To receive full credit for each construction you give, you must justify why your construction is correct unless the problem explicitly says otherwise.

Problem 1  Prove that the class of regular languages is closed under reversal. That is, show that given a regular language \( L \), show that \( L^R = \{ w^R \mid w \in L \} \) is regular.  

**Hint:** Given a DFA \( M = (Q, \Sigma, \delta, q_0, F) \) that recognizes \( L \), build an NFA \( N = (Q', \Sigma, \delta', q'_0, F') \) that recognizes \( L^R \).

Problem 2  Define

\[
\text{BackwardsAndForwards}(L) = \{ w \in L \mid w \in L \text{ and } w^R \in L \}.
\]

That is, given a language \( L \), \( \text{BackwardsAndForwards}(L) \) is a new language consisting of the elements of \( L \) whose reversal is also an element of \( L \). Using closure properties of regular languages, show that the class of regular languages is closed under the operation \( \text{BackwardsAndForwards} \).

Problem 3

a. Use closure properties of regular languages to show that regular languages are closed under set difference. That is, given regular languages \( L_1 \) and \( L_2 \), show that

\[ L_1 \setminus L_2 = \{ w \in L_1 \mid w \notin L_2 \} \]

is regular.

b. Show that regular languages are closed under symmetric set difference

\[ L_1 \triangle L_2 = \{ w \mid \text{either } w \in L_1 \text{ or } w \in L_2 \text{ but not both} \} \]

Problem 4

a. For any language \( L \), define

\[
\text{Prefix}(L) = \{ w \mid \exists x \in \Sigma^* \text{ s.t. } wx \in L \}.
\]

Prove that regular languages are closed under \( \text{Prefix} \).
b. For any language $L$, define

$$\text{Suffix}(L) = \{w \mid \exists x \in \Sigma^* \text{ s.t. } xw \in L\}.$$  

Using closure properties of regular languages and the result of part a, prove that regular languages are closed under $\text{Suffix}$.

**Problem 5** For languages $L_1$ and $L_2$, define

$$L_1 \odot L_2 = \{w \in L_1 \mid w \text{ does not contain any string in } L_2 \text{ as a substring}\}.$$  

Prove that regular languages are closed under $\odot$.\footnote{Hint: Think about what $\Sigma^* \circ L \circ \Sigma^*$ means for a language $L$. Write $L_1 \odot L_2$ in terms of set difference and concatenation and apply closure properties of regular languages.}

**Problem 6** Let $\Sigma$ and $\Gamma$ be alphabets and let $f : \Sigma \to \Gamma$ be a function that maps symbols in $\Sigma$ to symbols in $\Gamma$. One such example is $f : \{1, 2, 3, 4, 5\} \to \{a, b, c, d\}$ given by

- $f(1) = b$
- $f(2) = b$
- $f(3) = a$
- $f(4) = d$
- $f(5) = a$.

We can extend such an $f$ to operate on strings $w = w_1w_2 \cdots w_n$ by

$$f(w) = f(w_1)f(w_2) \cdots f(w_n).$$

Using the same example, $f(132254) = babbad$. We can extend $f$ to operate on languages by $f(L) = \{f(w) \mid w \in L\}$.  

Prove that if $L$ is a regular language and $f : \Sigma \to \Gamma$ is an arbitrary function—that is, it is not necessarily the example given above—then $f(L)$ is regular. \footnote{Hint: given a DFA $M$ that recognizes $L$, build an NFA $N$ that recognizes $f(L)$ by applying $f$ to the symbols on each transitions. To prove that this works, consider the states $M$ goes through on input $w$ and the states $N$ goes through on input $f(w)$.}

**Problem 7** A homomorphism is a function $f : \Sigma \to \Gamma^*$ that maps symbols in $\Sigma$ to strings over $\Gamma$. One example of a homomorphism is the function that maps every string to $\varepsilon$. A less-trivial example is $f : \{a, b\} \to \{a, b, c\}$ given by

- $f(a) = bacca$
- $f(b) = b$.\footnote{You can typeset $\odot$ in \LaTeX{} by putting the line \texttt{\usepackage{mathabx}} in the preamble and using \texttt{\S} in math mode.}
We can extend $f$ to operate on strings $w = w_1w_2 \cdots w_n$ by $f(w) = f(w_1)f(w_2)\cdots f(w_n)$ and languages by $f(L) = \{f(w) \mid w \in L\}$.

Prove that regular languages are closed under homomorphism. [Hint: As with your construction in Problem 6, you want to apply $f$ to the symbols on each transition but in this case you may need to add additional states if the length of $f(a)$ is not 1. Be sure to handle the case where $f(a) = \varepsilon$.]

**Problem 8** For each language in Exercise 1.6 in Sipser, give an equivalent regular expression. (You don’t need to prove that it’s correct.)

**Problem 9** Using the procedure given in Lemma 1.55 in Sipser, convert the regular expression $(0 \cup 1)^*01(00 \cup 1)^*$ to an NFA. Show each step.

**Problem 10** Using the procedure given in Lemma 1.60 in Sipser, convert the following DFA to a regular expression. Show each step.

Extra Credit This extra credit problem is worth as much as 1.5 other problems but is more difficult. Define the chop of languages $A$ and $B$ as

$$\text{Chop}(A, B) = \{w \mid \exists x \in B \text{ s.t. } wx \in A\}.$$ 

For example, if

$$X = \{aab, aba, bba, bbb\}$$
$$Y = \{ba, bbb\}$$
$$Z = \{a^n b^n \mid n \geq 0\}$$

are languages then,

$$\text{Chop}(X, Y) = \{\varepsilon, a, b\}$$
$$\text{Chop}(X, Z) = \{a, aab, aba, bba, bbb\}$$
$$\text{Chop}(Z, X) = \{a^n b^{n-3} \mid n \geq 3\}.$$ 

If you compare the definition of Chop to the definition of Prefix, you can see that they are similar. Indeed $\text{Prefix}(L) = \text{Chop}(L, \Sigma^*)$.

Prove that if $A$ is a regular language and $B$ is any language (in particular, it need not be regular), then $\text{Chop}(A, B)$ is regular. [Hint: Try modifying your proof that regular languages are closed under Prefix.]