We want to prove the following theorem.

**Theorem.** Let \( \Sigma \) be a finite alphabet and let \( w \in \Sigma^* \) be a word of length at least 2. If there exist nonempty words \( x, y \in \Sigma^+ \) such that \( w = xy = yx \), then there exists a nonempty word \( z \in \Sigma^+ \) and positive integers \( a \) and \( b \) such that \( x = z^a \) and \( y = z^b \).

**Proof.** We will prove the theorem by induction on the length of \( w \). The base case is \( |w| = 2 \). This case is trivial since it must be the case that \( |x| = |y| = 1 \). Therefore, \( x = y \) so \( z = x \) and \( a = b = 1 \).

For the inductive step, assume that the theorem holds for all words of length less than \( m \). Let \( w \) be a word of length \( m \) such that \( w = xy = yx \) for nonempty \( x \) and \( y \). Write \( x = x_1x_2 \cdots x_k \) and \( y = y_1y_2 \cdots y_n \). We can assume, without loss of generality, that \( k \leq n \) (otherwise swap \( x \) and \( y \)).

If \( k = n \), then since \( xy = yx \), we have \( x_1 = y_1 \), \( x_2 = y_2 \), \ldots, \( x_k = y_k \). Therefore, \( x = y \) and we can set \( z = x \) and \( a = b = 1 \).

Otherwise, \( k < n \) and we have a situation that looks like this.

\[
\begin{array}{c|c|c}
 & x & y \\
\hline
y & & \\
\hline
& x & \\
\end{array}
\]

From the picture, it’s clear that \( y \) starts with \( x \) and ends with the first \( n - k \) letters in \( y \). Formally, \( y = xy_1y_2 \cdots y_{n-k} \). Similarly, \( y \) ends with \( x \) so \( y = y_1y_2 \cdots y_{n-k}x \). Define \( y' = y_1y_2 \cdots y_{n-k} \). Thus,

\[
y = xy' = y'x
\]

and we can apply the inductive hypothesis since \( |y| = n < |w| = m \). In particular, there is a word \( z \) and integers \( a, c \) such that \( x = z^a \) and \( y' = z^c \). This gives

\[
x = z^a, \\
y = xy' = z^a z^c = z^{a+c}
\]

Setting \( b = a + c \) proves the theorem. \( \square \)