

Power of Randomization in Automata on Infinite Strings

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Joint work with Rohit Chadha and Mahesh Viswanathan

OutLine

Automata on Infinite Strings

Probabilistic Automata on Finite Strings

Probabilistic Buchi Automata

Finite State Probabilistic Monitors

Expressiveness results for FPMS

Decidability and Complexity results for FPMs

Decidability and Expressiveness for PBAs

Hierarchical PBAs

Automata on Infinite Strings (Rabin, Buchi, Muller, McNaughton)

- ▶ Finite State Automata on Finite Strings:
 $DFA \equiv NFA \equiv RExprs$

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- ▶ Finite State Automata on Finite Strings:
 $DFA \equiv NFA \equiv RExprs$
- ▶ Buchi Automata (BA) on Infinite Strings:
 $\mathcal{A} = (\Sigma, Q, \delta, q_0, F)$
 - ▶ Σ — input alphabet
 - ▶ Q — automaton states
 - ▶ $\delta \subseteq Q \times \Sigma \times Q$ — transition relation
 - ▶ q_0 — start state
 - ▶ $F \subseteq Q$ — accepting set

Buchi Automata

- ▶ A **run** r on an infinite input $a = a_0, \dots, a_i, \dots$ is a sequence (r_0, \dots, r_i, \dots) such that $r_0 = q_0$ and $(r_i, a_i, r_{i+1}) \in \delta$ for $i \geq 0$.

Buchi Automata

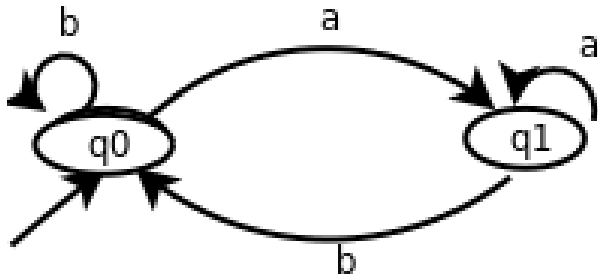
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- ▶ r is **accepting** if some accepting state appears infinitely often.
- ▶ $L(\mathcal{A})$ is the set of inputs on which \mathcal{A} has an accepting run.
- ▶ **An example BA:** Accepts strings in which every a is eventually followed by b .



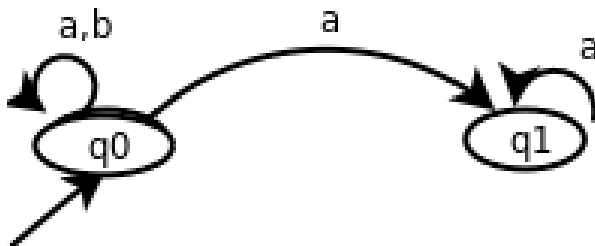
Accepting set = $\{q_0\}$

NonDet BA Vs. Det BA

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Accepting set = $\{q1\}$

Accepts strings in which eventually only a appears.

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- ▶ **Fact**: ω -Regular Langs = Boolean closure of Det- ω -Regular Langs.
- ▶ **Applications**: Verification of concurrent programs, decision procedures for logics: Linear Temporal Logic, S1S, Theory of Linear order, etc.

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Decision Problems

- ▶ **Emptiness**: Given BA \mathcal{A} , check if $L(\mathcal{A}) \neq \emptyset$. It is in NLOGSPACE and hence is in P.
- ▶ **Universality**; Given a BA \mathcal{A} , check if $L(\mathcal{A}) = \Sigma^\omega$. It is **PSPACE**-complete.
- ▶ **Language Containment**; Given a BAs \mathcal{A}, \mathcal{B} , check if $L(\mathcal{A}) \subseteq L(\mathcal{B})$. **PSPACE**-complete.

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Probabilistic Finite string Automata (PFA) (Rabin)

- ▶ $\mathcal{A} = (\Sigma, Q, \delta, q_0, F)$.
- ▶ $\delta : Q \times \Sigma \times Q \rightarrow [0, 1]$ so that $\sum_{q'} \delta(q, a, q') = 1$.

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- ▶ $L_{>\frac{1}{2}}(\mathcal{A})$ can be a non-regular set (Rabin 1960s).
- ▶ Determining non-emptiness of $L_{>\frac{1}{2}}(\mathcal{A})$ and $L_{\geq\frac{1}{2}}(\mathcal{A})$ are undecidable. Both are **R.E.**-complete. (Paz 1971, Soloma 1973).

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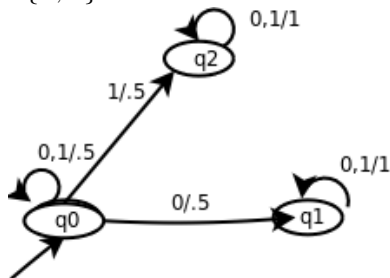
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- ▶ Example. $\Sigma = \{0, 1\}$.



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- ▶ Fix $a \in \Sigma^\omega$. Define the probability space (Q^ω, Δ, ϕ) where Δ is the event space and ϕ is the probability measure on it.

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If $r_0 \neq q_0$ then $\phi(S_u) = 0$.
- ▶ $PrOfAcc_{\mathcal{A}}(a) = \phi(Inf(F))$. (Note $Inf(F) \in \Delta$).

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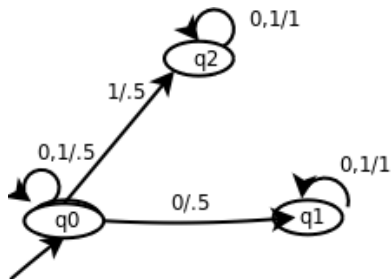
Hierarchical PBAs

Finite State Probabilistic Monitors (FPM)

- ▶ A **FPM** is a PBA with a designated **reject** state, an **absorbing** state. All other states are accepting states. Let $PrOfRej_{\mathcal{A}}(a) = 1 - PrOfAcc_{\mathcal{A}}(a)$.

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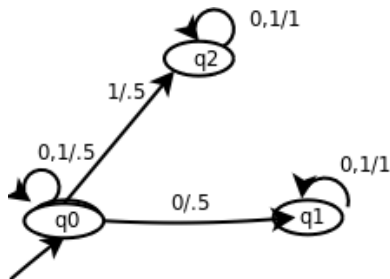
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Automaton B

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- ▶ Observe $PrOfRej_B(a) = val(a)$.

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- ▶ Modeling open systems that can fail.
- ▶ Model checking safety properties of open finite state probabilistic programs.

Properties of Infinite Executions

- ▶ Language $C \subseteq \Sigma^\omega$ is a **safety** property, if it is limit closed. That is, for any $a \in \Sigma^\omega$, if $\text{prefixes}(a) \subseteq (\text{Prefixes}(C))$ then $a \in C$.

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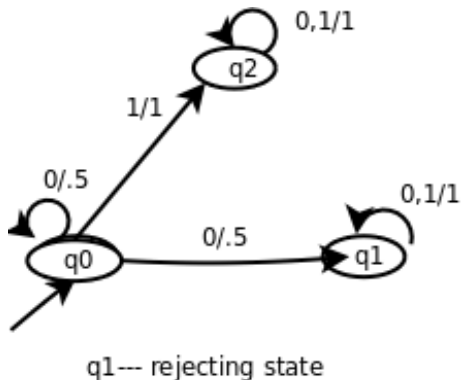
- ▶ Only Safety properties can be monitored using deterministic monitors.
- ▶ $C \subseteq \Sigma^\omega$ is an **almost safety** property if it is a countable union of safety properties.

Example: Set of sequences in which 1 appears at least 3 times.

Monitoring Non-safety Properties

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The following FPM monitors (i.e., accepts) the set of sequences in which 1 appears eventually. Not a safety property!



Classes of Monitorable Languages

Consider an alphabet Σ and a language $\mathcal{L} \subseteq \Sigma^\omega$.

- ▶ \mathcal{L} is **Monitorable with Strong Acceptance** if there is a FPM \mathcal{A} such that \mathcal{L} is the set of strings rejected by \mathcal{A} with probability 0. **MSA** is the class of all such languages.

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- ▶ \mathcal{L} is **Monitorable with Weak Acceptance** if there is a FPM \mathcal{A} such that \mathcal{L} is the set of strings rejected by \mathcal{A} with probability < 1 , i.e., accepted with non-zero prob. **MWA** is the class of all such languages.

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- ▶ \mathcal{L} is **Monitorable with Strict Cut-off** if there is a FPM \mathcal{A} such that \mathcal{L} is the set of strings rejected by \mathcal{A} with probability $< \frac{1}{2}$. **MSC** is the class of all such languages.

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- ▶ \mathcal{L} is **Monitorable with Non-strict Cut-off** if there is a FPM \mathcal{A} such that \mathcal{L} is the set of strings rejected by \mathcal{A} with probability $\leq \frac{1}{2}$. **MNC** is the class of all such languages.

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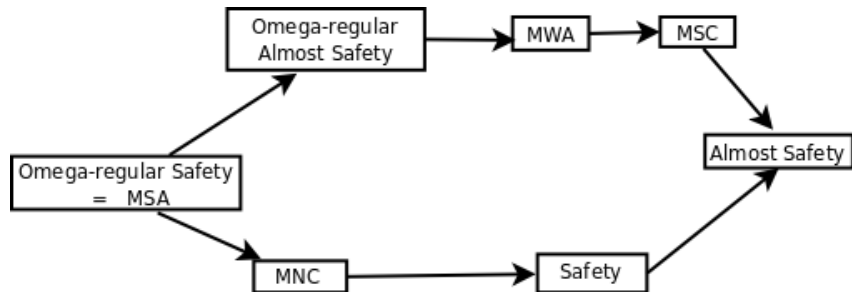
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Expressiveness results



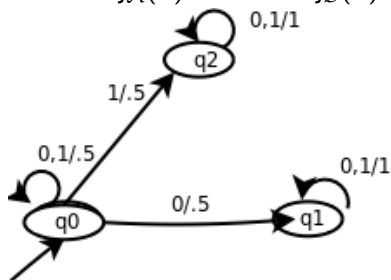
Containment Lattice among Language Classes

Monitoring Non- ω -regular Languages

- ▶ For any \mathcal{A}, \mathcal{B} , can construct \mathcal{C} such that for any input a ,
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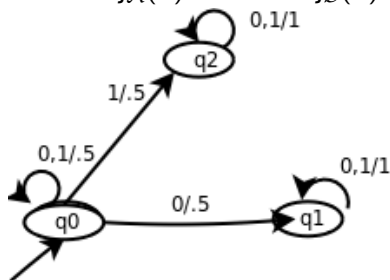
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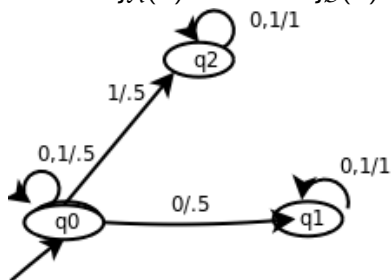
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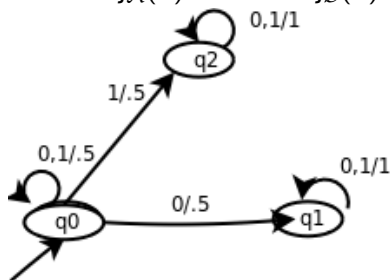
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- ▶ Let L be the set of inputs rejected by \mathcal{C} with prob $\leq \frac{1}{2}$.
- ▶ $L = \{a : val(a) \leq \frac{1}{\sqrt{2}}\}$. **$L \in MNC$ and not ω -regular.**

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Summary of Complexity and Decidability results

	EMPTINESS	UNIVERSALITY
Msa	PSPACE -complete	NL -complete
Mwa	PSPACE -complete	PSPACE -complete
Msc	co-R.E. -complete	Π_1^1 -complete
Mnc	R.E. -complete	co-R.E. -complete

OutLine

Automata on Infinite Strings

Probabilistic Automata on Finite Strings

Probabilistic Buchi Automata

Finite State Probabilistic Monitors

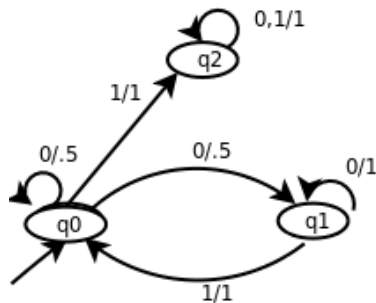
Expressiveness results for FPMS

Decidability and Complexity results for FPMs

Decidability and Expressiveness for PBAs

Hierarchical PBAs

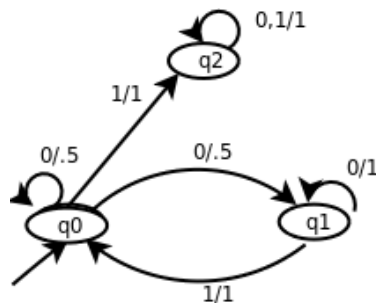
Decidability and Expressiveness for PBAs



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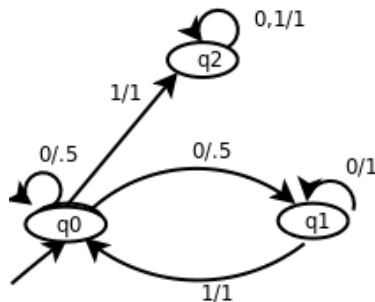
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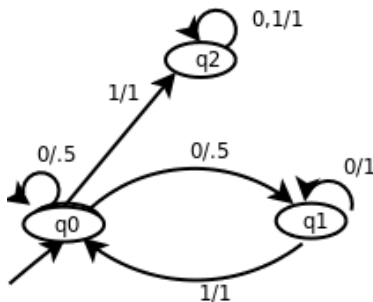
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- ▶ Acceptance probability $(1 - \frac{1}{2}) \times (1 - \frac{1}{2^2}) \times \dots (1 - \frac{1}{2^i}) \times \dots$

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- ▶ Reduce to emptiness and universality problems for FPMs.

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- ▶ Explore power of randomization in other computation models on infinite inputs.