To receive full credit for each construction you give, you must justify why your construction is correct unless the problem explicitly says otherwise.

**Problem 1** Prove that the class of regular languages is closed under reversal. That is, show that given a regular language $A$, show that $A^R = \{w^R \mid w \in A\}$ is regular. [Hint: Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes $A$, build an NFA $N = (Q', \Sigma, \delta', q'_0, F')$ that recognizes $A^R$.]

**Problem 2** Define

$$\text{BackwardsAndForwards}(A) = \{w \in A \mid w \in A \text{ and } w^R \in A\}.$$  

That is, given a language $A$, $\text{BackwardsAndForwards}(A)$ is a new language consisting of the elements of $A$ whose reversal is also an element of $A$. Using closure properties of regular languages, show that the class of regular languages is closed under the operation $\text{BackwardsAndForwards}$.

**Problem 3**

a. Use closure properties of regular languages to show that regular languages are closed under set difference. That is, given regular languages $A$ and $B$, show that

$$A \setminus B = \{w \in A \mid w \notin B\}$$

is regular.

b. Show that regular languages are closed under symmetric set difference

$$A \triangle B = \{w \mid \text{either } w \in A \text{ or } w \in B \text{ but not both}\}.$$  

**Problem 4** Recall the definitions of Prefix and Suffix

$$\text{Prefix}(A) = \{w \mid \exists x \in \Sigma^* \text{ s.t. } wx \in A\},$$

$$\text{Suffix}(A) = \{w \mid \exists x \in \Sigma^* \text{ s.t. } xw \in A\}.$$  

We showed in class that regular languages are closed under Prefix. Using closure properties of regular languages, show that regular languages are closed under Suffix.
Problem 5 For languages $A$ and $B$, define

$$A \circlearrowleft B = \{ w \in A \mid w \text{ does not contain any string in } B \text{ as a substring} \}.$$  

Prove that regular languages are closed under $\circlearrowleft$. [Hint: Think about what $\Sigma^* \circ L \circ \Sigma^*$ means for a language $L$. Write $A \circlearrowleft B$ in terms of set difference and concatenation and apply closure properties of regular languages.]

Problem 6 Let $\Sigma$ and $\Gamma$ be alphabets and let $f : \Sigma \to \Gamma$ be a function that maps symbols in $\Sigma$ to symbols in $\Gamma$. One such example is $f : \{1, 2, 3, 4, 5\} \to \{a, b, c, d\}$ given by

$$
\begin{align*}
    f(1) &= b \\
    f(2) &= b \\
    f(3) &= a \\
    f(4) &= d \\
    f(5) &= a.
\end{align*}
$$

We can extend such an $f$ to operate on strings $w = w_1w_2 \cdots w_n$ by

$$f(w) = f(w_1)f(w_2) \cdots f(w_n).$$

Using the same example, $f(132254) = \text{babbad}$. We can extend $f$ to operate on languages by $f(A) = \{ f(w) \mid w \in A \}$. 

Prove that if $A$ is a regular language and $f : \Sigma \to \Gamma$ is an arbitrary function—that is, it is not necessarily the example given above—then $f(A)$ is regular. [Hint: given a DFA $M$ that recognizes $A$, build an NFA $N$ that recognizes $f(A)$ by applying $f$ to the symbols on each transitions. To prove that this works, consider the states $M$ goes through on input $w$ and the states $N$ goes through on input $f(w)$.

Problem 7 A homomorphism is a function $f : \Sigma \to \Gamma^*$ that maps symbols in $\Sigma$ to strings over $\Gamma$. One example of a homomorphism is the function that maps every string to $\varepsilon$. A less-trivial example is $f : \{a, b\} \to \{a, b, c\}$ given by

$$
\begin{align*}
    f(a) &= \text{bacca} \\
    f(b) &= b.
\end{align*}
$$

We can extend $f$ to operate on strings $w = w_1w_2 \cdots w_n$ by $f(w) = f(w_1)f(w_2) \cdots f(w_n)$ and languages by $f(L) = \{ f(w) \mid w \in L \}$. 

Prove that regular languages are closed under homomorphism. [Hint: As with your construction in Problem 6, you want to apply $f$ to the symbols on each transition but in this case you may need to add additional states if the length of $f(a)$ is not 1. Be sure to handle the case where $f(a) = \varepsilon$.]

---

1You can typeset $\otimes$ in L\TeX by putting the line \usepackage{mathabx} in the preamble and using \textbackslash\textbackslash in math mode.
Problem 8 For each language below, give an equivalent regular expression. (You don’t need to prove that it’s correct.) In each case, $\Sigma = \{0, 1\}$.

$A = \{w \mid w$ begins with a 1 and ends with a 0}$
$B = \{w \mid w$ contains at least three 1s}$
$C = \{w \mid w$ contains the substring 0101}$
$D = \{w \mid w$ has length at least 3 and its third symbol is 0}$
$E = \{w \mid w$ starts with 0 and has odd length, or starts with 1 and has even length}$
$F = \{w \mid w$ doesn’t contain the substring 110}$
$G = \{w \mid$ the length of $w$ is at most 5}$
$H = \{w \mid w$ is any string except 11 and 111}$
$I = \{w \mid$ every odd position of $w$ is a 1}$
$J = \{w \mid w$ contains at least two 0s and at most one 1}$
$K = \{\varepsilon, 0\}$
$L = \{w \mid w$ contains an even number of 0s, or contains exactly two 1s}$
$M = \emptyset$
$N = \Sigma^* \setminus \{\varepsilon\}$

Problem 9 Using the procedure given in Lemma 1.55 in Sipser, convert the regular expression $(0 \cup 11)^*01(00 \cup 1)^*$ to an NFA. Show each step.