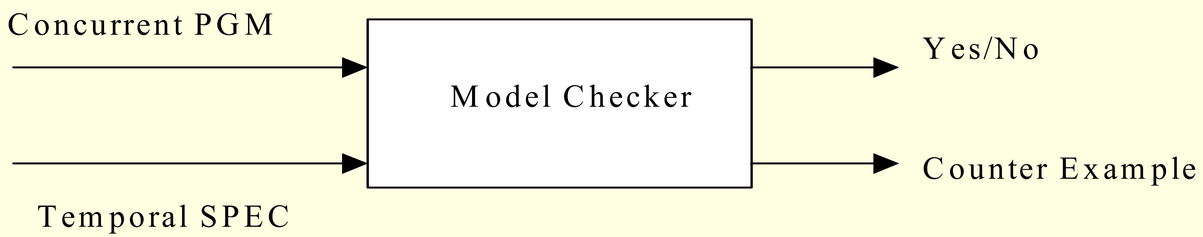
Symmetry Reductions.

A. Prasad Sistla
University Of Illinois at Chicago

Model-Checking



- Approach
  - Build the global state graph
  - Algorithm to check correctness
- Applications
  - Concurrent Programs.
  - Protocols.
  - Circuits.

Bottleneck: State Explosion

Has only been used for small size problems.

- number of states grows exponentially.

- Techniques to contain state explosion
  - Symbolic Model Checking (BDDs)
  - Stubborn Sets/Sleep Sets
  - Symmetry

    (Due to identical/similar processes)

Outline

- Symmetry & Quotient Structure
  - Program Symmetry
  - Formula Symmetry
  - Quotient Structure
  - State Symmetry
- Annotated Quotient Structure.
  - Fairness
- SMC – An implemented system
- Reduced Symmetry & Assymetry.
  - Guarded quotient Strucure.
  - Formula Decomposition
  - Subformula tracking.

Model : Shared Variable

- Notation
  - Variable name- $X_{i,j}$
    (denotes a variable shared by processes i, j)

- Program : A set of processes.
- Process : A set of guarded commands.

- Process $\mathcal{K}_i$ ::
  - [$LC_i = 0 \wedge F_{i,i+1} = 1 \rightarrow F_{i,i+1} := 0, LC_{i,} := 1;$
  - $LC_i = 1 \wedge F_{i,i-1} = 1 \rightarrow F_{i,i+1} := 0, LC_{i,i+1} := 2;$
  - $LC_i \rightarrow F_{i,i+1} := 1, F_{i,i+1} := 1, LC_i := 0;$]

- Program $\mathcal{K}$ :: $\mathcal{K}_1 \| \mathcal{K}_2 \| \ldots \| \mathcal{K}_n$

Program Symmetry

- $\mathcal{K}$:: $\mathcal{K}_1 \| \mathcal{K}_2 \| \ldots \| \mathcal{K}_n$
- $\|$ is commutative and associative
- $\mathcal{I}$ = Index set = $\{1,2,\ldots,n\}$
- For a permutation π on $\mathcal{I}$,
- define $\pi(\mathcal{K}_i)$ - Process obtained by changing indices of variables according to π ($X_{i,j}$ changed to $X_{\pi(i), \pi(j)}$)

- Aut $\mathcal{K}$ = $\{\pi \mid \pi(\mathcal{K}) = \mathcal{K}\}$

- Global State s : Assignment of values to variables.

- Globlal State Graph : $\mathcal{M} = (\mathcal{S},\mathcal{R})$

  $\mathcal{S}$ - Set of global states

  $(s,t) \in \mathcal{R}$ *iff* t is obtained from s by executing a single step of some process.

  (interleaved semantics)

- Interested in Symmetry of $\mathcal{M}$:

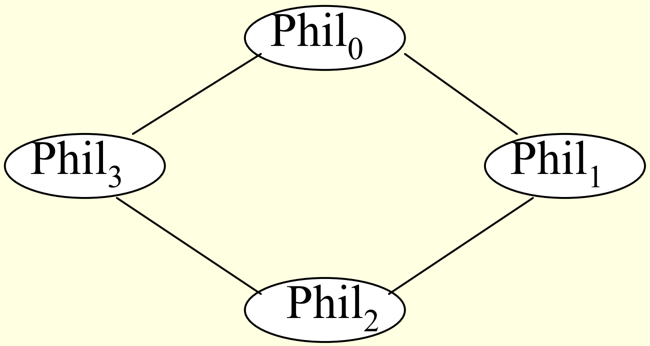
- s: global state.
- π : permutation on $\mathcal{I}$,
  - $\mathcal{I} = \{0, 1, \ldots, n\text{-}1\}$ = Process Indices.

- $\pi(s)$ is a global state in which
  - variable $X_{\pi(i),\ \pi(j)}$ gets the value of $X_{i,j}$ in s

- $Aut(\mathcal{M}) = \{\pi \mid \pi(\mathcal{M}) = \mathcal{M}\}$
  - Aut $\mathcal{M}$ is a group
- Lemma: Aut $\mathcal{K} \subseteq$ Aut $\mathcal{M}$

Examples:

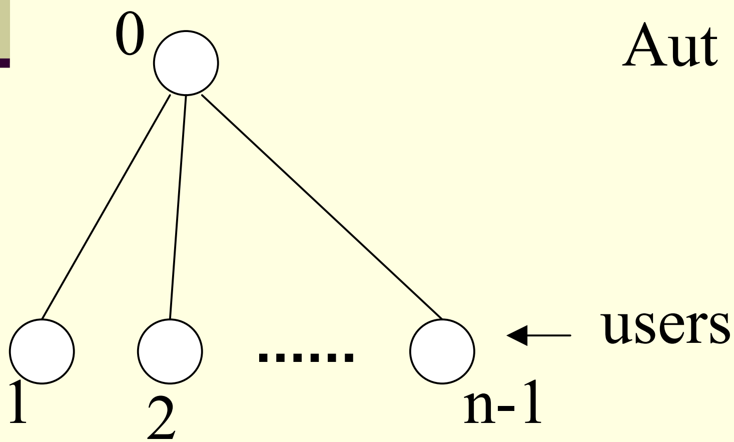
- For Symmetric soln of Dining Phil. Problem :



Aut $\mathcal{M}=\{\pi : \pi \text{ is a circular perm.}\}$

Aut $(\mathcal{K})$ = Aut $(\mathcal{M})$

- For the Resource controller Prog:



Aut $\mathcal{M} = \{\pi : \pi(0) = 0\}$

$\Downarrow$

π can permute the users.

0 is the controller process

Logics of Programs

CTL*:

Temporal operators: F, G, X, $\mathcal{U}$

F P : *eventually* P

G P : *always* P

X P : *nexttime* P

P $\mathcal{U}$ Q : P *until* Q

Path Quantifiers:

A – for all paths

E – for some path

AG(P) : Invariance

AF(P) : Inevitability of P

P : Basic assertion, uses indexed variables.

Ex: $LC_i = L$, $X_{i,j} > 0$

f : a formula in CTL*

$\pi(f)$ obtained by changing indices of variables according to π.

Symmetry of formulas:

$\text{Aut}\, f = \{\pi : \pi(f) \equiv f\}$

We use a subgroup of Aut f called Auto f.

$\text{Auto}\, f = \bigcap (\text{Aut p})$

p is a maximal prop. subformula of f.

Examples:

$f = AG\ ((T_1 \vee T_2) \rightarrow AF(C_1 \vee C_2))$

Global liveness for a mutual exclusion problem with two processes.

T_i : Process i is in trying mode.

C_i : Process i is in critical section.

Auto(f) = Sym $\mathcal{I}$ = Set of all permtns. $\mathcal{I} = \{1, 2\}$

$g = \bigwedge_{i=1,2} AG\ (T_i \rightarrow AF\ C_i)$

Aut(g) = Sym $\mathcal{I}$,   Auto(g) = {Id}

Quotient Structure:

$\mathcal{M} = (\mathcal{S}, \mathcal{R})$ – structure

f – CTL* formula

$G \subseteq \mathrm{Aut}\,\mathcal{M} \cap \mathrm{Auto}\,f$

<u>Equivalence relation</u> $\equiv_G$ on S

$s \equiv_G t$ *iff* $\exists\, \pi \in G$ such that $\pi(s) = t$

$\mathcal{M}/G = (\mathcal{S}^+, \mathcal{R}^+)$: Quotient Structure.

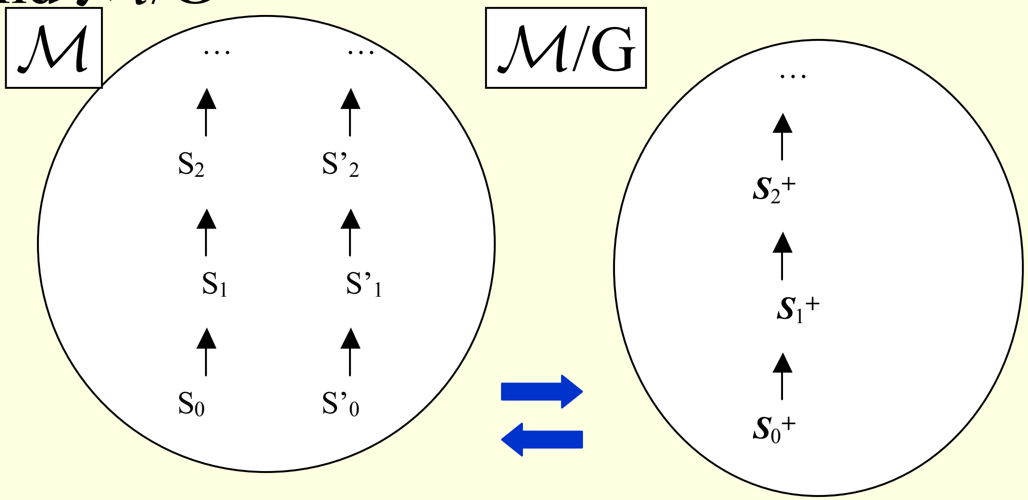
$\mathcal{S}^+$ has one representative for each equiv. class.

$(s^+,t^+) \in \mathcal{R}^+$ *iff* for some $s^{|} \equiv_G s^+$, $t^{|} \equiv_G t^+$

$(s^{|},t^{|}) \in \mathcal{R}$

Correspondence Lemma

- There is a bidirectional correspondence between paths of $\mathcal{M}$ and $\mathcal{M}/G$



- $(s_0, s_1, \ldots, s_i \ldots) \in \mathcal{M} \Rightarrow (\boldsymbol{s}_0^+, \boldsymbol{s}_1^+, \ldots, \boldsymbol{s}_i^+ \ldots) \in \mathcal{M}/G$
- $(\boldsymbol{s}_0^+, \boldsymbol{s}_1^+, \ldots, \boldsymbol{s}_i^+ \ldots) \in \mathcal{M}/G \Rightarrow \forall\, s_0' \equiv_G \boldsymbol{s}_0^+,\ \exists$ path in $\mathcal{M}$ $(s_0', s_1', \ldots, s_i' \ldots)$ such that $s_i' \equiv_G s_i^+$

Main Theorem

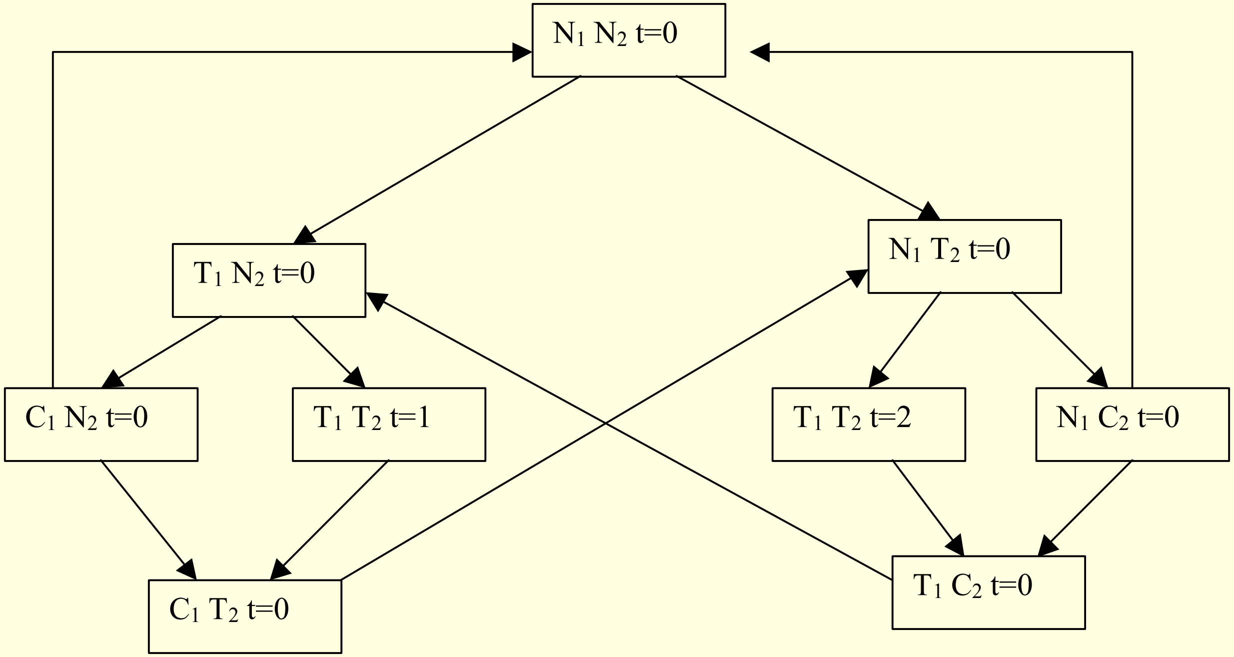
- For any $s \in \mathcal{S}$,
- $\mathcal{M}, s \models f$ *iff* $\mathcal{M}/G, \boldsymbol{s}^{+} \models f$
- (Ip &Dill 93,CFG 93,ES 93)
- <u>Proof</u> : Uses induction on *f* and the *corr.* Lemma.

- <u>Examples</u>

  Dining Phil. Problem

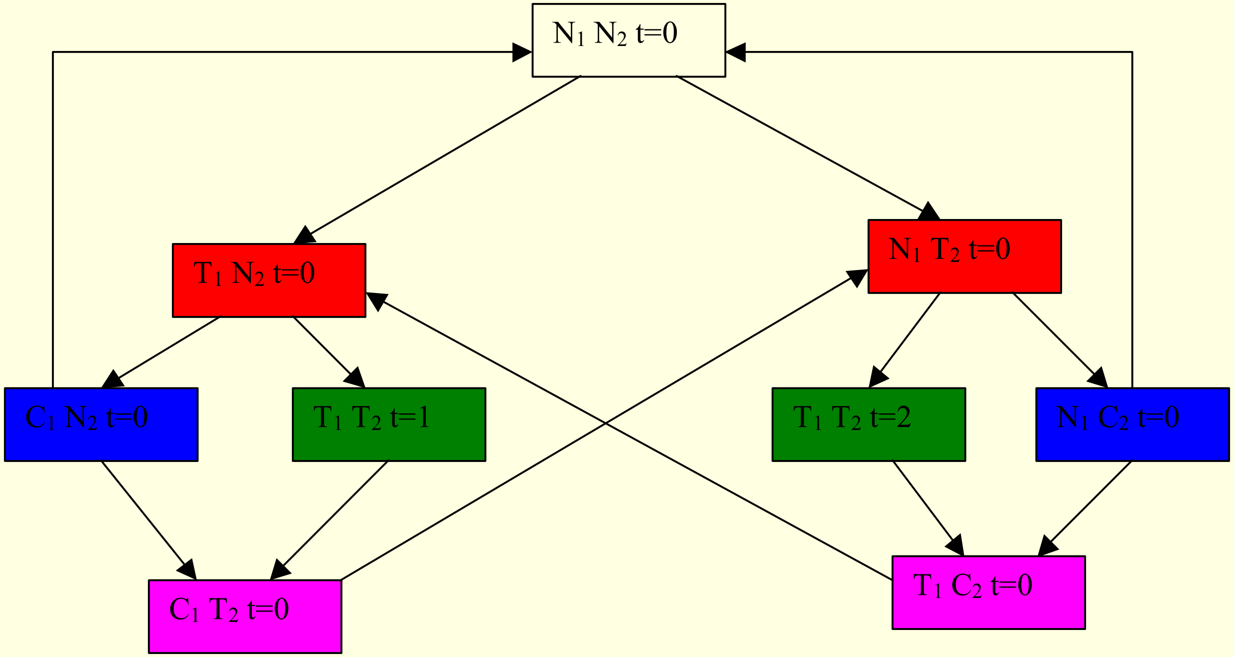
  - $f = AG(\,EX\ True)$ (absence of deadlock)
  - Auto f = Sym $\mathcal{I}$
  - Auto $\mathcal{M} \cap$ Auto f = all circ. permutations

Example



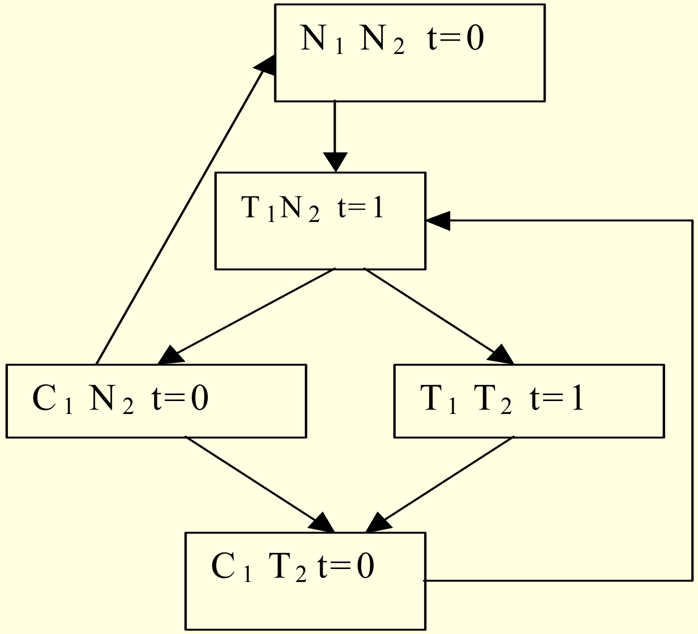
Two process Mutual Excl. $AG(\neg(C_1 \wedge C_2))$

Example contnd.



Two process Mutual Excl. $AG(\neg(C_1 \wedge C_2))$

Quotient Structure.



Fairness

- Correctness under <u>group fairness</u> is preserved
- $G \subseteq \text{Aut}\,\mathcal{M} \cap \text{Auto}\,f$
- Define index $i \equiv j$ *iff* $\exists\ \pi \in G$ such that $j=\pi(i)$
- $C_1, C_2, \ldots C_k$ are the equivalence classes of indices
- Group fairness: for $\ell = 1, \ldots, k$ some process $\in C_\ell$ executed infinitely often

Fairness Theorem

- ($\mathcal{M}$, s) satisfies f under group fairness *iff* ($\mathcal{M}$/G, s^+) satisfies f under group fairness

- Example:

  $f = AG\,((T_1 \vee T_2) \rightarrow AF\,(C_1 \vee C_2))$

  in the mutual exclusion example.

Incremental Computation of $\mathcal{M}/G$: $\mathcal{M} = (\mathcal{S}, \mathcal{R})$

$\mathcal{S}^+ = \{ s_0 \}$, s_0 -init. State

$\mathcal{Q} = \{ s_0 \}$

While $\mathcal{Q} \neq$ empty

s:= dequeue ($\mathcal{Q}$);

for each successor t of s

if ($\exists u \in \mathcal{S}^+$ such that $\boxed{u \equiv_G t}$)

then add (s, u) to $\mathcal{R}^+$

else $\mathcal{S}^+ = \mathcal{S}^+ \cup \{t\}$,

add (s, t) to $\mathcal{R}^+$

end for.

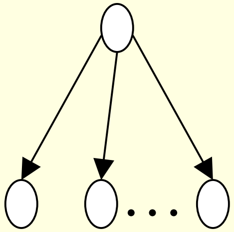
end while.

Checking $u \equiv_G t$ is a difficult problem.

Savings in the size of state space.

We can obtain exponential savings in some cases.

Resource Controller problem:



n- # of users

m - # of states of the controller

Assume each user has 3 states

$\mathcal{M}$ has $\mathcal{O}(m \cdot 3^n)$ states.

$\mathcal{M}/G$ has $\mathcal{O}(m \cdot n^3)$ states

Finding suitable G

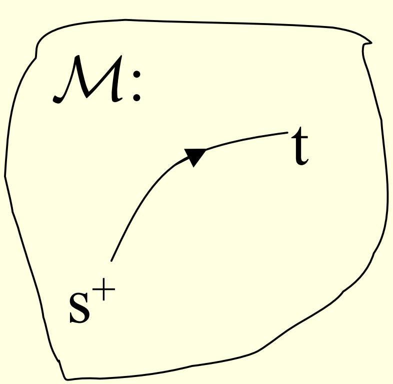
- Largest possible G $\subseteq$ Aut $\mathcal{M} \cap$ Auto f gives maximum compression.
- Difficulties:
  - Computing Aut $\mathcal{M}$ is difficult, as hard as graph isomorphism
    - use Aut $\mathcal{K}$ (can be determined from syntax)
  - Computing Auto f can be hard
  - Many times Aut $\mathcal{K}$, Auto f are known in advance
  - For isomorphic processes, Aut $\mathcal{K}$ = Aut CG

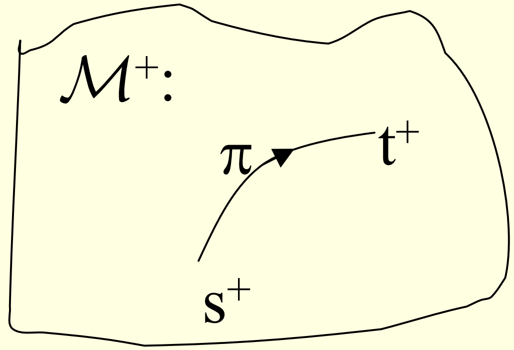
Automata Theoretic Approach & AQS

Don't need to consider formula symmetry.

Use an Annotated Quotient Structure. (AQS)

Take $G \subseteq \mathrm{Aut}\,\mathcal{M}$

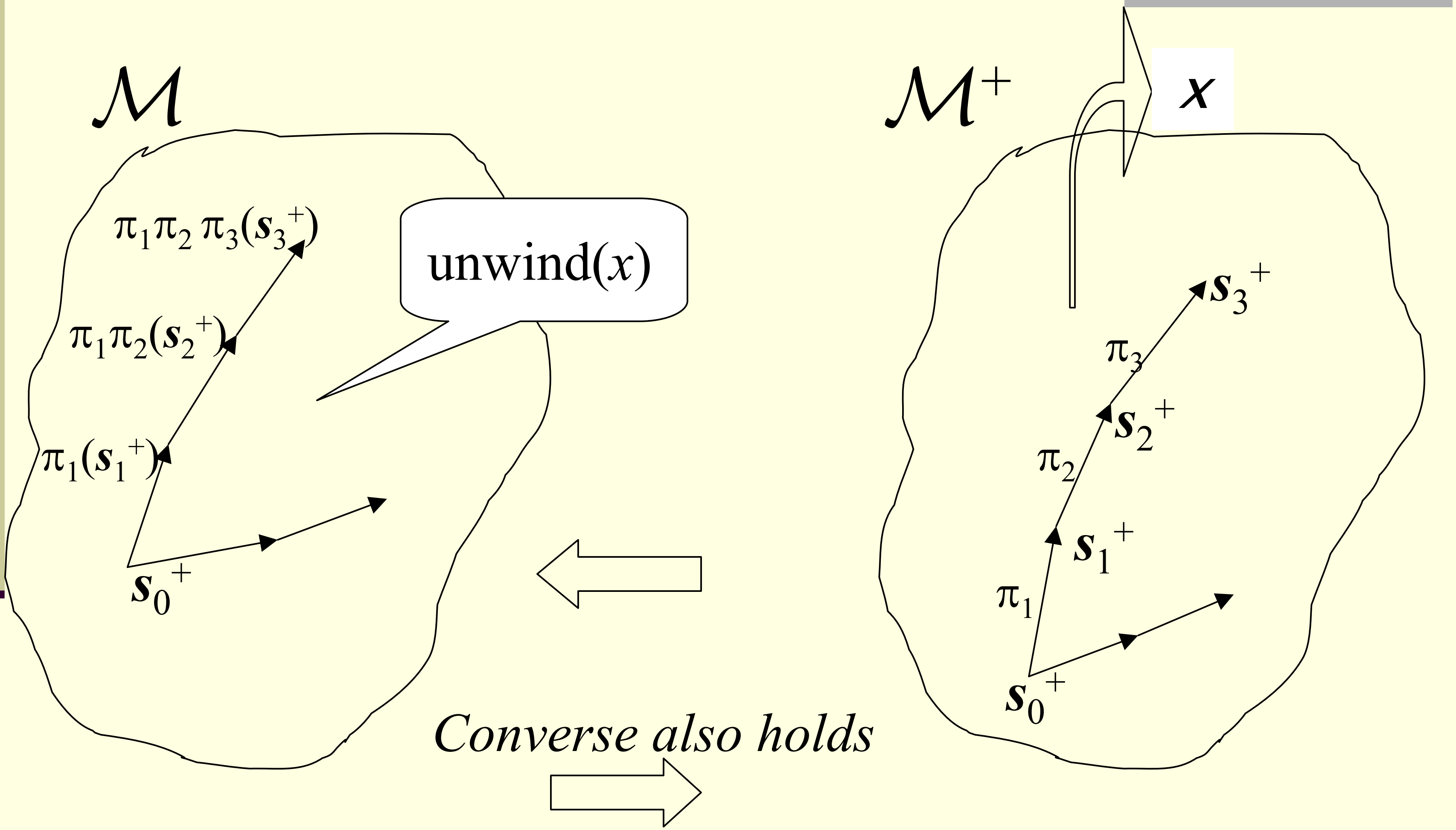




$t = \pi(t^+)$

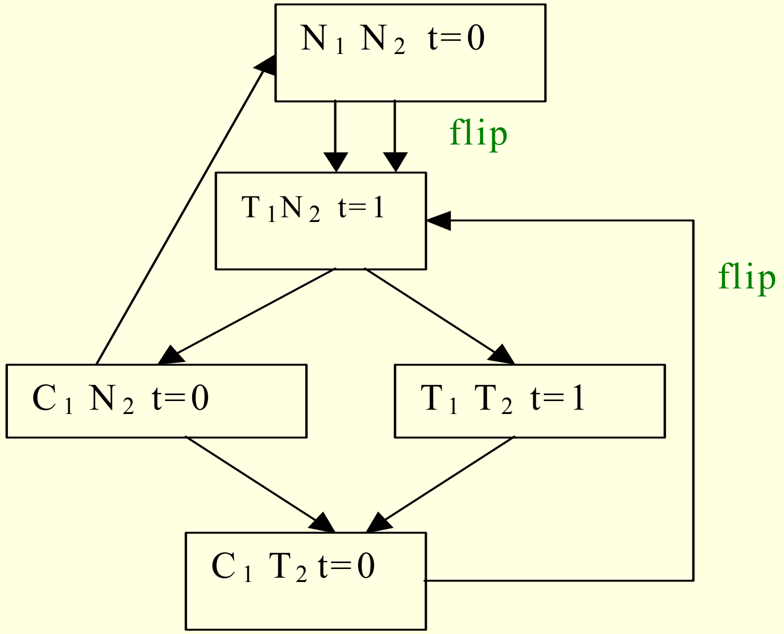
edges are labeled with permutations
if $(s^+, t) \in \mathcal{R}$ then
there is an edge from s^+ to t^+
labeled with π where $t = \pi(t^+)$

Correspondence between $\mathcal{M}$ & $\mathcal{M}^+$



$\mathcal{M}^+$ is a succinct encoding of $\mathcal{M}$.

Annotated Quotient:



G = {Id,Flip}

Correctness under fairness.

Want to check if $\mathcal{M}, s_0^{+} \models E\,(\phi \wedge f_i\ )$

where ϕ expresses weak fairness. $\mathcal{A}$ – Automaton for f.

Defn: $\mathcal{B}^{+} \;=\; \mathcal{M}^{+} \times \mathcal{A} \times \mathcal{I}$

$$(s^{+}, q, i) \xrightarrow{\pi} (t^{+}, r, j) \;\; \textit{iff}$$

$$s^{+} \xrightarrow{\pi} t^{+} \in \mathcal{M}^{+},$$

$$q^{+} \xrightarrow{(s^{+}\downarrow i)} r \in \mathcal{A} \;\text{ and}$$

$$\pi^{-1}(i) \;=\; j$$

Defn : A *SCC* C^{+} of $\mathcal{B}^{+}$ is green

if $\exists\ (s^{+}, q, i) \in C^{+}$ such that $q \in$ GREEN.

Theorem:

$\mathcal{M}, s_0^+ \models E(\phi \wedge f_i)$ *iff* $\mathcal{B}^+$ contains a subtly fair and green *SCC* C^+ that is reachable from (s_0^+, q_0, i).

How to check if C^+ is subtly fair ?

use a threaded graph C*.

Alg to mark all states in $\mathcal{M}$ that satisfy: $E(\phi \wedge f_i)$

1. Construct $\mathcal{A}$
2. Construct $\mathcal{B}^+ = \mathcal{M}^+ \times \mathcal{A} \times \mathcal{I}$
3. For each *SCC* C^+ of $\mathcal{B}^+$

   Check if C^+ is subtly fair

      Construct C

      Check if C* is plainly fair.

4. For each $s^+ \in \mathcal{M}$

   mark s^+ if a subtly fair and green *SCC* is reachable from (s^+, q_0, i) in $\mathcal{B}^+$

Complexity : $\mathcal{O}(|\mathcal{M}^{+}|.|\mathcal{A}|.n^2)$

Above approach can be extended to strong fairness

Complexity: $\mathcal{O}(|\mathcal{M}^{+}|.|\mathcal{A}|.n^3)$

On-the-Fly Algorithm is more subtle.

Implementation : SMC (Symmetry based Model – Checker)

- Developed at Univ. of Illinois at Chicago
- Uses AQS based approach
- Employs a variety of symmetries:
  - Program symmetry, State symmetry
- Uses variety of on-the-fly options
  - (AQS and/or product structure constructed on-the-fly)

Implementation : SMC(cont'd)

- Allows different fairness specifications
  (weak/strong features)

- Used for checking real world examples, found bugs in the *Fire-Wire* protocol.

Reduced Symmetry

- Symmetric system with asymmetric constructs.
  Ex: resource controller with priorities.
- Partially Symmetric system.

<u>Introduce</u>

1) Guarded Quotient Structures (GQS)

   Further extension of AQS

2) Two new techniques

   (a) Formula decomposition

   (b) Sub-formula tracking

Guarded Quotient Structures (GQS)

G =(S,E) -reachability graph.

Aut(G) – group of symmetries/automorphisms of G.

Aut(G) – may be small. Not much compression

Add edges to G and obtain an expanded graph

H = (S,F) so that

$F \supseteq E$

$\mathrm{Aut}(H) \supseteq \mathrm{Aut}(G)$

GQS – contd.

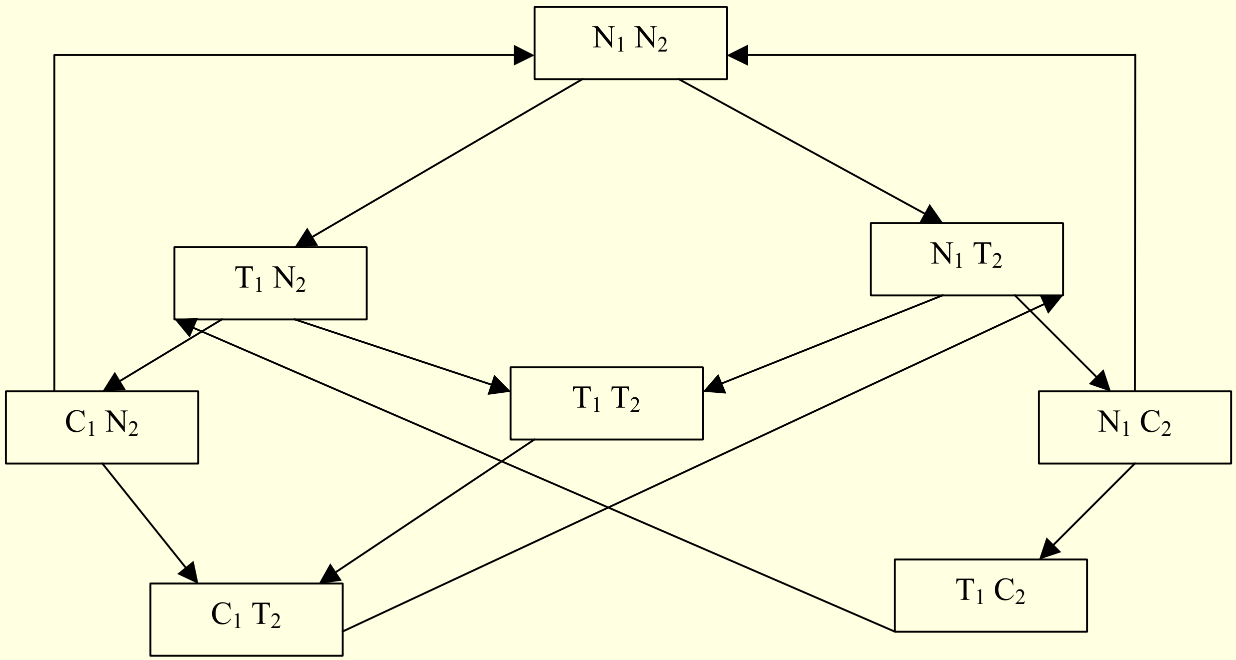
Construct the AQS of H

Add an edge condition with each edge of H (called guards)

Used during the unwinding process.

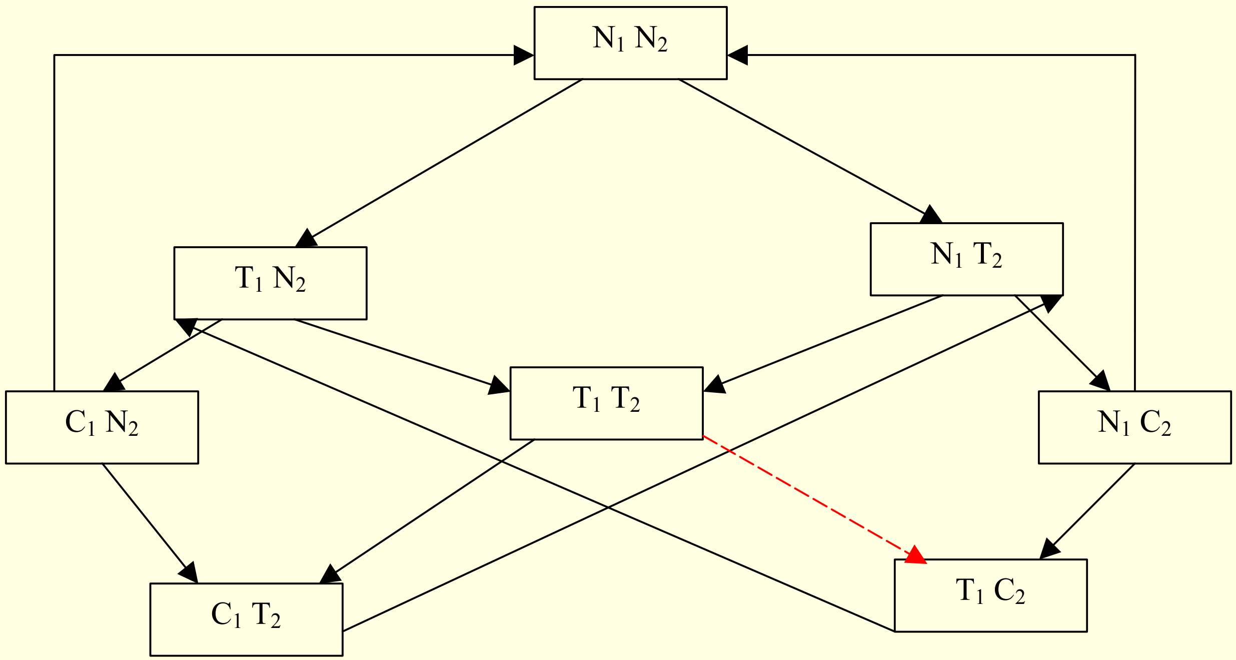
The resulting structure is GQS(G)

Example



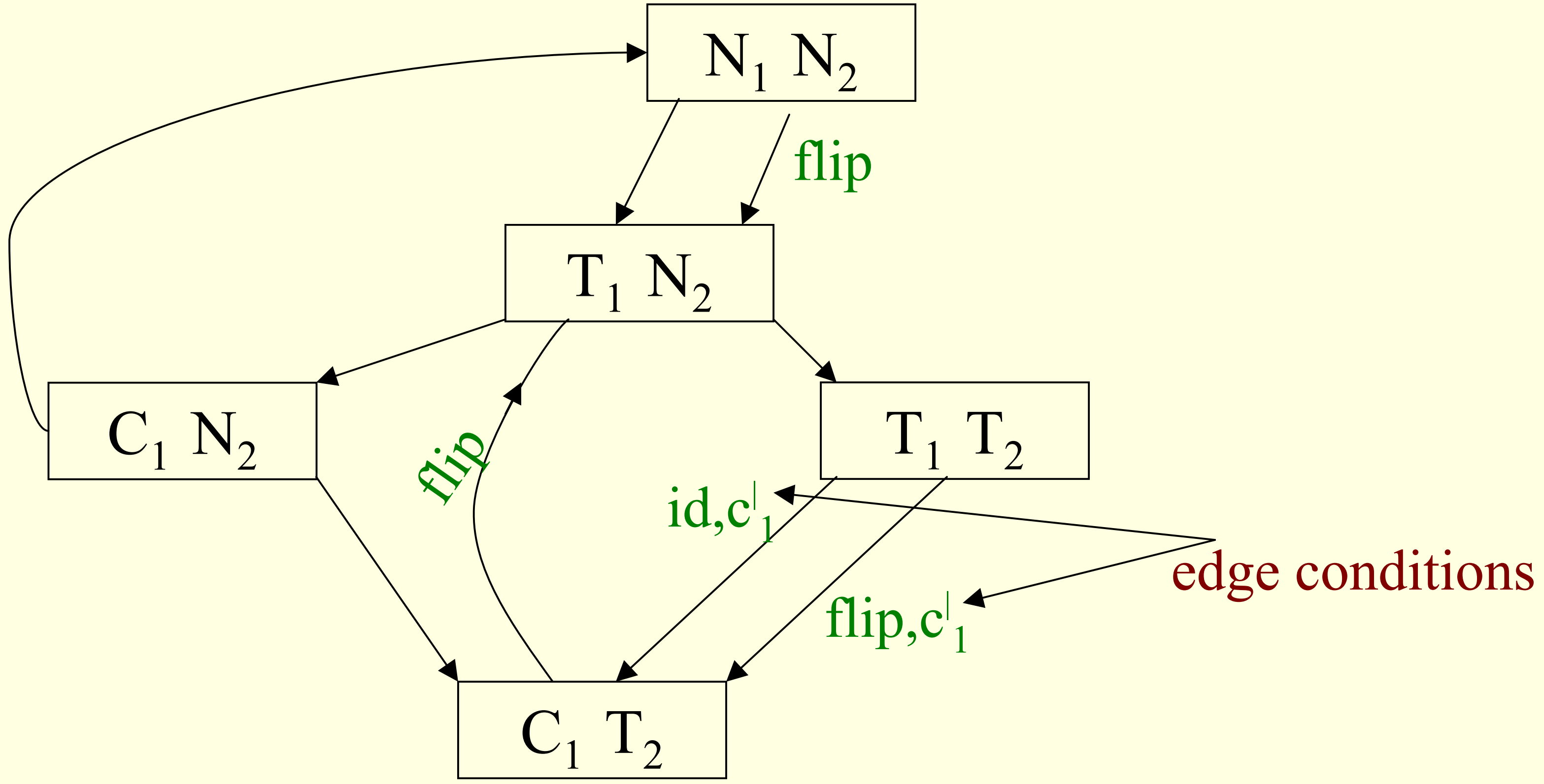
Mutual Exclusion with priority for Process 1 (The Graph G)

Example



Mutual Exclusion with priority for Process 1

Graph H



GQS for the system with priority.

Unwinding:

GQS:

$$s \xrightarrow{\phi, g_i} t$$

GQS-Struct

$$(t,\ \phi^{-1}(P_1),\ldots,\phi^{-1}(P_k),\ \phi^{-1}(\theta_1),\ldots,\phi^{-1}(\theta_\ell)\ )$$

$$\uparrow$$

$$(s, P_1,\ldots,P_k,\ \theta_1,\ldots,\theta_i,\ldots\theta_\ell\ )$$

correspond to & track atomic predicates in the formula

track edge conditions

To Check f

- Unwind the GQS to get GQS-struct
  - unwinding done w.r.t. P_1, P_2 … P_k and the edge conditions.
  - use the edge conditions to consider only those edges in G.
- Check f in GQS-struct
- When optimized: GQS-struct has
  - no more nodes than as $G/Aut(G)$
  - fewer nodes in many cases

- Important advantage of GQS:
  - Can use formula decomposition
  - Subformula tracking

Formula Decomposition

- Suppose $f = f_1 \wedge f_2$ (f_1, f_2 are state formulas)

  Only P_1 appears in f_1

  Only P_2 appears in f_2
- Unwind GQS w.r.t. P_1 and check f_1
- Unwind GQS w.r.t. P_2 and check f_2
- Avoids unwinding w.r.t P_1 and P_2 simultaneously.

Generalization.

- Group Top level subformulas into classes
  - all subformulas in a class contain same atomic propositions
  - Sets of atomic propositions of different classes are disjoint.
  - Unwind GQS once for each class.

- Can achieve exponential reduction in size of state space.

Subformula Tracking

Unwind the GQS w.r.t. non atomic state subformulas

<u>To check *f*</u>

- Choose a good maximal independent set $R = \{R_1, R_{2\ldots}\ R_k\}$ of state subformulas of f
- Replace each R_i in f by a new atomic proposition r_i and obtain $\overline{f}$
- Unwind GQS w.r.t. R to obtain GQS-struct
  - recursively determine which states of the GQS satisfy the subformulas in R
  - label a node in GQS-struct with r_i if the corresponding GQS node satisfies R_i (for $i = 1, \ldots k$)
- Check if $\overline{f}$ is satisfied in GQS-struct

Example

Consider a Resource Controller where process 0 is the controller and others are user processes

(i.e.processes 1,2,…n)

$f = E(P_1 \,\mathcal{U}\, g)$

where $\{\, g = \bigwedge_{2 \leqslant i \leqslant n} E(\, h(i)\,) \,\}$

$R = \{\, P_1, g \,\}$

Unwind GQS w.r.t. R to obtain GQS-struct

Recursively determine states that satisfy g (use formula decomposition)

Check $\overline{f}$ in GQS-struct

- In general, subformula tracking and formula decompostion are used recursively.

- Good independent sets have symmetric or partially symmetric subformulas.

Implementation

- Extended SMC to PSMC

  priorities can be specified in PSMC
- Used GQS together with

  formula decomposition and

  sub –formula tracking
- Checked *Fire-Wire* Protocol with priorities.

Conclusions.

- Symmetry reductions help in tackling the state explosion.
- Can also be used for systems with less symmetry.
- Have been implemented and used for verifying real-life protocols.

- <u>Future Work.</u>
  - Need to effectively combine with other methods.
  - Applications to software verification need to be explored.

Further Details @Home Page:

www.cs.uic.edu/~sistla