilde{y}_{k-1}, \tilde{t}_k^1, \ldots, \tilde{t}_k^n) \in Y_k\), we replace \(\tilde{t}_k^i\) by \(t^i(y)\), we obtain a point which is also in \(Y_k\).

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Proof of claim: \((\tilde{y}_{k-1}, \tilde{t}_k^1, \ldots, t^i(y), \ldots \tilde{t}_k^n) \in Y_k\)

- **Claim**: If at the point \((\tilde{y}_{k-1}, \tilde{t}_k^1, \ldots, \tilde{t}_k^n) \in Y_k\), we replace \(\tilde{t}_k^i\) by \(t^i(y)\), we obtain a point which is also in \(Y_k\).

- Note that all conditions concerning \(\tilde{t}_k^j, j \neq i\) are satisfied since \((\tilde{t}_k^1, \ldots, \tilde{t}_k^n) \in Y_k\).

- Since \(t^i(y)\) assigns probability 1 to \(t^i(\rho_{k-1}(y))\) it follows that \(t^i(\tilde{y}_{k-1}) = t^i(\rho_{k-1}(y))\). Therefore, the conditions concerning \(t^i(y)\) are satisfied.

This ends the proof.
### Type Space for the *Now-or-Never* game

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- \( p \leq \frac{1}{4} \)
- \( \frac{1}{4} < p < \frac{3}{4} \)
- \( \frac{3}{4} \leq p \)
Thank You! Any Questions?