MULTICHANNEL COMMUNICATION AND GRAPH VERTEX LABELING PROBLEMS

BY

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THESIS

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To All My Teachers

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Introduction

1.1 Information Theory

1.1.1 Background

Consider sending information over k independent unreliable channels. We want to partition the source information into k subsets so that if all k subsets are received, the original information can be completely reconstructed; if any of the channels fail, then the error, defined as the absolute (rather than Hamming) difference between the original information and the possible reconstruction of it, is minimized. Figure 1.1 shows the general setting of the problem. A trivial solution would be to divide the source information into k equal blocks, sending each over a separate channel. However, if any block fails to arrive, that part of the information is lost completely. The error in this case strongly depends on which particular channel has failed, generally an undesirable feature in this setting. Alternatively, we could send k complete copies, so that even if only one of the channels succeeds, all the information is still available. This scheme, while robust, utilizes resources poorly. Our goal is to partition the information in a way that allows us to recover as closely as possible the information originally sent, with the error depending only on the number of channels lost.

The problem described above is that of designing codes for a *diversity-based* (multichannel) communication system that guarantee a minimum fidelity at the user end based on the number of channels succeeding in transmitting information. This problem is known as the *Multiple Description problem* and was introduced by Gersho, Witsenhausen, Wolf, Wyner, Ziv, and Ozarow at the 1979 IEEE Information Theory Workshop. It is a generalization of the classical problem of source coding subject to a *fidelity criterion* [46].

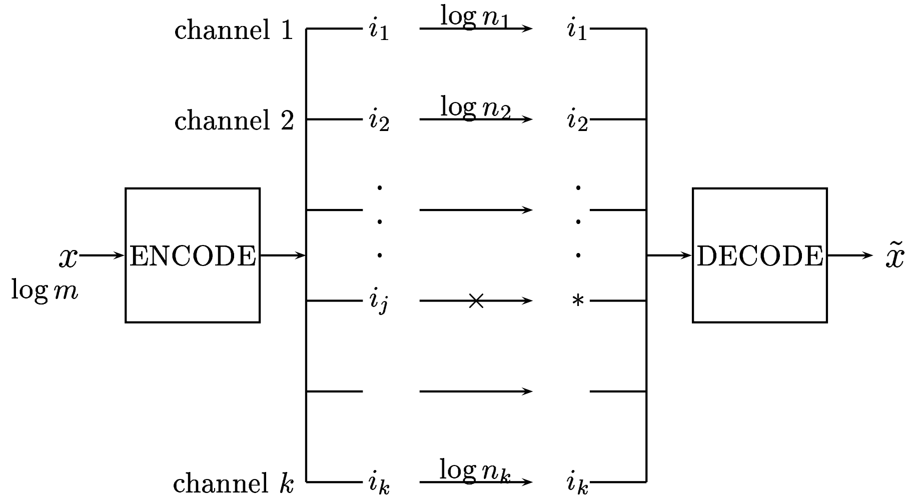

Figure 1.1: Schematic setting of the multichannel problem.

Initial progress on the Multiple Description problem was made by El-Gamal and Cover [15], who studied the achievable rate region for a memoryless source with a single-letter fidelity criterion. Ozarow [41] showed that the achievable region derived in [15] is the rate-distortion region for the special case of a memoryless Gaussian source with a square-error distortion criterion. Zhang and Berger [56] and Witsenhausen, Wolf, Wyner, and Ziv [53], [54] explored whether the achievable rate region is the rate-distortion region for other types of information sources.

The first constructive results for two channels with equal rates were presented by Vaishampayan [48], [47]. In [48] Vaishampayan designs Multiple Description Scalar Quantizers (MDSQs) with good asymptotic properties. We show, however, that this solution is not optimal.

An MDSQ is a *scalar quantizer* (mapping of the source to a finite integer point set) that is designed to work in a diversity-based communication system. The problem of designing an MDSQ consists of two main components: constructing an *index assignment* (a mapping of an integer source to a tuple to be transmitted) and optimizing the structure of the quantizer for that assignment. This paper focuses on the index assignment part of the problem. A general technique is presented for designing index assignments for an arbitrary number of channels. It gives upper and lower bounds on the information distortion for fixed channel rates. In case of two channels transmitting at equal rates, the bounds coincide, thus giving an optimal algorithm for the index assignment problem. In the case of three or more equal-rate channels, the bounds are within a multiplicative constant.

Real applications of the index assignment problem arise in video and speech communication over packet-switched networks. The information in this setting has to be split into several packets which can be lost in transmission resulting in poor signal quality ([29], [30], [55], [3]).

1.1.2 Problem Statement

We are given a communication system with k channels. Channel i transmits information reliably at a rate $\log n_i$ bits per second. Each channel either succeeds or fails to transmit the information. If a channel succeeds, the received information is assumed to be correct. If a channel fails, all the information transmitted over the channel is lost.

We assume the source to generate integers with uniform distribution, the result of a quantization process. We assume the numbers to be contiguous and refer to them by the indices 0 through $m-1$ for some m. Thus we consider the transmitted information be an integer x with at most $\lg m$ bits, where $1 \leq m \leq n_1 n_2 \cdots n_k$. An $(n_1, n_2, ..., n_k)$–level MDSQ maps this x to a unique k-tuple $(i_1, i_2, ..., i_k)$, where component i_j is sent over the jth channel. If all of the channels succeed, then we should be able to decode x from the k-tuple exactly. If some of the channels fail, we want the encoding to minimize the distortion between the original information and the reconstructed transmission. The system can be viewed as one encoder

$$f : \{0, ..., m-1\} \rightarrow \{0, ..., n_1 - 1\} \times \cdots \times \{0, ..., n_k - 1\}$$

and $2^k - 1$ decoders, $(g_0, ..., g_{2^k-2})$, each dealing with a unique subset of successful channels, with at least one succeeding channel. Let g_0 be the decoder with all channels succeeding and let D_t be the distortion rate of the set of channels represented in binary by t, a 1 corresponding to failure. The problem we are interested in, then, is describing the rate-distortion tuples

$$(\log n_1, \log n_2, ..., \log n_k; \quad 0, D_1, ..., D_{2^k-2}),$$

This is a generalization of the notation used for two channels, where the rate distortion tuples are specified by $(R_1, R_2; D_0, D_1, D_2)$. Here R_1 and R_2 are the transmission rates of the two channels; D_0 is the distortion in case both channels succeed; D_1 is the distortion in case of the first channel

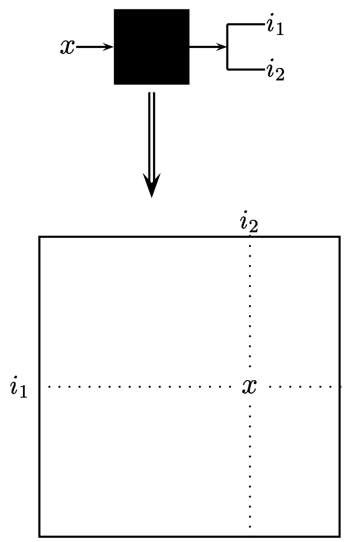

Figure 1.2: Correspondence between an encoding scheme and arrangement of numbers in a matrix.

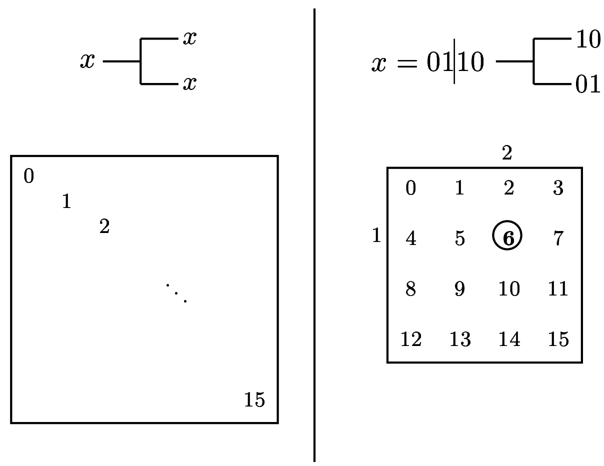

Figure 1.3: Examples of encoding schemes and the corresponding matrices: *left* - identical copies of the input number are sent over the channels; *right* - the input number is split into blocks of bits, which are sent over the channels.

failure; D_2 is the distortion in case of the second channel failure.

Consider the available information at the receiving end. If all of the channels succeed then we receive all the components of the tuple $x = (i_1, i_2, ..., i_k)$. If some of the channels fail, then the remaining successful channels imply upper and lower bounds on x, namely the largest and the smallest values among those consistent with the successful components of x. For example, when $k = 2$, x is mapped to a pair (i_1, i_2). If the first channel fails, we know that x is between the smallest and the largest numbers with i_2 as the second component of the encoding. Similarly, if the second channel fails, x is between the smallest and the largest numbers with i_1 as the first component. Thus, the following two problems are equivalent. On is designing a code to minimize distortion in case of complete failure of any l channels in a system with a contiguous source dictionary of size m and k channels, channel i transmitting reliably at rate $\log n_i$. This information theoretic problem translates to a combinatorial problem of putting numbers $X = \{0, ..., m-1\}$ into a k-dimensional matrix, dimension i being of size n_i, while minimizing the difference between the smallest and the largest number in each full l-dimensional submatrix. This correspondence is shown in Figures 1.2 and 1.3. In this paper we will be working with the combinatorial version of the problem.

1.2 Graph Theory

The problem of designing an encoding in a multiple channel system to minimize the error in case of channel failure can be represented as a graph problem. We associate a graph with each channel. The vertices of that graph correspond to numbers $0, ..., n_i - 1$, where $\log n_i$ is the capacity of channel i. There exists an edge (u, v) if the numbers u and v are indistinguishable in case of channel failure. That is, when the channel fails, we cannot say whether u or v was originally sent over the channel. More formally, given k channels of capacities $\log n_1, \log n_2, ..., \log n_k$ we build k graphs $G_1, ..., G_k$:

$$V_i = \{0, ..., n_i - 1\},$$

$$E_i = \{(u, v) : u, v \text{ Are possible decodings in case of channel failure}\}$$

For example, if in case of channel failure all the information sent over that channel is lost, then the corresponding graph is a clique, since all the numbers are possible decodings, regardless of what was originally sent over the channel. Another example is one bit corruption. That is in case of channel failure we cannot distinguish between numbers one bit apart. The corresponding channel graph in this case is a hypercube. Notice, that this model assumes that each channel can fail in only one way, and the type of failure is known a priori.

A *cartesian product* of graphs G and H, denoted by $G \times H$, is a graph whose vertices are $V_G \times V_H$, and there exists and edge $((x, y), (x', y'))$ if one of the coordinates is equal, and there exists a corresponding edge in the other graph, that is:

$$V(G \times H) := V(G) \times V(H),$$

$$E(G \times H) := \{((x, y), (x', y')) : (x = x' \wedge (y, y') \in E(H)) \vee (y = y' \wedge (x, x') \in E(G))\}.$$

Cartesian product of graphs is truly a cartesian product: it is commutative and associative up to graph isomorphism. See Figure 1.4 for an example of cartesian product of two cliques, a Hamming graph. Note that a hypercube can be viewed as both a cartesian product of cliques or paths of size 2.

A labeling of vertices of a graph $G(V, E)$ is a one-to-one function f from V onto the set

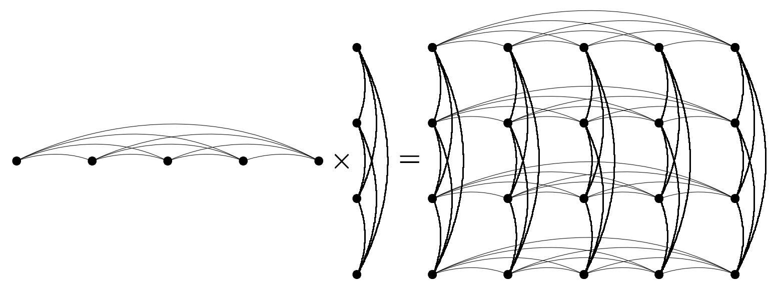

Figure 1.4: Cartesian product of two cliques, a Hamming graph.

$\{0, ..., |V|-1\}$. The *bandwidth of a labeling* f, denoted by $B_f(G)$, is the number

$$\max_{(u,v)\in E} |f(u) - f(v)|,$$

and the *bandwidth of* G, denoted by $B(G)$, is the number

$$\min_f \max_{(u,v)\in E} |f(u) - f(v)|.$$

The *wirelength (bandwidth sum, linear arrangement, edgesum) of a labeling* f, denoted by $L_f(G)$, is the number

$$\sum_{(u,v)\in E} |f(u) - f(v)|,$$

and the *wirelength (bandwidth sum, linear arrangement, edgesum) of* G, denoted by $L(G)$, is the number

$$\min_f \sum_{(u,v)\in E} |f(u) - f(v)|.$$

Notice, that divided by the number of the graph edges, $L(G)$ is the average of the label differences over all the edges.

Going back to the channel problem, designing an encoding scheme for sending numbers up to $\log m$ bits over k channels is equivalent to labeling an induced subgraph of order m of the cartesian product of channel graphs. Depending on whether we are interested in minimizing the maximum or the average decoding error, the corresponding graph optimization problem is bandwidth or wirelength, respectively. For example, designing an encoding for a system where channel failure

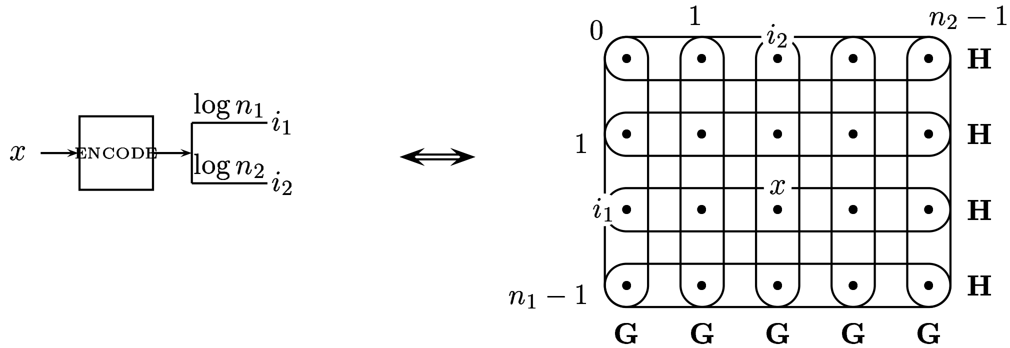

Figure 1.5: Correspondence between the channel communication problem with two channels and the graph labeling of a product of two graphs.

	Cliques: $(K_{n_i})^k$	Paths: $(P_{n_i})^k$
Bandwidth	*Harper [26]:* exact, $(K_2)^k$ *Hendrich & Stiebitz [28]:* exact, $(K_n)^2$ *Berger-Wolf & Reingold [7]:* exact, $(K_n)^2$, upper bound, $(K_n)^k$	$\equiv (P_2)^k$ *Chvátalová [14]:* exact, $(P_n \times P_m)$ *Fishburn & Wright [17]:* exact, $(P_n \times P_m)$ *Moghadam [39]:* exact, $(P_n)^k$ *Bollobas & Leader [10]:* exact, $(P_n)^k$
Wirelength	*Harper [25]:* exact, $(K_2)^k$ *Lindsey [36]:* exact, $(K_n)^k$ *Nakano [40]:* exact, $(K_n)^k$	$\equiv (P_2)^k$ *Nakano [40]:* bounds, $(P_n)^k$ *Fishburn, Tetali, Winkler [19]:* exact, $(P_n \times P_m)$

Table 1.1: The summary of results on the bandwidth and wirelength of Hamming and grid graphs.

constitutes loss of the entire information sent over the channel, for all channels, and the numbers that can be sent over the channels are $0, ..., \prod n_i - 1$, is equivalent to labeling all the vertices of a cartesian product of cliques, or Hamming graph. This correspondence is shown in Figure 1.5. We will concentrate on the bandwidth and wirelength of cartesian products of cliques (Hamming graph) and paths (grid graph). The up-to-date summary of results in the area is presented in Table 1.1.

Both bandwidth and wirelength problems for graphs originated in 1960's at the Jet Propulsion Laboratory at Pasadena. In fact it was motivated exactly by the coding theory problem that we are studying. L. H. Harper and A. W. Hales were looking for codes that minimized the error in a hypercube whose vertices were words of the code. In 1964 Harper [25] published his first paper on the wirelength of hypercubes. Harper proved the fundamental theorem for the wirelength problem. Let $A \subseteq V$,

$$\Theta_G(A) := |\{(u,v) \in E : u \in A, v \notin A\}|$$

$\Theta_G(A)$ is the number of boundary edges of A,

$$\Theta_G(m) := \min_{|A|=m} \Theta_G(A).$$

Let $S_f(m)$ be the set of vertices labeled $0, ..., m-1$

$$S_f(m) := f^{-1}\{0, ..., m-1\},$$

then

$$\begin{aligned}\sum_{(u,v)\in E} |f(u) - f(v)| &= \sum_{m=0}^{2^k} \Theta_G(S_f(m)) \\ &\geq \sum_{m=0}^{2^k} \Theta_G(m)\end{aligned}$$

That is, wirelength of a graph G is at least the sum of the of the number of the boundary edges of the optimal m-sets.

In 1966 Harper [26] published the corresponding result for the bandwidth problem. Let $A \subseteq V$,

$$\delta(A) := \{v \in A : \exists u \notin A, (u, v) \in E\}$$

$\delta(A)$ is the set of the boundary vertices of A, then for any connected graph G

$$\begin{aligned}B(G) &= \min_f \max_{(u,v)\in E} |f(u) - f(v)| \\ &\geq \max_m \min_{|A|=m} |\delta(A)|.\end{aligned}$$

That is, bandwidth of a graph G is at least the maximum number of the boundary vertices of the optimal m-sets.

Since the mid-sixties, there has been a continuous interest and a growing body of research in the graph bandwidth problem. For a survey on the topic of graph bandwidth up to 1982 see [13]. For a survey of the most recent results in bandwidth, wirelength, and other graph vertex ordering problems see Lai and Williams [34]. In general, the graph bandwidth problem is NP-complete [42]. It remains so even for trees. However, it seems to be tractable for the specific cases of Hamming and grid graphs that we are interested in. It is remarkable, that most of the results for these graphs were achieved using Harper's theorems. Harper's approach is called *isoperimetric problem*. It is a discrete analog of the continuous isoperimetric problem, a classical mathematical problem known to ancient Greeks. The original isoperimetric problem is to find among all geometric shapes with a given perimeter (isoperimetric shapes) the one with the largest area. The isoperimetric theorem states that, not surprisingly, the circle is the one. It can be generalized to solid bodies and higher dimensions. From the middle of last century the isoperimetric problem has been working its way into physics. The general form of the problems there is "for all the bodies with the given volume/surface area, which are the ones with the extremal values of a given function (elasticity, frequency, etc.)?". For more information on the subject see Polya and Szegö [43].

The edge/vertex isoperimetric problem on graphs is, given a simple connected graph $G(V, E)$, find induced subgraphs of order m that have minimum/maximum number of boundary edges/vertices. The subsets of vertices that induce the subgraphs that achieve these extremal values are called *isoperimetric*. If the isoperimetric sets $A_i \subseteq V, |A_i| = i$ are such that

$$A_1 \subset A_2 \subset \cdots \subset A_{|V|},$$

then the problem is said to have *nested solutions structure* (NSS). In the language of isoperimetric problem the above Harper's theorems are stated as:

If the edge (vertex) isoperimetric problem has the nested solution structure then the labeling corresponding to the isoperimetric sets is optimal for the wirelength (bandwidth) problem.

For a recent survey on edge isoperimetric problems on graphs see Bezrukov [9].

1.3 Organization

The rest of this thesis is organized as follows. In Chapter 2 we present the main results of the research. We restrict the multichannel communication problem in two different ways and model it as a graph labeling problem. In Section 2.1 we present a solution for no redundancy encoding for the case of complete information loss in case of channel failure. This corresponds to the graph problem of bandwidth optimization of Hamming graphs. Our solution is based on a labeling technique we introduce that we call Herringbone arrangement. While the solution is not optimal, the technique has good practical properties and is useful in other modifications of the communication problem. In Section 2.2 we present an optimal solution to the Hamming graph bandwidth problem. In Section 2.4 we explore the combinatorial properties of the Herringbone arrangement and its applications to other problems. In Section 2.3 we present an optimal solution to yet another restriction of the communication problem: no redundancy encodings with distance-1 type failure. We model it as a grid graph wirelength optimization problem. In Chapter 3 we discuss the related open problems and possible research extensions. We provide an appendix of the terminology we use throughout the thesis.

Chapter 2

Results

2.1 Maximum Error of Complete Channel Failure - Bandwidth of a Hamming Graph

In this section we consider the problem of designing no redundancy encodings for the case of complete information lost in case of channel failure and we are concerned with minimizing the maximum error. More formally, we are given k channels with channel i having capacity $\log n_i$. The numbers that can be sent over the k channels are 0 to $\prod n_i - 1$. In case of channel failure we loose the entire piece of information sent over that channel. The problem is to design an encoding-decoding pair that minimizes the maximum absolute difference between the information originally sent and the information decoded at the receiving end in case of the failure of any number of channels. When only one channel can fail, the problem is equivalent to minimizing graph bandwidth of a k-fold cartesian product of cliques (Hamming graph) $K_{n_1} \times K_{n_2} \times \cdots \times K_{n_k}$. As we have mentioned in the introduction, we represent the problem as that of the arrangement of numbers 0 to $\prod n_i - 1$ in an $n_1 \times n_2 \times \cdots \times n_k$ k-dimensional matrix to minimize the spread - maximum difference between any two numbers in any line. The correspondence between the three problems is shown in Figure 2.1.

We restrict our attention to the case of equal channel capacities, so that the corresponding k-dimensional matrix is a cube. First, the lower bound and the algorithm are derived for the case of failure of only one channel. (In Section 2.1.2 we show the results for the failure of an arbitrary number of channels.) Thus the k-channel problem is reduced to finding an arrangement of the $\prod n_i$ numbers in an $n \times \cdots \times n$ k-dimensional matrix that minimizes the maximum spread in a line.

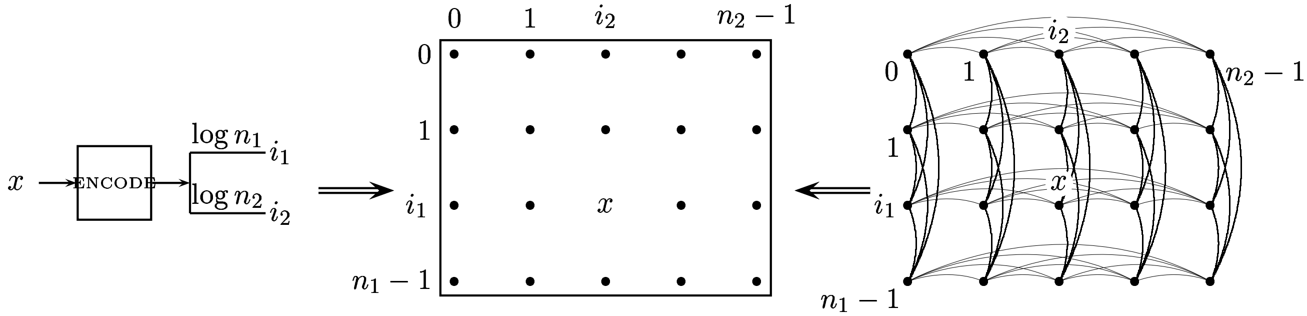

Figure 2.1: Correspondence between the channel communication with two channels, the bandwidth minimization of a product of two cliques, and the number arrangement of a two-dimensional matrix problems.

The idea of the lower bound proof is as follows:

1. For any possible arrangement A of $X = \{0, ...m-1\}$ in the matrix, consider the set of smalls of all the lines. Let $small(A)$ be that set sorted in ascending order. If a number is smallest in more than one line, then it appears in this list more than once. For example, zero always appears k times, and any $small(A)$ list starts with k zeros. The goal is to find a bounding sequence that is at least as large elementwise as any such smalls list. Then the jth smallest number in a line in any arrangement, $small(A)_j$, is at most the jth term of the bounding sequence. Let $\langle a \rangle = \langle a_1, a_2, ... \rangle$ be that bounding sequence; then the following must hold for all j:

$$a_j = \max_A \{small(A)_j\}.$$

   We will show in Lemma 1 that there exists an arrangement whose smalls list realizes the bounding sequence.

2. Just as with the smalls, the goal is to find a bounding sequence for bigs that elementwise is less than any bigs list, $big(A)$, produced by any arrangement. Let $\langle b \rangle = \langle b_1, b_2, ... \rangle$ be that bounding bigs sequence; then the following must hold for all j:

$$b_j = \min_A \{big(A)_j\}.$$

   Lemma 1 shows that there exists an arrangement whose bigs list realizes the bounding sequence.

3. Maximum pairwise difference of the bigs and smalls lists of an arrangement is a lower bound

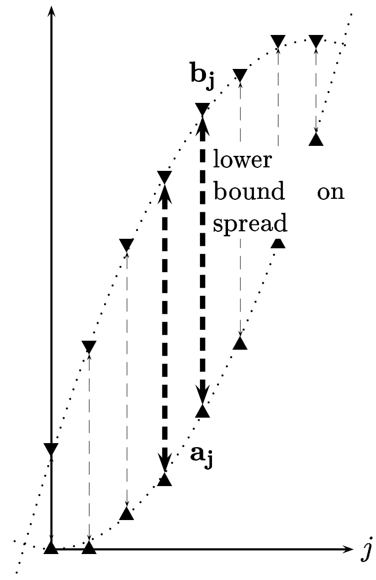

Figure 2.2: The process of derivation of the lower bound on spread.

on the maximum spread of that arrangement, $spread(A)$. That is, for all A

$$\max_j\{big(A)_j - small(A)_j\} \leq spread(A).$$

This statement is known as the Ski Instructor problem and the proof can be found in [35]. Since for the bounding sequences $\langle b\rangle$ and $\langle a\rangle$ we have

$$b_j \leq big(A)_j \text{ and } a_j \geq small(A)_j \text{ for all } A \text{ and } j,$$

pairing smaller b_j with smaller a_j gives a lower bound on the spread for all possible arrangements. That is, for all A:

$$\max_j(b_j - a_j) \leq \max_j\{big(A)_j - small(A)_j\} \leq spread(A).$$

The process is shown schematically in Figure 2.2.

Thus, the main focus of our proof is to find good bounding sequences. We shall build them inductively. Consider the smalls sequence. Suppose we have the initial segment of the bounding smalls sequence, $\langle a_1, a_2, ..., a_j\rangle$, and are now concerned with the next element in that sequence. We

x_1	x_2	$*$
$*$	$*$	

Figure 2.3: Let x_1 and x_2 be the elements less than x that already have been placed. All cells marked $*$ are intersections of two lines, one of which already has a number smaller than x. However putting x in the cell below x_1 or x_2 will produce one cell that is an intersection of two lines, both of which have a smaller number. Thus we favor those over the cell to the right of x_2, since placing x there results only in cells with at most one smaller number in their intersection.

place the elements of X in increasing order into the cells of the matrix. Notice that every cell is an intersection of k-lines. The key observation to maximizing the smalls sequence is that if x is a value in some cell, then x is the smallest number in every line that does not already have an element smaller than x. For example, if we put x into a cell that is an intersection of lines that do not currently have any elements in them, then x is the smallest number in all k lines, and thus appears in the smalls sequence k times. On the other hand, if all k lines already have numbers less than x, then it does not appear in the smalls sequence at all, and the next candidate for the sequence element is at least $x + 1$. In general, if s out of k lines have elements less than x, then x appears in the smalls sequence exactly $k - s$ times. Therefore, to maximize the next element of the smalls sequence, we put x into a cell with the largest number of lines in its intersection that already have elements less than x. Given a choice, we would also like to put x in a cell that reduces the number of lines without smaller values for the subsequent elements. An example of such placement is shown in Figure 2.3.

We now demonstrate the lower bound proof and give an arrangement for some special cases.

2.1.1 One Channel Failure

Assume that the matrix is a cube, that is $n_i = n$ for all i, and the number of elements to be placed in the matrix is $m = n^k$. The matrix being a cube corresponds to all channel capacities being equal. The number of elements being equal to n^k corresponds to having no redundancy in the system. That is, the size of the numbers to be transmitted equals to the total combined channel capacity. From the rate-distortion point of view, this corresponds to tuples of type $(\log n, \log n, ..., \log n; \mathbf{0}, D_1, ..., D_{2^k-1})$.

Herringbone Arrangement

We now define the arrangement that produces a bounding smalls sequence for the cubic matrix. In fact, this arrangement produces a bounding smalls sequence for a more general class of completely-filled rectangular matrices, the cube being a special case.

Definition 1

A *herringbone arrangement* of a k-dimensional $n_1 \times n_2 \times \cdots \times n_k$ completely filled matrix is defined recursively as follows. Assign an arbitrary order to the coordinates of the system $\langle i_1, i_2, ..., i_k \rangle$.

A herringbone arrangement of $0 \times \cdots \times 0$ k-dimensional matrix is empty. A herringbone arrangement of $1 \times \cdots \times 1$ k-dimensional matrix is the number 0 placed in the single cell.

A herringbone arrangement of a 0-dimensional matrix is also the number 0 placed in the single cell.

Given a herringbone arrangement of $t_1 \times t_2 \times \cdots \times t_k$ k-dimensional matrix (that is the cells of the matrix are filled up to the coordinate $t_i - 1$ in dimension i), a larger herringbone arrangement is produced by recursively filling the largest $(k-1)$-dimensional slice adjacent to the arrangement. More precisely, the following algorithm creates a larger arrangement.

- Project the existing arrangement onto the $(k-1)$-dimensional slices adjacent to it
- Calculate the $(k-1)$-dimensional volume of each projection
- Recursively fill the largest volume projection (using coordinate order to break ties) with the herringbone arrangement for $k-1$ dimensions

Examples of 2- and 3-dimensional arrangements are shown in Figure 2.4. The name "herringbone arrangement" was inspired by the herringbone-like pattern seen clearly in two dimensions. We denote the element in the cell $(i_1, ..., i_k)$ of the k-dimensional herringbone arrangement with $HB_k(i_1, ..., i_k)$. Let $i_p = \max_{1 \le j \le k} i_j$. If there is more than one coordinate with the maximum value, we take the largest coordinate. Then

$$HB_k(i_1, ..., i_k) = (i_p + 1)^{(p-1)} \cdot i_p^{(k-p+1)} + HB_{k-1}(i_1, ..., i_{p-1}, i_{p+1}, ..., i_k).$$

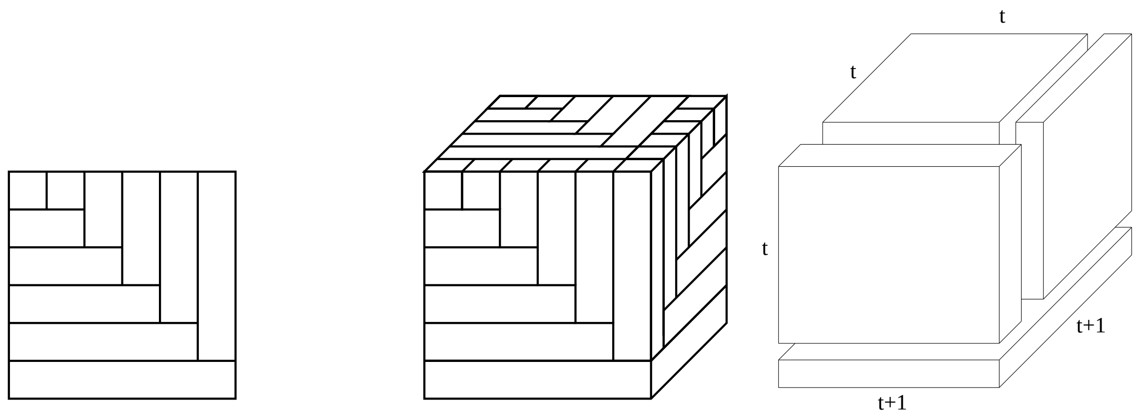

Figure 2.4: An example of herringbone arrangements in 2 and 3-dimensional matrices.

The last equality follows from the recursive definition of the herringbone arrangement. A herringbone arrangement fills the matrix in layers; the maximum coordinate value indicates the layer of the arrangement for the cell. Thus, if i_p is the maximum coordinate value, the value in a cell is in the i_p'th layer. That is, the first $i_p - 1$ layers are completely filled and the element is within a $(k-1)$-dimensional submatrix, recursively filled with the herringbone arrangement.

We can also write explicitly the inverse of the k-dimensional herringbone arrangement:

$$HB_k^{-1} : \mathcal{N} \to I^k$$

Let $t = \lceil \sqrt[k]{m} \rceil$. For $(t-1)^{k-j+1} t^{j-1} < m \leq (t-1)^{k-j} t^j$ for some $1 \leq j \leq k$

$$HB_k^{-1}(m) = (i_1, ..., i_j, ..., i_k),$$

$$\text{where } i_j = t \text{ and } (i_1, ..., i_{j-1}, i_{j+1}, ..., i_k) = HB_{k-1}^{-1}(m - (t-1)^k).$$

That is,

$$HB_k^{-1}(m) = \begin{cases} (t, HB_{k-1}^{-1}(m - (t-1)^k), & \text{if } (t-1)^k < m \leq (t-1)^{k-1} t \\ (\square, t, \square, ..., \square), & \text{if } (t-1)^{k-1} t < m \leq (t-1)^{k-2} t^2 \\ \vdots & \\ (\square, ..., \square, \square, t), & \text{if } (t-1) t^{k-1} < m \leq t^k \end{cases}$$

where the coordinates indicated by boxes form $HB_{k-1}^{-1}(m - (t-1)^k)$.

For example, when $k = 2$, the two-dimensional herringbone arrangement is defined as

$$HB_2^{-1} : \mathcal{N} \to I \times I$$

$$HB_2^{-1}(m) = \begin{cases} (\lceil\sqrt{m}\rceil, m - (\lceil\sqrt{m}\rceil - 1)^2), & \text{if } m \le (\lceil\sqrt{m}\rceil - 1)\lceil\sqrt{m}\rceil \\ (m - (\lceil\sqrt{m}\rceil - 1)\lceil\sqrt{m}\rceil, \lceil\sqrt{m}\rceil), & \text{otherwise} \end{cases}$$

Herringbone arrangements are unique up to the initial order of the coordinates.

The Lower Bound

Now we show that the herringbone arrangement realizes the bounding sequence for the smalls list.

Lemma 1 *A herringbone arrangement of values in a k-dimensional matrix maximizes the smalls sequence (the ascending list of lines' smallest numbers) for that matrix.*

Proof. The proof is a generalization of Harper's proof of the main theorem in [25]. We use induction on k and the dimension size n.

The base cases of $k = 0$ and $n = 1$ are trivial.

Suppose we have a matrix with the dimension size n. By the induction hypothesis, a herringbone arrangement maximizes the smalls sequence in the k-dimensional matrix up to the coordinate value $n - 2$ in every dimension; that is, it maximizes the initial segment of the smalls sequence for the entire matrix.

Consider the smallest element x which has not yet been used in the arrangement. As we have noted before, the goal is to put x into a cell that is an intersection of as many as possible lines with elements smaller than x. Since we have a complete herringbone arrangement of a smaller cube matrix, by induction hypothesis it has all the elements less than x. Thus any cell in the matrix has at most one line in its intersection with elements less than x. The cells that have one such line are precisely the cells that lie in the lines intersecting a face of the existing herringbone arrangement. Consider now all the lines that intersect a face. After placing the first element in any of these lines, there always exists a cell that is an intersection of at least two protected lines. Thus, once started, one must stay with the same face to ensure larger elements in the smalls list. Notice that the cells that are being filled are exactly the $(k-1)$-dimensional projection of a face of a herringbone arrangement, which is a $(k-1)$-dimensional matrix. Thus by induction hypothesis, it is filled with a herringbone arrangement.

Once a face is completed, we move on to another face. By a similar argument, it will be filled with a $(k-1)$-dimensional herringbone arrangement. However, there are several available faces that could be filled next. We now show that we must choose the face with largest volume projection. Notice, that one of the properties of the herringbone arrangement is that at any point the sizes of the available faces differ by at most 1 in any dimension, and they can differ in at most one dimension. Suppose we have one face F_1 of size $t_1 \times \cdots \times t_i \times \cdots \times t_k$ and another face F_2 of size $t_1 \times \cdots \times t_i + 1 \times \cdots \times t_k$. It is easy to see that the smalls sequence produced by filling the face F_1 agrees with the initial segment of the smalls sequence of F_2, assuming that they are filled with the same numbers. The first difference is in the number that comes after filling the face F_1. In the first case, this number goes onto a new face and thus appears in the smalls sequence $k-1$ times. However, since the face F_2 is bigger, the same element will still be within the F_2 face, albeit in a new $(k-2)$-dimensional submatrix, and hence will appear in the sequence only $k-2$ times. Therefore, to maximize the smalls sequence we must first fill the face with the largest volume of the projection where available.

The above arguments produce, by Definition 1, a herringbone arrangement. □

Consider now the complementary arrangement that starts with the largest value in the cell with the largest coordinate and then fills the cube with the herringbone arrangement using the numbers in the descending order. The following fact is a simple consequence of Lemma 1.

Corollary 1 *A complementary herringbone arrangement maximizes the bigs sequence for a k-dimensional cube.*

We are now ready to give a lower bound on the spread in a completely-filled cube.

Theorem 1 *The spread in a completely-filled cube is at least*

$$n^k - 1 - \left\lfloor \left(\frac{kn^{k-1}+2}{2k} \right)^{\frac{k}{k-1}} \right\rfloor - \left\lfloor \left(\frac{kn^{k-1}}{2k} \right)^{\frac{k}{k-1}} \right\rfloor .$$

Proof. By Lemma 1, a herringbone arrangement of a completely filled cube maximizes the smalls sequence. As we have noted, the complimentary arrangement minimizes the bigs sequences.

Since for any arrangement of the elements in the matrix we know the bounding sequences a_j and b_j, the spread for any arrangement is at least $\max_j\{b_j - a_j\}$. Consider the case of $j = kt^{k-1}$ for some t. In this case, a_j is the minimum in the first line after filling a subcube with sides of size t, that is $a_j = t^k = (j/k)^{\frac{k}{k-1}}$. Thus $\lfloor (j/k)^{\frac{k}{k-1}} \rfloor$ is a crude overestimate of any a_j (rounding up to the closest t^k) that coincides with a_j in infinitely many values. The sequence b_j is complementary of a_j. There are kn^{k-1} lines in a k-dimensional cube, hence there are kn^{k-1} elements in each of the smalls and the bigs sequences. Therefore the index complimentary to j in the sequence is $kn^{k-1} - j + 1$ and

$$b_j = n^k - 1 - a_{kn^{k-1}-j+1} \geq n^k - 1 - \lfloor ((kn^{k-1} - j + 1)/k)^{\frac{k}{k-1}} \rfloor.$$

Since the sequences b_j and a_j are complimentary and regular, $\max_j\{b_j - a_j\}$ is achieved in the middle of the sequence, that is, when $j = kn^{k-1}/2$.

$$\begin{aligned}
\max_j\{b_j - a_j\} &\geq b_{kn^{k-1}/2} - a_{kn^{k-1}/2} \\
&= \left(n^k - 1 - \left\lfloor \left(\frac{kn^{k-1} - \frac{kn^{k-1}}{2} + 1}{k}\right)^{\frac{k}{k-1}} \right\rfloor\right) - \left(\left\lfloor \left(\frac{kn^{k-1}}{2k}\right)^{\frac{k}{k-1}} \right\rfloor\right) \\
&= n^k - 1 - \left\lfloor \left(\frac{kn^{k-1} + 2}{2k}\right)^{\frac{k}{k-1}} \right\rfloor - \left\lfloor \left(\frac{kn^{k-1}}{2k}\right)^{\frac{k}{k-1}} \right\rfloor
\end{aligned}$$

□

Note, that this lower bound is weak. It is not sufficient to merely find the maximum difference between the ordered minima and maxima sequences. There are other constraints imposed by the embedding of the sequence in the cube that may give a higher lower bound. For example, the proof does not rely on the fact that the matrix is a cube. More generally, the proof does not take into consideration the relations between the dimension sizes. This means that the same technique gives a lower bound for any $n_1 \times n_2 \times ... \times n_k$ matrix. However for the case of two equal dimensions the lower bound of $\max_j b_j - a_j$ is tight and we can compute it exactly.

Corollary 2 *The spread in a completely filled 2-dimensional cube is at least*

$$\frac{n(n+1)}{2} - 1.$$

Proof. By definition of the 2-dimensional herringbone arrangement, the smalls sequence for a square is

$$\begin{aligned} a_j &= \begin{cases} \lfloor j/2 \rfloor^2 & \text{if } j \text{ is odd,} \\ (j/2-1)^2 + j/2 & \text{if } j \text{ is even.} \end{cases} \\ &= \left\lfloor \left(\frac{j-1}{2}\right)^2 \right\rfloor \end{aligned}$$

The bigs sequence is complimentary to the smalls sequence and is

$$b_j = n^2 - 1 - a_{2n-j+1} = n^2 - 1 - \left\lfloor \left(\frac{2n-j+1-1}{2}\right)^2 \right\rfloor.$$

Thus the difference $b_j - a_j$ is

$$n^2 - 1 - \left\lfloor \left(\frac{2n-j}{2}\right)^2 \right\rfloor - \left\lfloor \left(\frac{j-1}{2}\right)^2 \right\rfloor.$$

By doing the calculations for even and odd cases of j, we find that the integer value of $j = n$ maximizes the expression $b_j - a_j$ and

$$\max_j \{b_j - a_j\} = \frac{n(n+1)}{2} - 1.$$

□

The algorithm

The idea of the algorithm is to put the two complimentary herringbone arrangements together, without increasing the spread. Consider all the ways of merging the two arrangements in a cube. Assuming that the smalls arrangement starts at the $(0, 0, ..., 0)$ corner, and the complimentary bigs arrangement starts at the $(n-1, ..., n-1)$ corner, the possibilities are defined by the order of the coordinates for each arrangement. Thus there are $k!$ possibilities. First, assume for now that we can literally merge the two herringbone arrangements by putting two numbers in every cell of the matrix. In every line, to calculate the spread, the smallest number is taken from the smalls herringbone arrangement and the largest – from the bigs arrangement. Consider any line in the cube and the corresponding smallest and largest numbers defined by the merging permutation. Similar to an earlier argument, the maximum difference between the smallest and the largest numbers in a line occurs in the middle lines, that is the lines of the type $(\lceil (n-1)/2 \rceil, \ldots, *, \ldots, \lfloor (n-1)/2 \rfloor)$. We shall call this line Pr_p with p being the free coordinate. Let $HB_s(Pr_p)$ and $HB_b(Pr_p)$ denote the smallest and the largest elements, respectively, in the line Pr_p. Then to find the best permutation we calculate the following:

$$
\begin{aligned}
\min_{\pi \in Perm} \max_{1 \le p \le k} \{HB_b(Pr_p) - HB_s(Pr_p)\} &= \min_{\pi \in Perm} \max_{1 \le p \le k} \{n^k - 1 - (HB_s(Pr_{\pi(p)}) + HB_s(Pr_p))\} \\
&= n^k - 1 - \max_{\pi \in Perm} \min_{1 \le p \le k} \{HB_s(Pr_{\pi(p)}) + HB_s(Pr_p)\} \\
&= n^k - 1 - (HB_s(Pr_1) + HB_s(Pr_k))
\end{aligned}
$$

To see why the last equation is true, we look at the smallest number in line Pr_p of the herringbone arrangement as a function of p. The calculations for odd n are shown below (the algebra for even n is similar, and the result is the same).

$$
\begin{aligned}
HB_s(Pr_p) &= HB\Big(\frac{n-1}{2}, \ldots, 0, \ldots, \frac{n-1}{2}\Big), \qquad \text{where 0 is in coordinate } p \\
&= \left(\frac{n+1}{2}\right)^{k-1}\left(\frac{n-1}{2}\right) + \left(\frac{n+1}{2}\right)^{k-2}\left(\frac{n-1}{2}\right) + \ldots + \left(\frac{n+1}{2}\right)^{p}\left(\frac{n-1}{2}\right) \\
&\quad + \left(\frac{n+1}{2}\right)^{p-2}\left(\frac{n-1}{2}\right)^2 + \left(\frac{n+1}{2}\right)^{p-3}\left(\frac{n-1}{2}\right)^2 + \ldots + \left(\frac{n-1}{2}\right)^2 \\
&= \left(\frac{n-1}{2}\right)\sum_{i=p}^{k-1}\left(\frac{n+1}{2}\right)^i + \left(\frac{n-1}{2}\right)^2\sum_{i=0}^{p-2}\left(\frac{n+1}{2}\right)^i \\
&= \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)^p \frac{\left(\frac{n+1}{2}\right)^{k-p} - 1}{\left(\frac{n-1}{2}\right)} + \left(\frac{n-1}{2}\right)^2 \frac{\left(\frac{n+1}{2}\right)^{p-1} - 1}{\left(\frac{n-1}{2}\right)} \\
&= \left(\frac{n+1}{2}\right)^p\left(\left(\frac{n+1}{2}\right)^{k-p} - 1\right) + \left(\frac{n-1}{2}\right)\left(\left(\frac{n+1}{2}\right)^{p-1} - 1\right) \\
&= \left(\frac{n+1}{2}\right)^k - \left(\frac{n-1}{2}\right) - \left(\frac{n+1}{2}\right)^{p-1}
\end{aligned}
$$

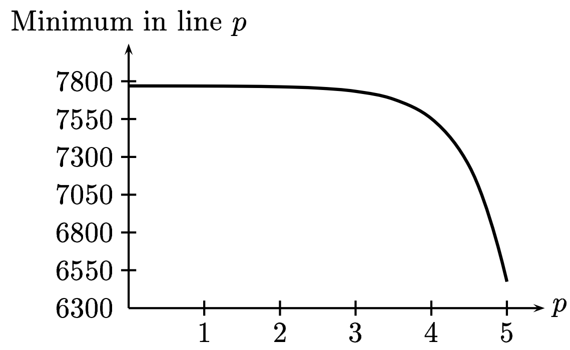

Figure 2.5: The minimum in line p of the Herringbone arrangement as a function of p, shown here for $n = 11$ and $k = 5$. We relax the integer requirement on p for the sake of representation.

This function is exponential in p, with a negative coefficient, thus it is minimal at $p = k$ (see Figure 2.5). The minimum is so small relative to the rest of the function, that $HB(Pr_k) + HB(Pr_i) < HB(Pr_p) + HB(Pr_q)$ for $p, q \neq k$ and any i. In fact, this is true for the extreme case of $p = q = k - 1$ and $i = 1$:

$$(HB(Pr_k) + HB(Pr_1)) - (HB(Pr_{k-1}) + HB(Pr_{k-1})) = -\left(\frac{n+1}{2}\right)^{k-1} + 2\left(\frac{n+1}{2}\right)^{k-2} - 1 < 0.$$

Thus for any permutation π,

$$\min_{1 \leq p \leq k} \{HB_s(Pr_{\pi(p)}) + HB_s(Pr_p)\} = \min \left\{ \begin{array}{l} HB_s(Pr_k) + HB_s(Pr_{\pi(k)}), \\ HB_s(Pr_k) + HB_s(Pr_{\pi^{-1}(k)}) \end{array} \right\},$$

and the maximum over all permutations π is

$$HB_s(Pr_1) + HB_s(Pr_k).$$

This means that one of the best permutations is the reverse permutation. The spread achieved by merging the minima and the maxima sequences using the reverse ordering of the coordinates when n is odd is

$$n^k - 1 - (HB_s(Pr_1) + HB_s(Pr_k)) = n^k - 1 - n\left(\left(\frac{n+1}{2}\right)^{k-1} - 1\right).$$

When n is even, the middle lines are of the type $(i_1, ..., *, ..., i_k)$ where all the coordinates equal

$\lfloor (n-1)/2 \rfloor$ or $\lceil (n-1)/2 \rceil$. Since the arrangement for the maxima sequence uses the reverse order of the coordinates,

$$HB_b(i_1, ..., *, ..., i_k) = n^k - 1 - HB_s(n - i_k - 1, ..., *, ..., n - i_1 - 1).$$

Thus the maximum difference occurs in lines with the first half of the coordinates (ignoring the $*$) being $\lceil (n-1)/2 \rceil$, and the last half of the coordinates being $\lfloor (n-1)/2 \rfloor$. (If k is even, when $*$ is in the first half of the coordinate values, there are more floors than ceilings, and when $*$ is in the last half, then there are more ceilings than floors). Thus the lower bound on the spread is

$$\begin{aligned} HB_b(\underbrace{\lceil \tfrac{n-1}{2} \rceil, ..., \lceil \tfrac{n-1}{2} \rceil}_{\lceil (k-1)/2 \rceil}, \underbrace{\lfloor \tfrac{n-1}{2} \rfloor, ..., \lfloor \tfrac{n-1}{2} \rfloor}_{\lfloor (k-1)/2 \rfloor}, n-1) - HB_s(\underbrace{\lceil \tfrac{n-1}{2} \rceil, ..., \lceil \tfrac{n-1}{2} \rceil}_{\lceil (k-1)/2 \rceil}, \underbrace{\lfloor \tfrac{n-1}{2} \rfloor, ..., \lfloor \tfrac{n-1}{2} \rfloor}_{\lfloor (k-1)/2 \rfloor}, 0) = \\ n^k - 1 - (HB_s(0, \underbrace{\lceil \tfrac{n-1}{2} \rceil, ..., \lceil \tfrac{n-1}{2} \rceil}_{\lfloor (k-1)/2 \rfloor}, \underbrace{\lfloor \tfrac{n-1}{2} \rfloor, ..., \lfloor \tfrac{n-1}{2} \rfloor}_{\lceil (k-1)/2 \rceil}) \\ + HB_s(\underbrace{\lceil \tfrac{n-1}{2} \rceil, ..., \lceil \tfrac{n-1}{2} \rceil}_{\lceil (k-1)/2 \rceil}, \underbrace{\lfloor \tfrac{n-1}{2} \rfloor, ..., \lfloor \tfrac{n-1}{2} \rfloor}_{\lfloor (k-1)/2 \rfloor}, 0)) \end{aligned}$$

Remembering that n is even, the above is equal to

$$\begin{aligned}
n^k - 1 \quad - &\quad (HB_s(0, \underbrace{\frac{n}{2}, \ldots, \frac{n}{2}}_{\lfloor (k-1)/2 \rfloor}, \underbrace{\frac{n-2}{2}, \ldots, \frac{n-2}{2}}_{\lceil (k-1)/2 \rceil}) + HB_s(\underbrace{\frac{n}{2}, \ldots, \frac{n}{2}}_{\lceil (k-1)/2 \rceil}, \underbrace{\frac{n-2}{2}, \ldots, \frac{n-2}{2}}_{\lfloor (k-1)/2 \rfloor}, 0)) \\
= &\quad n^k - 1 - \left(\left(\frac{n}{2}\right)^{\lceil \frac{k+1}{2} \rceil} \sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} \left(\left(\frac{n}{2}\right) + 1\right)^i + \left(\frac{n-2}{2}\right) \sum_{i=1}^{\lceil \frac{k-1}{2} \rceil} \left(\left(\frac{n-2}{2}\right) + 1\right)^i \right. \\
&\quad \left. + \left(\frac{n}{2}\right)^{\lfloor \frac{k+1}{2} \rfloor + 1} \sum_{i=0}^{\lceil \frac{k-1}{2} \rceil - 1} \left(\left(\frac{n}{2}\right) + 1\right)^i + \left(\frac{n-2}{2}\right)^2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor - 1} \left(\left(\frac{n-2}{2}\right) + 1\right)^i \right) \\
= &\quad n^k - 1 - \left(\left(\frac{n}{2}\right)^{\lceil \frac{k+1}{2} \rceil} \left(\frac{n+2}{2}\right) \frac{\left(\frac{n+2}{2}\right)^{\lfloor \frac{k-1}{2} \rfloor} - 1}{\left(\frac{n+2}{2}\right) - 1} + \left(\frac{n-2}{2}\right) \left(\frac{n}{2}\right) \frac{\left(\frac{n}{2}\right)^{\lceil \frac{k-1}{2} \rceil} - 1}{\left(\frac{n}{2}\right) - 1} \right. \\
&\quad \left. + \left(\frac{n}{2}\right)^{\lfloor \frac{k+1}{2} \rfloor + 1} \frac{\left(\frac{n+2}{2}\right)^{\lceil \frac{k-1}{2} \rceil} - 1}{\left(\frac{n+2}{2}\right) - 1} + \left(\frac{n-2}{2}\right)^2 \frac{\left(\frac{n}{2}\right)^{\lfloor \frac{k-1}{2} \rfloor} - 1}{\left(\frac{n}{2}\right) - 1} \right) \\
= &\quad n^k - 1 - \left(\left(\frac{n}{2}\right)^{\lceil \frac{k-1}{2} \rceil} \left(\frac{n+2}{2}\right) \left(\left(\frac{n+2}{2}\right)^{\lfloor \frac{k-1}{2} \rfloor} - 1 \right) + \left(\frac{n}{2}\right) \left(\left(\frac{n}{2}\right)^{\lceil \frac{k-1}{2} \rceil} - 1 \right) \right. \\
&\quad \left. + \left(\frac{n}{2}\right)^{\lfloor \frac{k+1}{2} \rfloor} \left(\left(\frac{n+2}{2}\right)^{\lceil \frac{k-1}{2} \rceil} - 1 \right) + \left(\frac{n-2}{2}\right) \left(\left(\frac{n}{2}\right)^{\lfloor \frac{k-1}{2} \rfloor} - 1 \right) \right)
\end{aligned}$$

When k is odd, this simplifies to

$$n^k + n - 2 - \left(\frac{n}{2}\right)^{\frac{k-1}{2}} \left((n+1) \left(\frac{n+2}{2}\right)^{\frac{k-1}{2}} - 2 \right),$$

and when k is even, this simplifies to

$$n^k + n - 2 - \left(\frac{n}{2}\right)^{\frac{k-2}{2}} \left(\frac{n+2}{2}\right) \left(n \left(\frac{n+2}{2}\right)^{\frac{k-2}{2}} - 1 \right).$$

We have shown that the merging of the two herringbone arrangements, the smalls and the bigs, gives the spread of:

$$n^k - 1 - n\left(\left(\tfrac{n+1}{2}\right)^{k-1} - 1\right), \qquad \text{if } n \text{ is odd,}$$

$$n^k + n - 2 - \left(\tfrac{n}{2}\right)^{\frac{k-1}{2}}\left((n+1)\left(\tfrac{n+2}{2}\right)^{\frac{k-1}{2}} - 2\right), \qquad \text{if } n \text{ is even, and } k \text{ is odd,}$$

$$n^k + n - 2 - \left(\tfrac{n}{2}\right)^{\frac{k-2}{2}}\left(\tfrac{n+2}{2}\right)\left(n\left(\tfrac{n+2}{2}\right)^{\frac{k-2}{2}} - 1\right), \qquad \text{if both } n \text{ and } k \text{ are even.}$$

However, we cannot exactly merge the two herringbone arrangements. We now present an algorithm that combines the two arrangements and preserves the spread calculated for the merging. For the case of 2-dimensional matrices, the lower bound and the merging bound coincide; thus, the bounds are tight and the algorithm is optimal.

Algorithm HERRINGBONE:

1. Fill the initial diagonal half of the matrix $(i_1, ..., i_k), \sum_{j=1}^{k} i_j \leq \lfloor k\frac{n-1}{2} \rfloor$ up to and including the bisecting hyperplane perpendicular to the main diagonal with the herringbone arrangement for the minima sequence.

2. Fill the rest of the matrix with the herringbone arrangement for the maxima sequence, skipping the cells already filled.

Theorem 2 *HERRINGBONE produces an arrangement of a k-dimensional cube with dimensions of size n with the spread of*

$$n^k - 1 - n\left(\left(\tfrac{n+1}{2}\right)^{k-1} - 1\right), \qquad \textit{if } n \textit{ is odd,}$$

$$n^k + n - 2 - \left(\tfrac{n}{2}\right)^{\frac{k-1}{2}}\left((n+1)\left(\tfrac{n+2}{2}\right)^{\frac{k-1}{2}} - 2\right), \qquad \textit{if } n \textit{ is even, and } k \textit{ is odd,}$$

$$n^k + n - 2 - \left(\tfrac{n}{2}\right)^{\frac{k-2}{2}}\left(\tfrac{n+2}{2}\right)\left(n\left(\tfrac{n+2}{2}\right)^{\frac{k-2}{2}} - 1\right), \qquad \textit{if both } n \textit{ and } k \textit{ are even.}$$

Proof. We shall assume for simplicity that n is odd (for an n even the argument works in

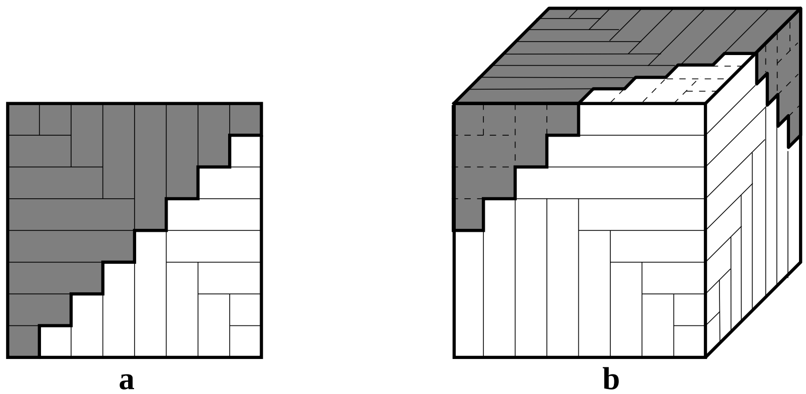

Figure 2.6: The pattern created by the algorithm in (a) two and (b) three dimensions. The shaded part is filled with the herringbone arrangement for the smalls sequence, the other half is filled with the arrangement for bigs sequence.

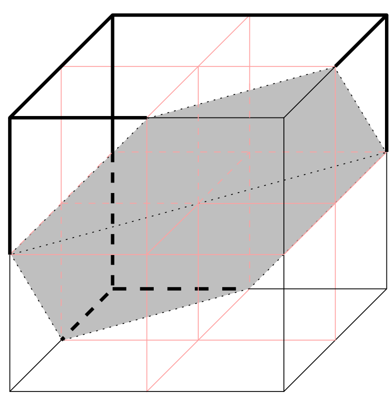

Figure 2.7: Dividing 3-dimensional matrix into 8 orthants. Thick lines show the parts where the border values come from the Herringbone arrangement for the minima sequence.

a similar way.) We use k axes-parallel hyperplanes to divide the matrix into 2^k orthants. Each hyperplane cuts one coordinate into half. We include the dividing hyperplane in both orthants it separates. (For $k = 3$ see Figure 2.7.) The first orthant containing the coordinate $(0, 0, ..., 0)$ lies entirely within the smalls arrangement, and the last orthant containing the largest coordinate $(n-1, n-1, ..., n-1)$ lies entirely within the bigs arrangement. Any line in the arrangement lies in two of the 2^k orthants. All central lines lie within the first and the last orthants. The spread in these lines is exactly the maximum difference between the smalls and the bigs herringbone arrangements, as calculated above. We show that the spread in any other line does not exceed the spread in the central lines.

Besides the central lines, there are three other types of lines:

(a) lines that do not cross the bisecting hyperplane separating the smalls and the bigs arrangements,

(b) lines that cross the bisecting hyperplane and the endpoints are neither in the first nor the last orthant of the cube,

(c) lines that cross the bisecting hyperplane and one of the endpoints is either in the first or the last quadrant of the cube.

If a line does not cross the bisecting hyperplane, then either its smallest number lies in the first orthant or its largest number lies in the last orthant, since these are the only orthants that do not have the bisecting plane cutting through them. Without loss of generality, consider the case where the line lies completely in the smalls arrangement half and its smallest number lies in the first orthant. Then the smallest number in the lines is at most $\lfloor n/2 \rfloor^k$ away from the smallest number in the parallel central line, while its largest number is at least $\lfloor (n/2)^k \rfloor - \lfloor n/2 \rfloor^k$ away from the largest number in the central line. That is, the difference between the large‘st numbers is greater than the difference between the smallest numbers in the two lines. Thus, the spread in a line of type (a) is not greater than the spread in a central line.

The smallest number in a line of type (b) does not lie in the first orthant; therefore, one of the coordinates of the smallest number is greater than $\lceil n/2 \rceil$. One of the properties of the herringbone arrangement is that the values in any line increase in the direction of increasing coordinates.

Therefore the smallest number in the line is greater than the smallest number in the parallel central line. Similarly, the largest number in the line is less than the largest number in the parallel central line. Thus the spread in the line, is less than that in the parallel central line.

We now consider a line of type (c). Without loss of generality, we assume that the smallest number in the line lies in the first orthant. Thus the largest number is not in the last orthant, since all the fixed coordinates of the points on the line are less than $\lceil n/2 \rceil$. Recall that the values in a line of the herringbone arrangement increase in the direction of increasing coordinates. Therefore, both the smallest and the largest numbers in the line are less than those in the parallel central line. We show that the difference between the central smallest number and the smallest number in the line is at least the difference between the largest numbers; thus, the spread in the line is no greater than the spread in the central line. Moreover, we show that this is true for any two lines of type (c) that differ in a single coordinate; that is, the spread in the line closer to the central is at least the spread in the other line. Let a_t and a_{t+1} be the minima in these lines, and b_t, b_{t+1} be the maxima, $t < \lfloor n/2 \rfloor$. Let $Corner(k, t)$ be the number of cells cut off the corner of size t of a k-dimensional cube. Then

$$a_{t+1} - a_t \leq (t+1)^k - t^k,$$

and

$$\begin{aligned} & b_{t+1} - b_t \\ & \quad \geq (n-t)^k - Corner\left(k, \left(\left\lfloor \frac{n}{2} \right\rfloor - (t+1)\right) k\right) - \left((n-(t+1))^k - Corner\left(k, \left(\left\lfloor \frac{n}{2} \right\rfloor - t\right) k\right)\right) \\ & \quad = (n-t)^k - (n-t-1)^k - \left(Corner\left(k, \left(\left\lfloor \frac{n}{2} \right\rfloor - t - 1\right) k\right) - Corner\left(k, \left(\left\lfloor \frac{n}{2} \right\rfloor - t\right) k\right)\right). \end{aligned}$$

Notice that

$$Corner(k, t) = \sum_{i_1=1}^{t} \sum_{i_2}^{i_1} \cdots \sum_{i_{k-1}}^{i_{k-2}} i_{k-1} = \binom{t+k-1}{k},$$

therefore,

$$
\begin{aligned}
b_{t+1} - b_t &\geq (n-t)^k - (n-t-1)^k - \left(\binom{\left(\lfloor \frac{n}{2} \rfloor - t\right) k - k + k - 1}{k} - \binom{\left(\lfloor \frac{n}{2} \rfloor - t\right) k + k - 1}{k} \right) \\
&= (n-t)^k - (n-t-1)^k - \left(\binom{\left(\lfloor \frac{n}{2} \rfloor - t\right) k - 1}{k} - \binom{\left(\lfloor \frac{n}{2} \rfloor - t + 1\right) k - 1}{k} \right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(b_{t+1} - b_t) - (a_{t+1} - a_t) \\
&\quad \geq (n-t)^k - (n-t-1)^k - \left(\binom{\left(\lfloor \frac{n}{2} \rfloor - t\right) k - 1}{k} - \binom{\left(\lfloor \frac{n}{2} \rfloor - t + 1\right) k - 1}{k} \right) - ((t+1)^k - t^k) \\
&\quad = ((n-t)^k - (n-t-1)^k) - ((t+1)^k - t^k) - \left(\binom{\left(\lfloor \frac{n}{2} \rfloor - t\right) k - 1}{k} - \binom{\left(\lfloor \frac{n}{2} \rfloor - t + 1\right) k - 1}{k} \right)
\end{aligned}
$$

$$
t < \left\lfloor \frac{n}{2} \right\rfloor, \text{ therefore } ((n-t)^k - (n-t-1)^k) - ((t+1)^k - t^k) > 0
$$

$$
\binom{\left(\lfloor \frac{n}{2} \rfloor - t\right) k - 1}{k} < \binom{\left(\lfloor \frac{n}{2} \rfloor - t + 1\right) k - 1}{k}, \text{ therefore } - \binom{\left(\lfloor \frac{n}{2} \rfloor - t + 1\right) k - 1}{k} > 0.
$$

Thus,

$$
(b_{t+1} - b_t) - (a_{t+1} - a_t) \geq 0.
$$

That is, the largest numbers increase faster than the smallest numbers in lines of type (c) as they get closer to the center. Therefore, the spread in a line of type (c) is not greater than the spread in a central line.

We have shown that the maximum spread is achieved in the center lines and is as calculated above. □

Corollary 3 *HERRINGBONE produces an arrangement of an n by n square with the spread of*

$$
\frac{n(n+1)}{2} - 1.
$$

This upper bound on the spread of an n by n square matches the lower bound given in Corollary 2. Therefore, for a square, the bounds are tight and the algorithm HERRINGBONE is optimal.

In general, for a completely-filled k-dimensional cube the spread is between the lower bound LB and the upper bound UB, where

$$\mathbf{LB} = n^k - 1 - \left\lfloor \left(\frac{kn^{k-1} + 2}{2k} \right)^{\frac{k}{k-1}} \right\rfloor - \left\lfloor \left(\frac{kn^{k-1}}{2k} \right)^{\frac{k}{k-1}} \right\rfloor$$

and

$$\mathbf{UB} = \begin{cases} n^k - 1 - n\left(\left(\frac{n+1}{2}\right)^{k-1} - 1\right), & \text{if } n \text{ is odd,} \\ n^k + n - 2 - \left(\frac{n}{2}\right)^{\frac{k-1}{2}}\left((n+1)\left(\frac{n+2}{2}\right)^{\frac{k-1}{2}} - 2\right), & \text{if } n \text{ is even, and } k \text{ is odd,} \\ n^k + n - 2 - \left(\frac{n}{2}\right)^{\frac{k-2}{2}}\left(\frac{n+2}{2}\right)\left(n\left(\frac{n+2}{2}\right)^{\frac{k-2}{2}} - 1\right), & \text{if both } n \text{ and } k \text{ are even.} \end{cases}$$

This means that in a multiple description system with all channels of equal capacity the distortion in case of failure of a single channel is between LB and UB. We address the case of failure of multiple channels in the next section.

2.1.2 Arbitrary Number of Channel Failures

In the previous section we have obtained the distortion in a multiple description system with equal capacity channels for the case of failure of a single channel. We now consider the possibility of failure of more than one channel. That is, for k channels, we are concerned with the distortions D_{k+1} through D_{2^k-2} in the rate-distortion tuples $(\log n, ..., \log n; 0, D_1, ..., D_k, D_{k+1}, ..., D_{2^k-2})$. In the number arrangement domain, we are concerned with designing an arrangement that minimizes the spread in any slice of any dimension.

Notice, however, that the herringbone arrangement maximizes and minimizes the smalls and bigs sequences, respectively, for any slice. Thus we can use the same construction for the algorithm.

The maximum error guaranteed by the algorithm in case of l channel failures is

$$b(\underbrace{\lceil\tfrac{n-1}{2}\rceil,\dots,\lceil\tfrac{n-1}{2}\rceil}_{\lfloor(k-l)/2\rfloor},\underbrace{\lfloor\tfrac{n-1}{2}\rfloor,\dots,\lfloor\tfrac{n-1}{2}\rfloor}_{\lceil(k-l)/2\rceil},\underbrace{*,\dots,*}_{l})-a(\underbrace{\lceil\tfrac{n-1}{2}\rceil,\dots,\lceil\tfrac{n-1}{2}\rceil}_{\lfloor(k-l)/2\rfloor},\underbrace{\lfloor\tfrac{n-1}{2}\rfloor,\dots,\lfloor\tfrac{n-1}{2}\rfloor}_{\lceil(k-l)/2\rceil},\underbrace{*,\dots,*}_{l}),$$

which if n is odd, equals

$$n^k-1-\left(\left(\frac{n+1}{2}\right)^{k-l}-1\right)\frac{(n+1)^l+(n-1)^l}{2^l},$$

and if n is even, equals

$$n^k+n-2-\left(\left(\frac{n}{2}\right)^{\lceil\frac{k-l}{2}\rceil}\left(\frac{n+2}{2}\right)^l\left(\left(\frac{n+2}{2}\right)^{\lceil\frac{k-l}{2}\rceil}-1\right)+\left(\frac{n}{2}\right)^l\left(\left(\frac{n}{2}\right)^{\lfloor\frac{k-l}{2}\rfloor}-1\right)+\right.$$
$$\left.\left(\frac{n}{2}\right)^{\lfloor\frac{k+l}{2}\rfloor}\left(\left(\frac{n+2}{2}\right)^{\lfloor\frac{k-l}{2}\rfloor}-1\right)+\left(\frac{n-2}{2}\right)^l\left(\left(\frac{n}{2}\right)^{\lceil\frac{k-l}{2}\rceil}-1\right)\right).$$

This means that in a multiple description system with all channels of equal capacity the distortion D_t is at most as calculated above with l being the number of 1s in the binary representation of t.

2.2 Optimal Hamming Graph Bandwidth

In the previous section we considered the setting of no redundancy communication with complete information loss channel failure. We derived upper and lower bounds on the information distortion for that problem for the case of equal channel capacities. We have noted that when restricted to only one channel failure, the problem is equivalent to the bandwidth optimization problem of Hamming graphs. In this section we present a technique that provides a lower bound for the bandwidth of the Hamming graph with arbitrary clique sizes (arbitrary channel capacities in the communication problem). The technique also suggests an algorithm which provides an almost matching upper bound and is thus nearly optimal. The minimal bandwidth is

$$B(K_{n_1}\times K_{n_2}\times\cdots\times K_{n_k})=\Theta(B(K_2^k)\times\prod\frac{n_i}{2})\approx\frac{\prod n_i}{2\sqrt{k-1}},$$

where $B(K_2^k) = \sum_{t=0}^{k-1} \binom{t}{\lfloor t/2 \rfloor}$ is the bandwidth of the product of k 2-cliques. Both the lower and upper bounds are stronger than the ones derived in the previous section.

The formal statement of the channel problem is the following. Given k channels of capacities $\log n_1, \log n_2, ..., \log n_k$, we need to create and encoding-decoding pair for sending numbers up to $\sum_{t=1}^{k} \log n_t$ bits long in a way that minimizes the maximum absolute difference between the sent and the received numbers in case of one channel failure. All the information sent over a channel that has failed is assumed to be lost. In the setting of graph bandwidth, we need to find an optimal labeling that minimizes the bandwidth of $K_{n_1} \times K_{n_2} \times \cdots \times K_{n_k}$. Again, we will represent both problems as the number arrangements of $\{0 \cdots \prod n_t - 1\}$ into an $n_1 \times n_2 \times \cdots \times n_k$ matrix in a way that minimizes the maximum difference between the largest and the smallest number in any *line* – a full one-dimensional submatrix. We first show a lower bound for the problem and then present an algorithm that achieves that lower bound with a small additive factor. We conjecture, however, that the algorithm is optimal and the lower bound can be tightened slightly. After giving the fundamental lemmata we demonstrate the approach for the two-dimensional case, where we achieve a tight lower bound and give an optimal algorithm, and then generalize it to arbitrary dimensions.

Recall, that the spread of a matrix is the maximum of the differences between any two numbers in any line. First, we note that the lower bound on the spread for *any* line is the lower bound on the spread in the entire matrix, therefore the maximum of the line bounds is also a lower bound for the matrix spread. Thus we can deal with one line at a time. We then restrict our attention to a special kind of arrangement showing that this restriction does not eliminate optimal arrangements. Then for these arrangements it is easier to find a lower bound on the spread in any line.

Definition 2 *An arrangement is* monotonic *if the values in any line ascend with the increase of the changing coordinate. That is, an arrangement A is monotonic if for all $0 \leq i_t, j_t \leq n_t - 1, 1 \leq s \leq k$*

$$((t \neq s \rightarrow i_t = j_t) \wedge i_s < j_s) \rightarrow A(i_1, ..., i_s, ..., i_k) < A(j_1, ..., j_s, ..., j_k).$$

Lemma 2 *Given any arrangement of any set of $n_1 n_2 \cdots n_k$ numbers, sorting it to become monotonic one coordinate at a time, one line at a time, does not increase the spread. That is, for any*

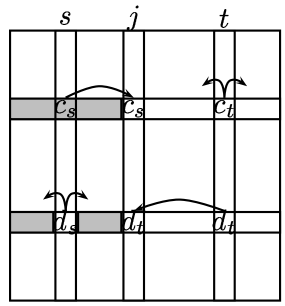

Figure 2.8: Sorting the values within the rows causes the spread in columns to increase to $d_t - c_s$, occurring now in column j. Before sorting, d_t was in column t and c_s was in column s. Shaded are the c's less than c_s but not equal to c_t and d's that are less than d_t and not equal to d_s. Note that necessarily $s < t$, but j can be any column relative to s and t.

arrangement A,

$$\text{spread}(\text{sorted}(A)) \leq \text{spread}(A).$$

Proof. We first show that given any arrangement, sorting the numbers to become monotonic in *one* coordinate does not increase the overall spread. It is obvious that rearranging the numbers in any way within the same line does not change the spread in that line, thus sorting within a coordinate does not change the spread in that coordinate. Suppose the spread has increased in another coordinate. The situation is illustrated in Figure 2.8. Let the maximum spread in that coordinate *after* sorting be $d_t - c_s$ appearing in line j (where d_t was in line t *before* the rearrangement, and c_s was in line s). Then

$$|d_t - c_t| < d_t - c_s, \text{ thus } c_s < c_t, \text{ and}$$

$$|d_s - c_s| < d_t - c_s, \text{ thus } d_s < d_t.$$

Then there are $j - 2$ (since d_t and c_s are now in line j) d's less than d_t and not equal to d_s. There are $j - 1$ c's less than c_s and not equal to c_t. Therefore, by the pigeonhole principle, there exists $c_p < c_s$ that was paired up with $d_p > d_t$ before the rearrangement. But then $d_p - c_p > d_t - c_s$, which contradicts the assumption that the spread increased after sorting. Thus sorting in one coordinate does not increase the spread in *any* coordinate.

Gale and Karp [20] show that if the arrangement was monotonic in any coordinate then it will remain so after the numbers are sorted in any other coordinate. Thus the matrix can be sorted to have a monotonic arrangement one coordinate at a time, one line at a time, without increasing the spread. □

This allows us to restrict attention to monotonic arrangements. From these arrangements we can more easily find a general structure of the lower bound on the spread.

Consider some number x in a cell of a monotonic arrangement. The k axis-parallel hyperplanes that pass through that cell divide the matrix into 2^k orthants. For any monotonic arrangement of any set of $n_1 n_2 \cdots n_k$ numbers, all the numbers in the orthant containing the coordinate $(0, 0, ..., 0)$ are necessarily less than x, and all the numbers in the orthant containing the last coordinate are necessarily greater than x. Besides the first and the last orthants, the other $2^k - 2$ orthants may contain both numbers less than and greater than x. Any line passing through x necessarily has both numbers less and greater than x by the nature of monotonicity of the arrangement. However any other line can be filled entirely with only smaller or larger numbers.

Lemma 3 *For the optimal arrangement A of numbers $0, ..., \prod_{i=1}^{k} n_i - 1$ in an $n_1 \times n_2 \times ... \times n_k$ matrix, there exists a line $(i_1, i_2, ..., i_{j-1}, *, i_{j+1}, ..., i_k)$ (all the coordinates but the jth are fixed) in that arrangement and there exists a cell in that line $(i_1, i_2, ..., i_j, ..., i_k)$ such that the spread in that line is at least the volume of any minimal set of orthants (as defined by the cell) that separates between the orthant containing the cell $(0, .., 0)$ (the first orthant) and the orthant containing the cell $(n_1 - 1, ..., n_k - 1)$ (the last orthant).*

Proof. First, we will note several facts:

- Removing any *minimal* separating set of orthants leaves only two *connected* sets of orthants: the set containing the first orthant (we shall call this set "small" orthants) and the set containing the last orthant ("large" orthants).
- Since the set is a minimal separating set, *any* cell within any of the separating orthants is contained in lines that intersect the "large" orthants and in lines that intersect the "small" orthants.
- No line passes through both the "small" and "large" orthants, since otherwise they would not be separated.

Now we are ready to prove the lemma. Let $V_{small}(cell)$ be the volume (number of cells) of the "small" orthants, $V_{large}(cell)$ be the volume of the "large" orthants, and $V_{sep}(cell)$ be the volume of the separating orthants. Note that $V_{small} + V_{sep} + V_{large} = \prod_{i=1}^{k} n_i = V$. For the optimal arrangement A let $l = (i_1, i_2, ..., *, ..., i_k)$ be the line with the largest spread (that is, the spread of the arrangement is the spread in this line). Here are the two possible cases:

- there exists a cell $(i_1, i_2, ..., i_j, ..., i_k)$ such that the smallest number in the line, $\min_l$, is at most $V_{small}(cell)$ (for any separating set defined by the cell), and the largest number in the line, $\max_l$, is at least $V - V_{large}(cell)$. Then

$$\begin{aligned} \text{spread}(A) &\geq \max_l - \min_l \\ &\geq V - V_{large}(cell) - V_{small}(cell) \\ &= V_{sep}(cell) \end{aligned}$$

  and the statement of the lemma holds.

- for all cells in the line, for some separating set for each cell, either $\max_l < V - V_{large}$ or $\min_l > V_{small}$.

  Let i_j be such that V_{sep} as defined by $(i_1, ..., i_j, ..., i_k)$ is the smallest. Without loss of generality we assume that $\min_l > V_{small}(i_1, ..., i_j, ..., i_k)$, while $\max_l$ can be either less or greater or equal to $V - V_{large}(i_1, ..., i_j, ..., i_k)$.

  Since $\min_l > V_{small}(i_1, ..., i_j, ..., i_k)$ there must be at least one element less than $\min_l$ in the separating orthants defined by the cell $(i_1, ..., i_j, ..., i_k)$. Suppose there are s elements that are less than $\min_l$ total in the separating orthants. Then there are at most $s - 1$ elements greater than $\min_l$ in the small orthants. Each of the s elements in the separating orthants must be in a line that intersects large orthants (as defined by the cell $(i_1, ..., i_j, ..., i_k)$). Since $\max_l - \min_l$ is the largest spread, all elements in those lines must be less than $\max_l$. Let there be l of those elements. One possible way this can happen is shown in Figure 2.9. We use a switching idea similar to Fishburn, Tetali, and Winkler [19]. Suppose there are $t \geq l$ elements total less than max_l in the large orthants. Then $max_l \leq V - V_{large} + t$. Replace

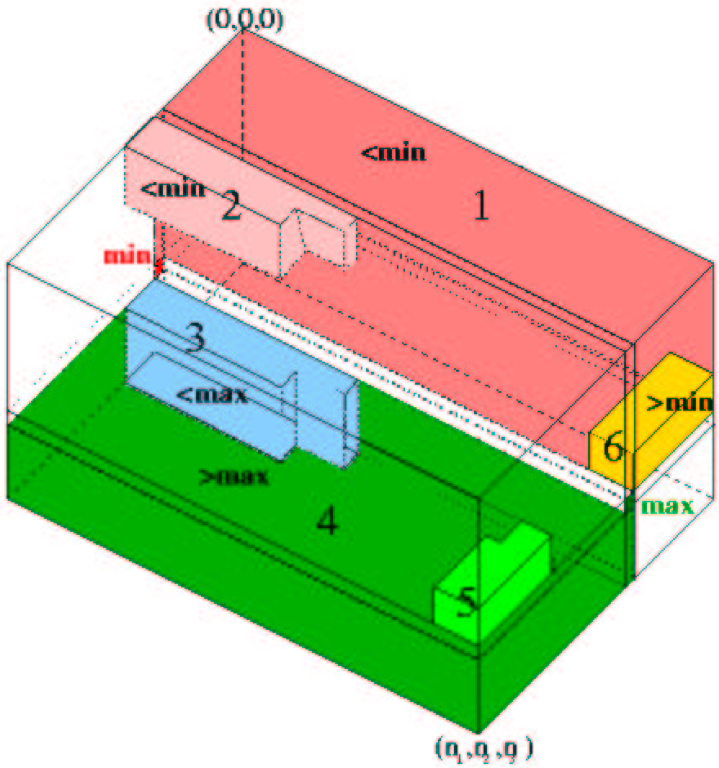

Figure 2.9: Areas (1) and (6) are the small orthants, areas (4) and (3) are the large orthants, and the uncolored area, with (2) and (5), are the separating orthants. The indicated min and max are the minimum and maximum in the line l. (1) are the numbers less than $\min_l$ in the small orthants. (2) are the s numbers less than $\min_l$ in the separating orthants. (3) are the l numbers less than $\max_l$ in the large orthants. Those are the intersection of the lines that have a light red minimum with the large orthants. (4) are the numbers greater than $\max_l$ in the large orthants. (5) are the p numbers greater than $\max_l$ in the separating orthants. (6) are the matching numbers greater than $\min_l$ in the small orthants.

the largest of those t elements l_1 with max_l, then replace the next largest element l_2 with l_1 and so on, trickling down until we get to the smallest of those t elements. Put that element instead of max_l. We have not violated the monotonicity. We have increased each of the t elements by at least 1 and decreased max_l by t. Therefore the new $max_l \leq V - V_{large}$ and all the elements in the large orthants are greater than max_l. Similarly, we can replace the largest of the s small elements, s_1, with min_l, then replace the next largest element s_2 with s_1, and so on until we either reach the tth largest or the smallest of the s elements. We replace min_l with that element. We have increased each of the s elements by at most 1, so the relative spread has not increased. The minimum min_l has decreased by at most t, so the spread in line l has not increased.

If $s > t$ then we have stopped after replacing t of the s elements and there are still some elements less than the new min_l in the separating orthants in the lines with elements greater than the new max_l. We have not increased the spread in line l or anywhere else, but the spread in those lines is greater than the new $max_l - min_l$ which equals the old spread since both the minimum and the maximum decreased by the same amount. This is a contradiction to the fact the l was the line with the maximum spread.

If $s \leq t$ then there are no elements less than min_l in the separating orthants and the new $min_l \leq V_{small}$. If there are no elements greater than max_l in the separating orthants, then $max_l - min_l \geq V_{sep}$, which means the initial spread was also at least V_{sep}, which is a contradiction. Suppose there are p elements greater than the new max_l in the separating orthants. Those elements must be in lines that intersect small orthants and the elements in the small orthants in those lines must be greater than the new min_l. Suppose there are q of those elements. We can perform the same replacement procedure and if $p > q$ we will get the same contradiction as in case of $s > t$. Otherwise, $p \leq q$. Since there are no elements less than the new min_l in the separating orthants and there are q elements in the small orthants that are greater than min_l, then the new $min_l = V_{small} - q$ and therefore the original $min_l = V_{small} - q + s$. Similarly, since there are p elements greater than the new max_l in the separating orthants and no elements less than it in the large orthants, the new $max_l = V - V_{large} - p$ and thus the original $max_l = V - V_{large} - p + t$. So the original spread is the difference between the original $\max_l$ and $\min_l$ which is

$$(V - V_{large} - p + t) - (V_{small} - q + s) = V_{sep} + q - p + s - t.$$

Since $p \leq q$ and $s \leq t$, the original spread was at least V_{sep}, which is, again, a contradiction.

Therefore, the spread in an optimal arrangement is at least V_{sep} for any minimal set of the separating orthants as defined by some cell in the maximum spread line.

□

We have proved that the spread in an optimal arrangement is at least the volume of *any* minimal

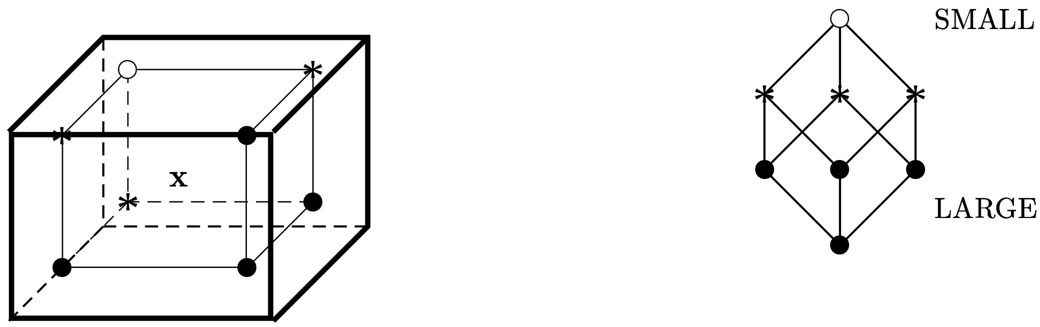

Figure 2.10: Correspondence between the orthants and a hypercube in 3 dimensions. The $small(x)$ and $large(x)$ numbers meet only if a line passes through two adjacent orthants.

set of orthants separating between the "small" and "large" orthants, for some cell in the largest spread line. Therefore, there is a cell such that the spread is at least the volume of the *largest* minimal separating set of orthants, the one that has the largest number of orthants. In fact, if we associate a super vertex with each orthant and have an edge between any two vertices if the corresponding orthants are adjacent, then we get a k-dimensional hypercube representing the orthants. Figure 2.10 shows this in 3 dimensions. By definition of bandwidth, the largest minimum separating set of orthants is exactly the bandwidth of the k-dimensional hypercube. So for an optimal arrangement, for any line in the arrangement, the spread is at least the minimum over all cells of the volume of the bandwidth-separating set of orthants. Thus the spread in the optimal arrangement is at least

$$\max_{\text{all lines}} \{ \min_{\text{cells in a line}} \{ \min_{\substack{\text{separating set} \\ \text{of } B(K_2^k) \text{ orthants}}} \{\text{volume of the separating set of orthants}\}\}\}.$$

Using this lemma we can calculate the lower bound on the spread in the optimal arrangement. We will first demonstrate our approach in two dimensions and then generalize is to arbitrary number of dimensions.

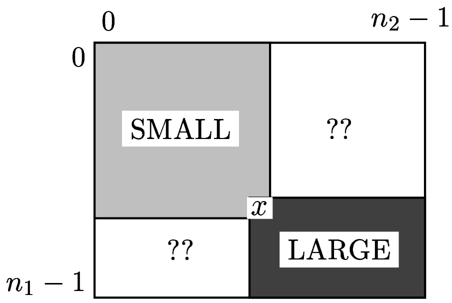

Figure 2.11: The location of numbers less and greater than a given cell value in a monotonic arrangement.

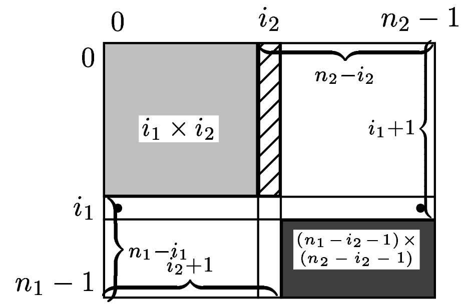

Figure 2.12: Spread in a row as the volume of the unfilled separating orthants (as defined by the cell (i_1, i_2)). To minimize the spread, the larger of the top and bottom parts of the column is filled.

2.2.1 Two Dimensions

Theorem 3 *Without loss of generality assume $n_1 \leq n_2$. The spread in any arrangement of an n_1 by n_2 matrix is at least*

$$\frac{(n_1+1)n_2}{2} - 1 \textit{ if } n_1 \textit{ is odd,}$$

$$\frac{n_1(n_2+1)}{2} - 1 \textit{ if } n_1 \textit{ is even.}$$

Proof. In two dimensions, there is only one set of orthants separating between the first and the last orthants, which is the other two of the four orthants. This is also consistent with $B(K_2^2) = 2$. Thus the spread in a two dimensional arrangement is at least

$$\max_{\text{all lines}} \{ \min_{\text{cells in a line}} \{\text{area of the two separating orthants}\}\} =$$

$$\max \left\{ \begin{array}{l} \max\limits_{\text{all rows}} \{ \min\limits_{\text{cells in a row}} \{\text{area of the two separating orthants}\}\}, \\ \max\limits_{\text{all columns}} \{ \min\limits_{\text{cells in a column}} \{\text{area of the two separating orthants}\}\} \end{array} \right\}.$$

Let $0 \leq i_1 \leq n_1 - 1$, $0 \leq i_2 \leq n_2 - 1$. The lower bound on the spread in an n_1 by n_2 matrix is

(see Figure 2.12 for illustration of the calculations)

$$\max\left\{\begin{array}{l}\max\limits_{\text{row } i_1}\left\{\min\limits_{i_2}\{(i_1+1)(n_2-i_2)+(n_1-i_1)(i_2+1)-2-\max\{i_1,\ n_1-i_1-1\}\}\right\},\\ \\ \max\limits_{\text{col } i_2}\left\{\min\limits_{i_1}\{(i_1+1)(n_2-i_2)+(n_1-i_1)(i_2+1)-2-\max\{i_2,\ n_2-i_2-1\}\}\right\}\end{array}\right\}$$

The spread in a line is a symmetric unimodal function in the free coordinate, with the maximum occurring in the middle, thus we separate at the half point and evaluate at endpoints:

$$= \max\left\{\begin{array}{l}\max\limits_{\text{row } i_1}\left\{\begin{array}{lll} i_1 \le \lceil\frac{n_1}{2}\rceil-1 & \overset{\Longrightarrow}{i_2=0} & (i_1+1)n_2+(n_1-i_1)-(n_1-i_1)-1\\ i_1 > \lceil\frac{n_1}{2}\rceil-1 & \overset{\Longrightarrow}{i_2=n_2-1} & (i_1+1)+(n_1-i_1)n_2-i_1-2\end{array}\right\},\\ \max\limits_{\text{col } i_2}\left\{\begin{array}{lll} i_2 \le \lceil\frac{n_2}{2}\rceil-1 & \overset{\Longrightarrow}{i_1=0} & (n_2-i_2)+n_1(i_2+1)-(n_2-i_2)-1\\ i_2 > \lceil\frac{n_2}{2}\rceil-1 & \overset{\Longrightarrow}{i_1=n_1-1} & n_1(n_2-i_2)+(i_2+1)-i_2-2\end{array}\right\}\end{array}\right\}$$

$$= \max\left\{\max\limits_{\text{row } i_1}\left\{\begin{array}{l} i_1 \le \lceil\frac{n_1}{2}\rceil-1 \Rightarrow (i_1+1)n_2-1\\ i_1 > \lceil\frac{n_1}{2}\rceil-1 \Rightarrow (n_1-i_1)n_2-1\end{array}\right\},\ \max\limits_{\text{col } i_2}\left\{\begin{array}{l} i_2 \le \lceil\frac{n_2}{2}\rceil-1 \Rightarrow n_1(i_2+1)-1\\ i_2 > \lceil\frac{n_2}{2}\rceil-1 \Rightarrow n_1(n_2-i_2)-1\end{array}\right\}\right\}$$

$$= \max\left\{\max\left\{\left\lceil\frac{n_1}{2}\right\rceil n_2-1,\ \left\lfloor\frac{n_1}{2}\right\rfloor n_2-1\right\},\ \max\left\{n_1\left\lceil\frac{n_2}{2}\right\rceil-1,\ n_1\left\lfloor\frac{n_2}{2}\right\rfloor-1\right\}\right\}$$

$$= \max\left\{\left\lceil\frac{n_1}{2}\right\rceil n_2-1,\ n_1\left\lceil\frac{n_2}{2}\right\rceil-1\right\}$$

$$= \left\{\begin{array}{ll} n_1 \text{ even} \Rightarrow \frac{n_1n_2}{2}-1 \le n_1\lceil\frac{n_2}{2}\rceil-1 \Rightarrow & \begin{array}{l} n_2 \text{ odd} \Rightarrow \frac{n_1(n_2+1)}{2}-1\\ n_2 \text{ even} \Rightarrow \frac{n_1n_2}{2}-1\end{array}\\ n_1 \text{ odd} \Rightarrow \frac{(n_1+1)n_2}{2}-1 \ge n_1\lceil\frac{n_2}{2}\rceil-1 \Rightarrow & \frac{(n_1+1)n_2}{2}-1\end{array}\right\}$$

Thus in case of n_1 odd the spread in the matrix is at least $(n_1+1)n_2/2-1$ and if n_1 is even and n_2 is odd then the spread is at least $n_1(n_2+1)/2-1$. We will present arrangements that achieve these bounds thus the lower bound is sharp. We now show, however, that the lower bound of $n_1n_2/2-1$ for the case of both n_1 and n_2 even is not sharp.

The minimum spread for the column $n_2/2-1$ is achieved when $i_1=0$, while the minimum spread for the column $n_2/2$ is achieved when $i_1=n_1-1$. All the arrangements consistent with

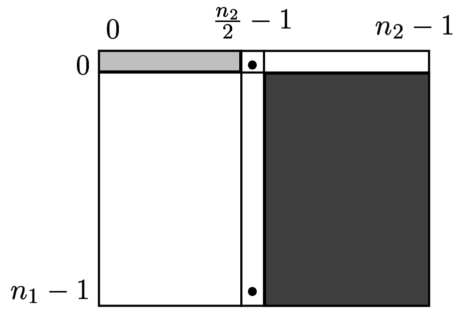

Figure 2.13: Minimum spread for the column $n_2/2-1$ occurs when $i_1=0$. The elements in the the light gray area are less than the minimum in the column, and the elements in the dark gray area are greater than the maximum.

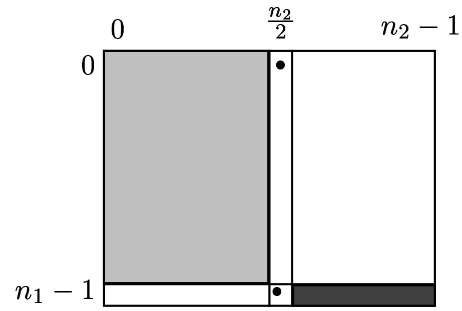

Figure 2.14: Minimum spread for the column $n_2/2$ occurs when $i_1=n_1-1$. The elements in the the light gray area are less than the minimum in the column, and the elements in the dark gray area are greater than the maximum.

both spreads have the form shown in Figure 2.15. However, it is not difficult to see that for any arrangement of this type the spread in any row i_1 passing through the areas 2 and 5 the spread is at least

$$(n_1-i_1)\frac{n_2}{2}+(i_1+1)\frac{n_2}{2}-1=\frac{(n_1+1)n_2}{2}-1.$$

The spread in rows passing through areas 1 and 3 or 4 and 6 is at most the spread in any column, which is at least

$$\frac{n_1n_2}{2}-1<\frac{(n_1+1)n_2}{2}-1.$$

Similarly, for any first coordinate i_1, Figure 2.16 shows all the arrangements consistent with the spread achieved in the column $n_2/2-1$ with the first coordinate being i_1 and the column $n_2/2$ with the first coordinate being n_1-i_1. Again, for any row passing through the areas 2 and 5 is at least $\frac{(n_1+1)n_2}{2}-1$. The spread in columns is at least

$$\begin{aligned}(n_1-i_1)\frac{n_2}{2}+i_1(\frac{n_2}{2}+1)-1 &= \frac{n_1n_2}{2}+i_1-1\\ &\le \frac{(n_1+1)n_2}{2}-1.\end{aligned}$$

The best spread is achieved when there is no areas 2 and 5, that is, $i_1=n_1/2$. The row spread

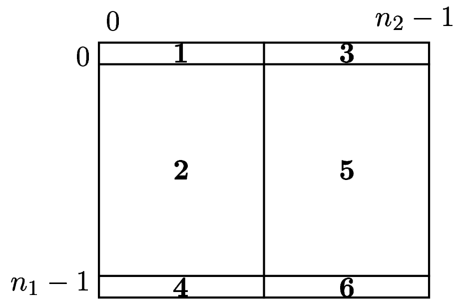

Figure 2.15: Arrangements that maintain the minimum spreads in the the two central columns. The areas are filled monotonically in the order shown in the figure.

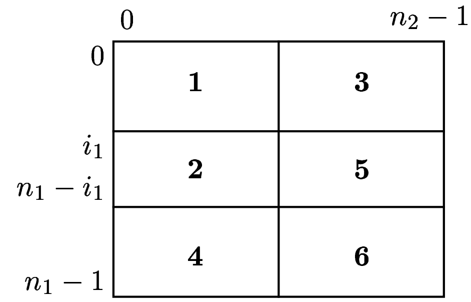

Figure 2.16: Arrangements that maintain the spreads achieved for first coordinate i_1 in column $n_2/2-1$ and first coordinate $n_1 - i_1$ in column $n_2/2$. Since the spread in a line is a symmetric unimodal function with the maximum for the middle coordinate, those spreads are equal.

now is at most the column spread for any row and the column spread is at least

$$
\begin{aligned}
\frac{n_1 n_2}{2} + \frac{n_1}{2} - 1 &= \frac{n_1(n_2+1)}{2} - 1 \\
&\leq \frac{(n_1+1)n_2}{2} - 1.
\end{aligned}
$$

Thus, for any monotonic arrangement with both n_1 and n_2 even the spread must be at least

$$\frac{n_1(n_2+1)}{2} - 1.$$

□

We now present an arrangement that achieves this lower bound and is thus optimal. The algorithm is slightly different for odd and even n_1 therefore we will state them separately.

Theorem 4 *The following algorithm produces an arrangement of spread*

$$\frac{n_1(n_2+1)}{2} - 1$$

if $n_1 \leq n_2$ and n_1 is even and is thus optimal:

Fill consecutively, left to right, the upper half-columns of the matrix, then fill the lower half-

columns of the matrix in the same manner:

1. *Fill consecutively, column by column, the upper half of each column i_2.*

   That is, fill the cells

$$(0, i_2),\ (1, i_2),\ \ldots,\ (\frac{n_1}{2} - 1, i_2)$$

   with numbers

$$i_2\frac{n_1}{2}, i_2\frac{n_1}{2} + 1, \ldots, (i_2 + 1)\frac{n_1}{2} - 1.$$

2. *Fill consecutively, column by column, the lower half of each column i_2.*

   That is, fill the cells

$$(\frac{n_1}{2}, i_2),\ (\frac{n_1}{2} + 1, i_2),\ \ldots,\ (n_1 - 1, i_2)$$

   with numbers

$$\frac{n_1 n_2}{2} + i_2\frac{n_1}{2}, \frac{n_1 n_2}{2} + i_2\frac{n_1}{2} + 1, \ldots, \frac{n_1 n_2}{2} + (i_2 + 1)\frac{n_1}{2} - 1.$$

The arrangement is shown schematically in Figure 2.17.

The proof is simply an algebraic verification of the spread in all the rows and columns.

Proof. The difference between the largest and the smallest number in any column i_2 is the difference between the elements in the last row and the first row of the column:

$$\left(\frac{n_1 n_2}{2} + (i_2 + 1)\frac{n_1}{2} - 1\right) - i_2\frac{n_1}{2} = \frac{n_1 n_2}{2} + \frac{n_1}{2} - 1 = \frac{n_1(n_2 + 1)}{2} - 1.$$

The difference between the largest and the smallest number in any row i_1 is the difference between the elements in the last column and the first column of the row:

$$\left((n_2 - 1)\frac{n_1}{2} + i_1\right) - \left(0\frac{n_1}{2} + i_1\right) = \frac{n_1(n_2 - 1)}{2} < \frac{n_1(n_2 + 1)}{2} - 1.$$

Thus the overall spread of the arrangement is $n_1(n_2 + 1)/2 - 1$, which is the lower bound for the case of n_1 even, and the arrangement is optimal. □

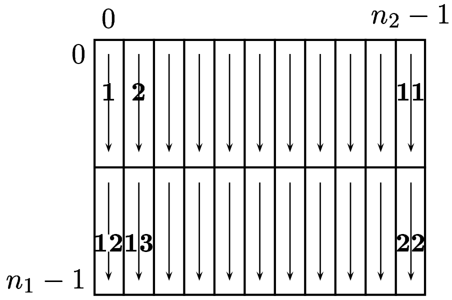

Figure 2.17: The arrangement in case of n_1 even.

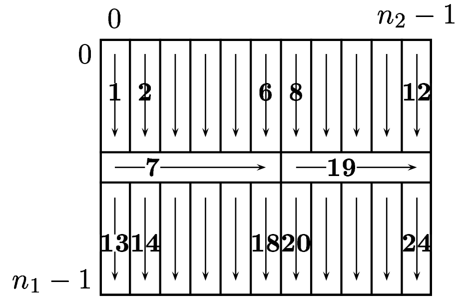

Figure 2.18: The arrangement in case of n_1 odd.

Theorem 5 *The following algorithm produces an arrangement of spread*

$$\frac{(n_1+1)n_2}{2} - 1$$

if $n_1 \le n_2$ and n_1 is odd and is thus optimal:

1. *Fill consecutively, column by column, the upper $\lfloor n_1/2 \rfloor$ cells of columns* 0 *through* $\lceil n_2/2 \rceil - 1$.

   That is, fill the cells

   $$(0, i_2),\ (1, i_2),\ \ldots,\ (\left\lfloor \frac{n_1}{2} \right\rfloor - 1, i_2)$$

   with numbers

   $$i_2 \left\lfloor \frac{n_1}{2} \right\rfloor, i_2 \left\lfloor \frac{n_1}{2} \right\rfloor + 1, \ldots, (i_2+1) \left\lfloor \frac{n_1}{2} \right\rfloor - 1.$$

2. *Fill consecutively the left $\lceil n_2/2 \rceil$ cells of the $(\lfloor n_1/2 \rfloor)$th row.*

   That is, fill the cells

   $$(\left\lfloor \frac{n_1}{2} \right\rfloor, 0),\ (\left\lfloor \frac{n_1}{2} \right\rfloor, 1),\ \ldots,\ (\left\lfloor \frac{n_1}{2} \right\rfloor, \left\lceil \frac{n_2}{2} \right\rceil - 1)$$

   with numbers

   $$\left\lceil \frac{n_2}{2} \right\rceil \left\lfloor \frac{n_1}{2} \right\rfloor, \left\lceil \frac{n_2}{2} \right\rceil \left\lfloor \frac{n_1}{2} \right\rfloor + 1, \ldots, \left\lceil \frac{n_2}{2} \right\rceil \left(\left\lfloor \frac{n_1}{2} \right\rfloor + 1 \right) - 1.$$

3. *Fill consecutively, column by column, the upper $\lfloor n_1/2 \rfloor$ cells of columns $\lceil n_2/2 \rceil$ through $n_2 - 1$.*

That is, fill the cells

$$(0, i_2),\ (1, i_2),\ \ldots,\ (\left\lfloor \frac{n_1}{2} \right\rfloor - 1, i_2)$$

with numbers

$$i_2 \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lceil \frac{n_2}{2} \right\rceil, i_2 \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lceil \frac{n_2}{2} \right\rceil + 1, \ldots, (i_2 + 1) \left\lfloor \frac{n_1}{2} \right\rfloor - 1 + \left\lceil \frac{n_2}{2} \right\rceil.$$

4. *Fill consecutively, column by column, the lower* $\lfloor n_1/2 \rfloor$ *cells of columns* 0 *through* $\lceil n_2/2 \rceil - 1$.

   That is, fill the cells

$$(\left\lceil \frac{n_1}{2} \right\rceil, i_2),\ (\left\lceil \frac{n_1}{2} \right\rceil + 1, i_2),\ \ldots,\ (n_1 - 1, i_2)$$

   with numbers

$$i_2 \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_1}{2} \right\rfloor n_2 + \left\lceil \frac{n_2}{2} \right\rceil, i_2 \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_1}{2} \right\rfloor n_2 + \left\lceil \frac{n_2}{2} \right\rceil + 1, \ldots, (i_2 + 1) \left\lfloor \frac{n_1}{2} \right\rfloor - 1 + \left\lfloor \frac{n_1}{2} \right\rfloor n_2 + \left\lceil \frac{n_2}{2} \right\rceil.$$

5. *Fill consecutively the right* $\lfloor n_2/2 \rfloor$ *cells of the* $(\lfloor n_1/2 \rfloor)$*th row.*

   That is, fill the cells

$$(\left\lfloor \frac{n_1}{2} \right\rfloor, \left\lceil \frac{n_2}{2} \right\rceil),\ (\left\lfloor \frac{n_1}{2} \right\rfloor, \left\lceil \frac{n_2}{2} \right\rceil + 1),\ \ldots,\ (\left\lfloor \frac{n_1}{2} \right\rfloor, n_2 - 1)$$

   with numbers

$$\left(n_2 + \left\lceil \frac{n_2}{2} \right\rceil\right) \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lceil \frac{n_2}{2} \right\rceil, \left(n_2 + \left\lceil \frac{n_2}{2} \right\rceil\right) \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lceil \frac{n_2}{2} \right\rceil + 1, \ldots, \left(n_2 + \left\lceil \frac{n_2}{2} \right\rceil\right) \left\lfloor \frac{n_1}{2} \right\rfloor + n_2 - 1.$$

6. *Fill consecutively, column by column, the lower* $\lfloor n_1/2 \rfloor$ *cells of columns* $\lceil n_2/2 \rceil$ *through* $n_2 - 1$.

   That is, fill the cells

$$(\left\lceil \frac{n_1}{2} \right\rceil, i_2),\ (\left\lceil \frac{n_1}{2} \right\rceil + 1, i_2),\ \ldots,\ (n_1 - 1, i_2)$$

with numbers

$$(n_2+i_2)\left\lfloor\frac{n_1}{2}\right\rfloor+n_2,(n_2+i_2)\left\lfloor\frac{n_1}{2}\right\rfloor+n_2+1,\ldots,(n_2+i_2+1)\left\lfloor\frac{n_1}{2}\right\rfloor-1+n_2.$$

The arrangement is shown schematically in Figure 2.18.

The proof is algebraic and is similar to the case of n_1 even.

Proof. The difference between the largest and the smallest number in columns 0 through $\lceil n_2/2\rceil-1$ is the difference between the elements in the last and first rows of that column i_2:

$$\begin{aligned}\left((i_2+1)\left\lfloor\frac{n_1}{2}\right\rfloor-1+\left\lfloor\frac{n_1}{2}\right\rfloor n_2+\left\lceil\frac{n_2}{2}\right\rceil\right)-i_2\left\lfloor\frac{n_1}{2}\right\rfloor &= \left\lfloor\frac{n_1}{2}\right\rfloor n_2+\left\lfloor\frac{n_1}{2}\right\rfloor+\left\lceil\frac{n_2}{2}\right\rceil-1\\ &< \left\lfloor\frac{n_1}{2}\right\rfloor n_2+\left\lfloor\frac{n_2}{2}\right\rfloor+\left\lceil\frac{n_2}{2}\right\rceil-1\\ &= \frac{(n_1+1)n_2}{2}-1,\end{aligned}$$

since $n_1\le n_2$.

The difference between the largest and the smallest number in columns $\lceil n_2/2\rceil$ through n_2-1 is, again, the difference between the elements in the last and first rows of that column i_2:

$$\left((n_2+i_2+1)\left\lfloor\frac{n_1}{2}\right\rfloor-1+n_2\right)-\left(i_2\left\lfloor\frac{n_1}{2}\right\rfloor+\left\lceil\frac{n_2}{2}\right\rceil\right)=\left\lfloor\frac{n_1}{2}\right\rfloor n_2+\left\lfloor\frac{n_1}{2}\right\rfloor+\left\lceil\frac{n_2}{2}\right\rceil-1,$$

which is the same as in the other columns, and thus less than $(n_1+1)n_2/2-1$.

The difference between the largest and the smallest number in rows 0 through $\lfloor n_1/2-1\rfloor$ is the difference between the elements in the last and first columns in that row i_1:

$$\left((n_2-1)\left\lfloor\frac{n_1}{2}\right\rfloor+\left\lceil\frac{n_2}{2}\right\rceil+i_1\right)-\left(0\left\lfloor\frac{n_1}{2}\right\rfloor+i_1\right)=\left\lfloor\frac{n_1}{2}\right\rfloor(n_2-1)+\left\lceil\frac{n_2}{2}\right\rceil<\frac{(n_1+1)n_2}{2}-1.$$

The difference between the largest and the smallest number in rows $\lfloor n_1/2\rfloor+1$ through n_1-1 is, again, the difference between the elements in the last and first columns in that row i_1:

$$\left((2n_2-1)\left\lfloor\frac{n_1}{2}\right\rfloor+n_2+i_1\right)-\left(0\left\lfloor\frac{n_1}{2}\right\rfloor+\left\lfloor\frac{n_1}{2}\right\rfloor n_2+\left\lceil\frac{n_2}{2}\right\rceil+i_1\right)=\left\lfloor\frac{n_1}{2}\right\rfloor(n_2-1)+\left\lfloor\frac{n_2}{2}\right\rfloor<\frac{(n_1+1)n_2}{2}-1.$$

The difference between the largest and the smallest number in the $\lfloor n_1/2 \rfloor$th row is

$$\left(\left(n_2 + \left\lceil \frac{n_2}{2} \right\rceil\right) \left\lfloor \frac{n_1}{2} \right\rfloor + n_2 - 1\right) - \left\lceil \frac{n_2}{2} \right\rceil \left\lfloor \frac{n_1}{2} \right\rfloor = n_2 \left(\left\lfloor \frac{n_1}{2} \right\rfloor + 1\right) - 1 = \frac{(n_1+1)n_2}{2} - 1.$$

Thus the overall spread of the arrangement is $(n_1+1)n_2/2 - 1$, which is the lower bound for the case of n_1 odd, and so the arrangement is optimal. □

We have shown that the proposed algorithm produces an arrangement with the spread that matches the lower bound of

$$\frac{(n_1+1)n_2}{2} - 1 \text{ if } n_1 \text{ is odd,}$$

$$\frac{n_1(n_2+1)}{2} - 1 \text{ if } n_1 \text{ is even,}$$

and thus is optimal.

2.2.2 Generalization to Arbitrary Dimensions

Given a $n_1 \times n_2 \times \cdots \times n_k$ matrix, where $n_1 \leq n_2 \leq \cdots \leq n_k$, the goal is to arrange the numbers $\{0, \ldots, \prod n_i - 1\}$ in a way that minimizes the maximum difference between the largest and the smallest number in any line of the matrix.

Using techniques very similar to the two-dimensional case, it is possible to show a lower bound of roughly

$$B(K_2^k) \times \prod \frac{n_i}{2},$$

where $B(K_2^k)$ is the bandwidth of the product of k 2-cliques, and give an optimal arrangement that achieves it.

Theorem 6 *The spread in any arrangement of an $n_1 \times \cdots \times n_k$ matrix, where $n_1 \leq n_2 \leq \cdots \leq n_k$, is at least*

$$B(K_2^k) \times \prod_{i=1}^{k} \left\lfloor \frac{n_i}{2} \right\rfloor.$$

Proof. By Lemma 3 the spread in the optimal arrangement is at least

$$\max_{\text{all lines}} \{ \min_{\text{cells in a line}} \{ \min_{\substack{\text{separating set} \\ \text{of } B(K_2^k) \text{ orthants}}} \{\text{volume of the separating set of orthants}\}\}\}.$$

Just like in two dimensions, the smallest volume of the separating orthants for any line $(i_1, ..., *, ..., i_k)$ occurs either for the cell $i_j = 0$ or $i_j = n_j - 1$, depending on which of the coordinates $i_t, t \neq j$ are at most $\lfloor n_t/2 \rfloor$ and which ones are greater. Thus the maximum over all lines of the minimum volume of the separating orthants is achieved for one of the extreme cells of the central lines $(\lfloor n_1/2 \rfloor, \lfloor n_2/2 \rfloor, ..., *, ..., \lfloor n_k/2 \rfloor)$. This in itself immediately gives a lower bound of

$$B(K_2^k) \times \prod_{i=1}^{k} \left\lfloor \frac{n_i}{2} \right\rfloor.$$

□

The optimal arrangement in k dimensions is constructed similarly to the 2-dimensional one.

Theorem 7 *The following algorithm produces an arrangement A of spread at most*

$$B(K_2^k) \times \prod \left\lceil \frac{n_i}{2} \right\rceil + \frac{n_1}{2} - 1$$

in case $n_1 = \min_i \{n_i\}$ is even and is thus nearly optimal:

1. *Divide the matrix into 2^k orthants by dividing each coordinate n_i into two halves of size $\lfloor n_i/2 \rfloor$ and $\lceil n_i/2 \rceil$*

2. *Fill the first orthant (containing the coordinate $(0, ..., 0)$) in the following way:*

   (a) $A(0) = (0, ..., 0)$

   (b) *The $1st$ coordinate of $A(m)$ is the $1st$ coordinate of $A(m-1)$ plus 1 modulo $\lfloor n_1/2 \rfloor$. If the ith coordinate becomes 0 then the $(i+1)st$ coordinate increases by 1 modulo $\lfloor n_{i+1}/2 \rfloor$.*

3. *Fill the orthants one after another in a way similar to the first orthant. The orthants are filled in the order corresponding to the optimal numbering of K_2^k [26]: each step number a neighbor of the smallest already numbered vertex, taking care that the maximum bandwidth difference occurs between the vertices in K_2^k adjacent along the n_1 coordinate. To ensure that, after numbering the vertex corresponding to the first orthant with number 0, number the orthant adjacent to it along the $(k-i+1)th$ coordinate with number i.*

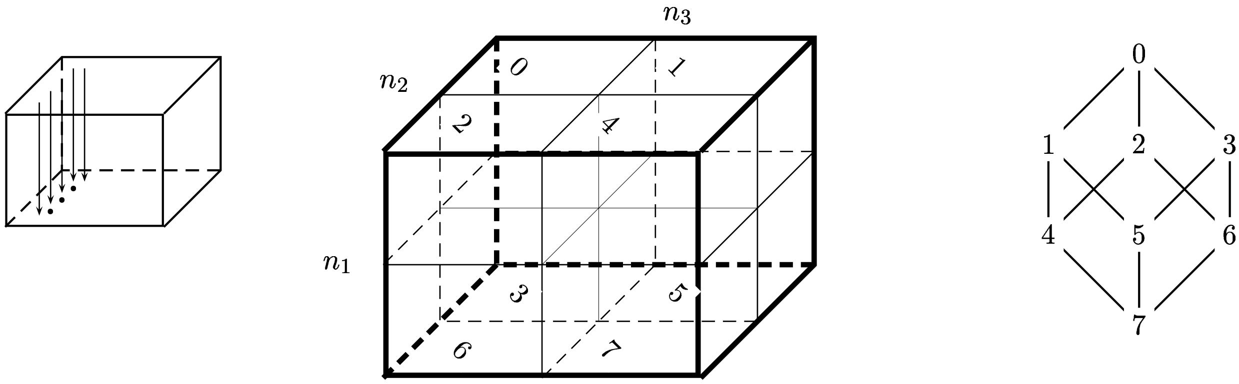

Figure 2.19: Schematic representation of 3-dimensional arrangement in case of n_1 even.

The algorithm is shown schematically for 3 dimensions in Figure 2.19.

Proof. Any line in the arrangement is contained within two orthants, therefore the spread in any line is the difference between the labels of the corresponding orthants times the volume of the larger-volume orthant plus the difference between the smallest and the largest number of the line within the smaller-volume orthant. Notice that for any two orthants with the label difference less than the bandwidth of the k dimensional hypercube the spread in the line passing through them is at most the bandwidth times the volume of the larger-volume orthant. Therefore the maximum spread occurs in a line passing through two orthants with the label difference equal to the bandwidth of the k dimensional hypercube. By construction all such orthants align in the direction of the first dimension, that is along the i_1 coordinate. Thus the spread in that line is

$$B(K_2^k) \times \prod \left\lceil \frac{n_i}{2} \right\rceil + \frac{n_1}{2} - 1$$

□

Theorem 8 *The following algorithm produces an arrangement A of spread at most*

$$B(K_{n_2} \times \cdots \times K_{n_k}) + B(K_{n_2}^k) \left\lfloor \frac{n_1}{2} \right\rfloor \prod_{i=2} \left\lceil \frac{n_i}{2} \right\rceil$$

in case $n_1 = \min_i \{n_i\}$ is odd and is thus nearly optimal:

1. *Divide the matrix into 2^k orthants by dividing each coordinate $n_i \neq n_1$ into two halves of size*

$\lfloor n_i/2 \rfloor$ *and* $\lceil n_i/2 \rceil$. *For* n_1 *use the coordinates* $i_1 < \lceil n_1/2 \rceil - 1$ *and* $i_1 > \lceil n_1/2 \rceil - 1$ *to define the rest of the orthants. The submatrix* $(\lceil n_1/2 \rceil - 1, *, *, ..., *)$ *is left out.*

2. *Fill the first orthant (containing the coordinate* $(0, ..., 0)$*) in the following way:*

    (a) $A(0) = (0, ..., 0)$

    (b) *The* $1st$ *coordinate of* $A(m)$ *is the* $1st$ *coordinate of* $A(m-1)$ *plus* 1 *modulo* $\lfloor n_1/2 \rfloor$. *If the* ith *coordinate becomes* 0 *then the* $(i+1)st$ *coordinate increases by* 1 *modulo* $\lfloor n_{i+1}/2 \rfloor$.

3. *Fill the orthants one after another in a way similar to the first orthant. The orthants are filled in the same way and the same order as in the case of* n_1 *even up to and including the* $small(x)$ *orthant corresponding to the first edge in the hypercube that gives the maximum bandwidth.*

4. *Recursively fill the* $small(x)$ *half of the* $k-1$*-dimensional submatrix* $(\lceil n_1/2 \rceil - 1, *, *, ..., *)$ *with the optimal arrangement.*

5. *Fill the orthants up to the* $large(x)$ *orthant that corresponds to the other vertex on the first edge in the hypercube with the maximum bandwidth.*

6. *Filling the rest of the orthants in the same manner as in case of* n_1 *even.*

Proof. Similar to the case of n_1 even, each line passes through either two orthants above the submatrix $(\lceil n_1/2 \rceil - 1, *, *, ..., *)$, below that submatrix, through an orthant above, an orthant below, and the submatrix, or lies entirely within the submatrix. It is easy to see that the maximum spread occurs in a line of the last type and is therefore,

$$B(K_{n_2} \times \cdots \times K_{n_k}) + B(K_{n_2}^k) \left\lfloor \frac{n_1}{2} \right\rfloor \prod_{i=2} \left\lceil \frac{n_i}{2} \right\rceil$$

□

Thus we have shown that the bandwidth of a Hamming graph $K_{n_1} \times \cdots \times K_{n_k}$ is between the lower bound LB and the upper bound UB, where LB and UB are as follows:

$$\begin{aligned}
\mathbf{LB} &= B(K_2^k) \times \prod_{i=1}^{k} \left\lfloor \frac{n_i}{2} \right\rfloor \\
\mathbf{UB} &= \left\{ \begin{array}{ll} B(K_2^k) \times \prod \lceil \frac{n_i}{2} \rceil \frac{n_1}{2} - 1 & \text{if } n_1 \text{ is even} \\ B(K_{n_2} \times \cdots \times K_{n_k}) + B(K_{n_2}^k) \lfloor \frac{n_1}{2} \rfloor \prod_{i=2} \lceil \frac{n_i}{2} \rceil & \text{if } n_1 \text{ is odd} \end{array} \right\}
\end{aligned}$$

Notice, if all n_is are even, then the difference between the LB and UB is $n1/2 - 1$, which is very small compared to the order of magnitude of the LB of $O((n_1/2)^k)$. The difference between LB and UB is largest when all n_is are odd. Let all n_i be equal n. Noting that

$$B(K_{n_2}^k) = \sum_{i=0}^{k-1} \binom{i}{\lfloor i/2 \rfloor} \approx \frac{2^{k-1}}{\sqrt{k-1}},$$

the upper bound approximately equals

$$\mathbf{UP} \approx \sum_{i=2}^{k} \frac{n^i}{2\sqrt{i-1}} + n - 1,$$

while the lower bound is approximately

$$\mathbf{LB} \approx \frac{n^k}{2\sqrt{k-1}}.$$

Thus, the difference between LB and UB in case n is odd is the order of $O(n^{k-1})$. Therefore, overall, the upper and lower bounds nearly coincide in infinitely many points. We believe that the upper bound is the correct bandwidth of the Hamming graph and the lower bound needs to be tightened.

2.3 Average Error of Distance-1 Failure - Wirelength of a Grid Graph

In this section we focus on yet another special case of the channel communication problem. We assume that if a channel fails then not all the information is lost, and the possible decodings are within absolute distance of 1 from what was originally sent over the channel. We are interested in minimizing the average error. As was mentioned in the introduction, we can associate a graph with each channel. The vertices of the graph for channel i are $\{0, ..., n_i - 1\}$, and there is an edge (u, v) if $|u - v| = 1$. That is the graph associated with each channel is a path P_{n_i}, and the graph corresponding to all k channels is a grid – a cartesian product of paths. Since we are interested in average error, the graph optimization problem is the wirelength problem, also known as bandwidth sum, edgesum, or linear arrangement. Thus the channel coding problem with the dictionary of size m and a single channel failure is equivalent to determining which induced subgraphs of order m of a $P_{n_1} \times P_{n_2} \times ... \times P_{n_k}$ have the smallest wirelength. Again, both problems can be represented as number arrangements of the elements of X in a k-dimensional $n_1 \times n_2 \times ... \times n_k$ matrix that minimizes the sum of differences of adjacent cells. The coordinates of that matrix in each dimension i can have values from 0 to $n_i - 1$, where n_i is the size of dimension i. This is the setting that we will consider from now on. In this section we present a solution for all $n_i = n$ and $m = n^k$.

Recently Fishburn, Tetali, and Winkler [19] gave an optimal labeling for a two–dimensional grid $P_n \times P_m$. In this section we suggest how their labeling can be extended to arbitrary dimensions using the Herringbone arrangement introduced in Section 2.1.

Fishburn, Tetali, and Winkler prove that the minimum wirelength of a n by m grid, $m \geq n \geq 2$, is

$$m(n^2 + n - 1) - n - t^* \frac{2(t^*)^2 - 6nt^* + 3n^2 + 3n - 2}{3},$$

where $t^* \in \{1, 2, ..., \lfloor n/2 \rfloor\}$ that maximizes $t(2t^2 - 6nt + 3n^2 + 3n - 2)$. The labeling that achieves this optimum is shown schematically in Figure 2.20.

Notice, that the corner pieces of the labeling look very much like a Herringbone arrangement, which is not a coincidence. In general, we suggest that the optimal labeling in k dimensions looks as following:

Given $n_1 \times n_2 \times \cdots \times n_k$ grid, $n_i \geq n_{i+1}$, divide the matrix into 3^k pieces by dividing each dimension into 3 parts. Fill the corner pieces using the Herringbone arrangement, recursively filling each tile of the arrangement with the optimal labeling for the $(k-1)$–dimensional grid. Then fill in the entire central part of the first dimension using $(k-1)$–dimensional slats that are recursively filled with the optimal arrangement for the lower dimension. Then fill the central part of the second

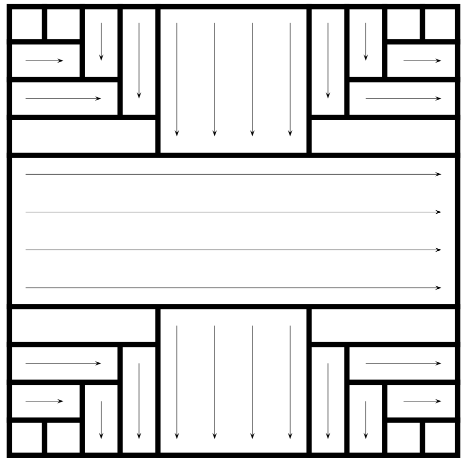

Figure 2.20: Schematic representation of an optimal labeling.

dimension, omitting the part already filled – the intersection with the first dimension, and so on. The example for 3 dimensions is shown schematically in Figure 2.21.

We now outline the proof, following closely the proof of Fishburn, Tetali, and Winkler.

2.3.1 Monotonic Arrangements

Fishburn, Tetali, and Winkler introduce the idea of *doubly-monotonic assignments* – those which increase as both coordinates increase. In Section 2.2 we gave a definition of monotonic arrangements which is a generalization of doubly monotonic assignments to arbitrary dimensions. With this definition, Lemma 1 from [19] holds for monotonic assignments with the same proof.

Lemma 4 *Wirelength of a grid graph is minimized by a monotonic arrangement.*

Thus we need to consider only monotonic arrangements from now on. It follows then that the wirelength is fully determined by the border values of the assignment, since the sum is telescopic in each coordinate. Only the values in cells of the form $(..., 0, ...)$ or $(..., n_i - 1, ...)$ appear in the wirelength expression. The coefficient with which any value appears in the expression is [# of $n_i - 1$'s minus # of 0s in its coordinate]. Figure 2.22 shows the coefficients for each cell in a $6 \times 6 \times 6$ arrangement.

Thus as in [19], subject to monotonicity, we are looking for arrangements with relatively small values in cells of the k "bigs" sides and relatively large values in cells of the other k "smalls" sides.

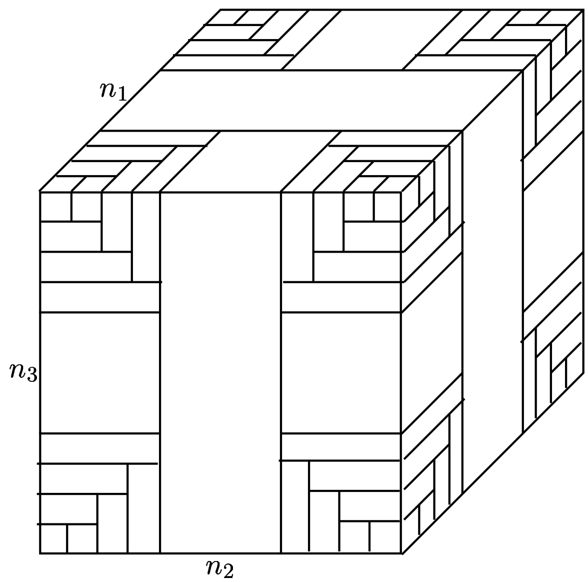

Figure 2.21: Schematic representation of the suggested generalization in 3 dimensions

$i_3 = 0$

-3	-2	-2	-2	-2	-1
-2	-1	-1	-1	-1	0
-2	-1	-1	-1	-1	0
-2	-1	-1	-1	-1	0
-2	-1	-1	-1	-1	0
-1	0	0	0	0	1

$i_3 = 1, 2, 3, 4$

-2	-1	-1	-1	-1	0
-1	0	0	0	0	1
-1	0	0	0	0	1
-1	0	0	0	0	1
-1	0	0	0	0	1
0	1	1	1	1	2

$i_3 = 5$

-1	0	0	0	0	1
0	1	1	1	1	2
0	1	1	1	1	2
0	1	1	1	1	2
0	1	1	1	1	2
1	2	2	2	2	3

Figure 2.22: Coefficients in the wirelength expression of each cell in a $6 \times 6 \times 6$ arrangement.

Notice that the Herringbone arrangement and its complement do exactly that.

2.3.2 Creating the Assignment

The authors of [19] use induction and case-by-case analysis to show the optimality of their assignment. Generalizing the statement to arbitrary dimensions follows the same argument.

Notice that since the coefficient with which a number contributes to the wirelength depends solely on the number of 0s and $n_i - 1$s in its coordinates, the arrangement has recursive structure. The wirelength for a k-dimensional arrangement is made up of n $(k-1)$-dimensional arrangements and the wirelength in the fixed coordinate. For example, in 3 dimensions, for a 6×6 arrangement shown in Figure 2.22, the wirelength for the entire arrangement is the sum of the 2-dimensional arrangement wirelengths for each value of $i_3 = 0, 1, 2, 3, 4, 5$ plus the sum of the differences of the values in cells $(i_1, i_2, 5)$ and $(i_1, i_2, 0)$. Thus we can use induction on the number of dimensions and assume that all $(k-1)$-dimensional submatrices $(*, ..., *, i_j, *, ..., *)$ are optimal $(k-1)$-dimensional arrangements of the appropriate numbers in those submatrices.

Given such strong structure from $k-1$ dimensions we get the complementarity of the arrangement as a consequence. A is a *complimentary* arrangement if

$$A(i_1, ..., i_k) + A(n_1 - 1 - i_1, n_2 - 1 - i_2, ..., n_k - 1 - i_k) = \prod_{i=1}^{k} n_i - 1$$

More generally, for any set of ascending numbers $x_0, ..., x_{\prod_{i=1}^{k} n_i - 1}$, the indices obey the equality

$$\text{The index of } A(i_1, ..., i_k) + \text{ the index of } A(n_1 - 1 - i_1, n_2 - 1 - i_2, ..., n_k - 1 - i_k) = \prod_{i=1}^{k} n_i - 1$$

This is a center-point symmetry: the arrangement is symmetric with respect to the point with the center coordinates.

Square Grids

Let $n_i = n$ for all i. We show by induction on k, the number of dimensions, the structure of the optimal arrangement. The base case for $k = 2$ has been shown in [19]. For any k and $n = 2$ Harper [25] gave an optimal arrangement with the wirelength of $2^{k-1}(2^k - 1)$.

Define $P_t(s)$ as a $(t+s) \times t^{k-1}$ matrix whose first t^k section is a Herringbone arrangement and then s initial t^{k-1} corners of $(k-1)$–dimensional submatrices.

Let $R_t(n)$ be $P_t(n-2t)$ with a Herringbone arrangement of the next numbers $(n-t)t^{k-1}$ through $nt^{k-1}-1$ in order, starting in the corner of the matrix and maintaining monotonicity. A corner of the matrix is a cell with all the coordinates being either 0 or $n_i - 1$. Let the corner have the coordinates $c_j = \{0, n_j - 1\}^k$, then the smallest number in the Herringbone addition of $R_t(n)$, $(n-t)t^{k-1}$, is in the cell with each coordinate $i_j = \max\{0, c_j - t\}$, and the largest number, $nt^{k-1}-1$, is in the cell with all the coordinates $i_j = \max\{t-1, c_j\}$.

Lemma 5 *Suppose $n \geq r \geq 1$ and assume without loss of generality that if the arrangement has a t^k Herringbone arrangement in its coordinate-wise smallest corner, with $1 \leq t < r$, then $A(t, 0, 0, ..., 0) = t^k$ (due to monotonicity, t^k must be in one of the cells of the form $(0, ..., 0, t, 0, ..., 0)$). Then every optimal arrangement includes a $P_t(r-2t)$ for some $t = 2, 3, ..., \lfloor r/2 \rfloor$.*

Proof. We prove the lemma by induction on r. The first valid value of r is $r = 3$, since $1 \leq t \leq \lfloor r/2 \rfloor$. In this case $t = 1$ and $P_t(r-2t)$ is the two cells $(0, ..., 0)$ and $(1, 0, ..., 0)$ with 0 and 1 in them, correspondingly. Similarly, for $r = 4$.

Assume that the conclusion of Lemma 5 holds for arbitrary $r \geq 4$. As we consider cases for $r+1$, we repeatedly use the switching technique of [19]: "If the cell that should hold c is currently occupied by p and $p > c$, increase each of the entries $c, c+1, ..., p-1$ by 1 and enter c in the position previously occupied by p. This preserves the monotonicity of the arrangement and produces an arrangement with smaller wirelength." We abbreviate this statement by "replace p by c" and show that it holds when it is not obvious. The induction hypothesis assumes that an optimal arrangement for $n \geq r$ includes $P_t(r-2t)$ for some $1 \leq t \leq \lfloor r/2 \rfloor$. Suppose that $n \geq r+1$, and we will show that the arrangement also includes $P_t(r+1-2t)$ for some $\leq t \leq \lfloor (r+1)/2 \rfloor$. Similar to [19] we consider three different cases of the relative values of r and t and prove that in each case the arrangement includes $P_t(r+1-2t)$:

$$r - 2t = 0 \;\Rightarrow\; \text{the arrangement includes } P_t(1)$$
$$r - 2t = 1 \;\Rightarrow\; \text{the arrangement includes } P_{t+1}(0) \text{ or } P_t(2)$$
$$r - 2t \geq 2 \;\Rightarrow\; \text{the arrangement includes } P_t(r+1-2t)$$

- $\mathbf{r - 2t = 0}$. If we have a t^k Herringbone arrangement then by monotonicity, t^k must be adjacent to it in one of the cells $(0, ..., 0, t, 0, ..., 0)$, without loss of generality, $A(t, 0, ..., 0) = t^k$. By the induction hypothesis on the lower dimension k the arrangement that minimizes the wire-length for the $(k-1)$-dimensional slat $(t, *, *, ..., *)$ includes a t^{k-1} Herringbone arrangement in its coordinate-wise smallest corner. Therefore the slat adjacent to the initial t^k Herringbone arrangement is a t^{k-1} Herringbone arrangement of the next t^{k-1} numbers $t^k, ..., t^k + t^{k-1} - 1$. This is by definition $P_t(1)$.

- $\mathbf{r - 2t = 1}$. We have a t^k Herringbone arrangement with one t^{k-1} Herringbone arrangement slat adjacent to it. By monotonicity, the element $t^k + t^{k-1}$ must be adjacent to the whole construction either in a cell $(t+1, 0, ..., 0)$ or a cell of the type $(0, ..., 0, t, 0, ..., 0)$ (with 0 in the first coordinate).

  If $t^k + t^{k-1}$ is in a cell of the type $(0, ..., 0, t, 0, ..., 0)$ then since we just want to maximize the values of the slat that appear in cells with at least one 0 among the coordinates, we must use the $(k-1)$-dimensional Herringbone arrangement. Therefore the slat is a $(t+1) \times t^{k-2}$ Herringbone arrangement. We can continue the argument in this way and we either finish the $(t+1)$st layer of the the $(t+1)^k$ Herringbone arrangement, which will give us $P_{t+1}(0)$, or a first element in some slat will be adjacent to the slat just finished. If this happens for the first slat, then we have the case of $t^k + t^{k-1}$ in the $(t+1, 0, ..., 0)$ cell, and then by the surface values maximizing argument the whole slat must be a t^{k-1} Herringbone arrangement, which produces $P_t(2)$ by definition. The question is, can we have part of the t^{k+1} Herringbone arrangement and then instead of continuing to the next coordinate recursively, just have the next element adjacent to the slat we just finished. Suppose yes, and we have $(t+1)^i t^{k-i}$

for some $i \geq 2$ and the element $(t+1)^i t^{k-i}$ is in a cell $(0, ..., 0, t+1, 0, ...0)$ where $t+1$ is the ith coordinate and the entire submatrix $(\underbrace{\leq t, ..., \leq t}_{i-1}, t+1, \leq t-1, ..., \leq t-1)$ is a $(k-1)$-dimensional Herringbone arrangement (which it has to be by the induction hypothesis on k. We can replace the submatrix $(\underbrace{\leq t, ..., \leq t}_{i}, t, \leq t-1, ..., \leq t-1)$, that is the next slat in the k-dimensional Herringbone arrangement, with this slat and the next $(t+1)^{i-2} t^{k-i}$ consecutive elements sufficient to complete the proper Herringbone slat. We have not violated the monotonicity. We have decreased the wirelength since among the initial $(t+1)^{i-1} t^{k-i}$ elements, if they where on the surface before the replacement, they will stay on the surface, and among the next $(t+1)^{i-2} t^{k-i}$ elements, fewer of them will be on the surface after the switch (by the property of the Herringbone arrangement). Thus we must have either $P_{t+1}(0)$ or $P_t(2)$.

- $\mathbf{r - 2t \geq 2}$. By monotonicity, the element $t^{k-1} r$ is adjacent to the arrangement so far. There are two cases: either it is in the cell $(r, 0, ..., 0)$ or it is in a cell of the type $(0, ..., 0, t, 0, ..., 0)$. If $A(r, 0, ..., 0) = t^{k-1} r$, then the whole slat $(r, \leq t-1, ..., \leq t-1)$ is a $(k-1)$-dimensional Herringbone arrangement and by definition this is $P_t(r - 2t + 1)$.

  If $A(0, ..., 0, t, 0, ..., 0) = t^{k-1} r$ then the whole slat $(\leq r-1, \leq t-1, ..., \leq t-1, t, \leq t-1, ..., \leq t-1)$ is the initial segment of an optimal n^{k-1} $(k-1)$-dimensional arrangement with numbers $t^{k-1} r, ..., t^{k-1} r + n^{k-1} - 1$. Notice that this partial arrangement has the same dimensions and the same entries as a partial Herringbone arrangement of $(t+1)^2 t^{k-i}$ and the element $(t+1)^2 t^{k-i}$ in cell $(0, t+1, ...0)$, which has better contribution to wirelength. However, we have shown in the previous case that even this partial arrangement is suboptimal. Therefore, if the optimal arrangement includes $P_t(r - 2t)$ with $r \geq 2t + 2$, it includes $P_t(r - 2t + 1)$.

□

The inductive hypothesis on the lower dimension $k-1$, the monotonicity constraints, and the complimentary symmetry overall and within each $k-1$ dimensional submatrix fix the k-dimensional structure of the arrangement given the initial $P_t(n-t)$ segment:

- The first nt^{k-1} numbers in an $R_t(n)$ segment in coordinates $(< n, < t, ..., < t)$.

- The next $n(n-2t)t^{k-2}$ numbers in $(k-1)$-dimensional recursively built slats in coordinates $(<n, t \text{ to } n-t-1, <t, ..., <t)$.

- The next nt^{k-1} numbers in coordinates $(<n, n-1 \text{ to } n-1, <t, ..., <t)$ in a $R_t(n)$ like structure: each of the corners is a Herringbone arrangement of t^k numbers with consecutive slats in between.

- We continue inductively. Once we have the first $n^i t^{k-i}$ numbers, for $0 \leq i \leq k$, in coordinates $(\underbrace{<n, ..., <n}_{i}, <t, ..., <t)$, we add $n-2t$ consecutive recursive $(k-1)$-dimensional $n^i t^{k-i-1}$ slats in coordinates $(\underbrace{<n, ..., <n}_{i}, t \text{ to } n-t-1, <t, ..., <t)$. Then complete the submatrix $(\underbrace{<n, ..., <n}_{i}, n-t \text{ to } n-1, <t, ..., <t)$ with an $R_t(n)$ like structure of two Herringbone arrangements in the corners with consecutive slats between them. This completes the arrangement of the first $n^{i+1}t^{k-i-1}$ numbers in coordinates $(\underbrace{<n, ..., <n}_{i+1}, <t, ..., <t)$.

The schematic representation of a 3-dimensional arrangement produced this way is shown in Figure 2.21.

It is obvious, by the inductive hypothesis on the dimension and the recursive structure of the arrangement, that the optimal t for k dimensions is the same as the optimal t for 2 dimensions, which is the value that maximizes

$$2t^3 - 6nt^2 + (3n^2 + 3n - 2)t$$

Figure 2.23 shows an example of the optimal arrangement for a $6 \times 6 \times 6$ grid ($t = 2$) with the wirelength of 6930.

2.4 Properties of the Herringbone Arrangement

In this section we focus on the Herringbone arrangement. We have used this construction for the bandwidth of Hamming graphs in Section 2.1 and wirelength of grid graphs in Section 2.3. However, Herringbone arrangement is an interesting combinatorial object in its own right. In this section we investigate its properties and look at the applications.

$i_3 = 0$

0	1	8	12	16	17
2	3	9	13	18	19
24	25	28	30	32	33
36	37	40	42	44	45
48	49	56	60	64	65
50	51	57	61	66	67

$i_3 = 3$

108	109	112	114	116	117
110	111	113	115	118	119
120	121	122	123	124	125
126	127	128	129	130	131
132	133	136	138	140	141
134	135	137	139	142	143

$i_3 = 1$

4	5	10	14	20	21
6	7	11	15	22	23
26	27	29	31	34	35
38	39	41	43	46	47
52	53	58	62	68	69
54	55	59	63	70	71

$i_3 = 4$

144	145	152	156	160	161
146	147	153	157	162	163
168	169	172	174	176	177
180	181	184	186	188	189
192	193	200	204	208	209
194	195	201	205	210	211

$i_3 = 2$

72	73	76	78	80	81
74	75	77	79	82	83
84	85	86	87	88	89
90	91	92	93	94	95
96	97	100	102	104	105
98	99	101	103	106	107

$i_3 = 5$

148	149	154	158	164	165
150	151	155	159	166	167
170	171	173	175	178	179
182	183	185	187	190	191
196	197	202	206	212	213
198	199	203	207	214	215

Figure 2.23: Optimal $6 \times 6 \times 6$ arrangement.

Recall that a square Herringbone arrangement is defined as follows.

$$HB_k : \mathcal{N} \rightarrow I^k$$

Let $t = \lceil \sqrt[k]{m} \rceil$. For $(t-1)^{k-j+1}t^{j-1} < m \leq (t-1)^{k-j}t^j$ for some $1 \leq j \leq k$

$$HB_k(m) = (i_1, ..., i_j, ..., i_k),$$

$$\text{where } i_j = t \text{ and } (i_1, ..., i_{j-1}, i_{j+1}, ..., i_k) = HB_{k-1}(m - (t-1)^k).$$

That is,

$$HB_k(m) = \begin{cases} (t, HB_{k-1}(m-(t-1)^k), & \text{if } (t-1)^k < m \leq (t-1)^{k-1}t \\ (\Box, t, \Box, ..., \Box), & \text{if } (t-1)^{k-1}t < m \leq (t-1)^{k-2}t^2 \\ \cdots & \\ (\Box, ..., \Box, \Box, t), & \text{if } (t-1)t^{k-1} < m \leq t^k \end{cases}$$

where the coordinates indicated by boxes form $HB_{k-1}(m-(t-1)^k)$.

For example, when $k = 2$, the two-dimensional Herringbone Arrangement is defined as

$$HB_2 : \mathcal{N} \rightarrow I \times I$$

$$HB_2(m) = \begin{cases} (\lceil\sqrt{m}\rceil, m - (\lceil\sqrt{m}\rceil - 1)^2), & \text{if } m \leq (\lceil\sqrt{m}\rceil - 1)\lceil\sqrt{m}\rceil \\ (m - (\lceil\sqrt{m}\rceil - 1)\lceil\sqrt{m}\rceil, \lceil\sqrt{m}\rceil), & \text{otherwise} \end{cases}$$

Figure 2.24 shows two and three-dimensional Herringbone arrangements.

Definition 3 *Given some diagonal step path from* $(0,0)$ *to* $(i_1 - 1, i_2 - 1)$*, a* Skew-Herringbone arrangement SHB *of the* i_1 *by* i_2 *rectangle is defined as following:*

1. *Divide the rectangle into two halves along the path, including the path either into the upper or the lower half.*

2. *Divide the upper half into vertical* slats *– a piece of a line.*

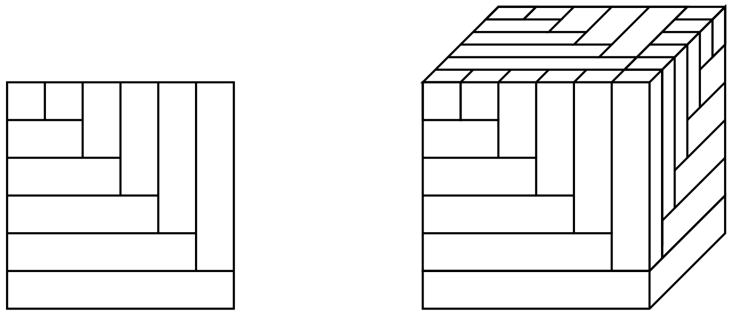

Figure 2.24: An example of herringbone arrangements in 2 and 3-dimensional matrices.

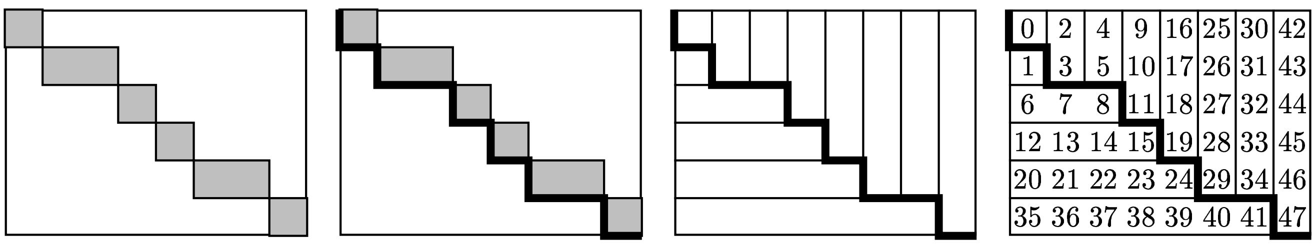

Figure 2.25: An example of a Skew-Herringbone arrangement.

3. *Divide the lower half into horizontal slats.*

4. $SHB(0,0) = 0$.

5. *Given a partial Skew-Herringbone arrangement, consider the two unfilled slats (vertical and horizontal) adjacent to it. Fill the slat that does not extend beyond the existing arrangement with the next consecutive ascending set of available numbers.*

Figure 2.25 shows the process of creating a 6 by 8 Skew-Herringbone arrangement.

Given this definition of a combinatorial object we can explore its properties.

2.4.1 Properties of the Herringbone Arrangement

- **Spiral.** Consider a two-dimensional spiral on an infinite discrete grid. It is made up of one-dimensional slats, each the size of the maximum of the two sides of the space filled so far. Figure 2.26 shows this spiral.

  Now consider the same spiral limited to half-space. It is made up of the same one-dimensional slats. Rather than continuing around the space filled so far, the spiral now wraps around half

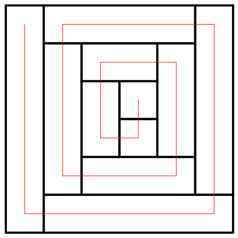

Figure 2.26: Full two-dimensional discrete spiral.

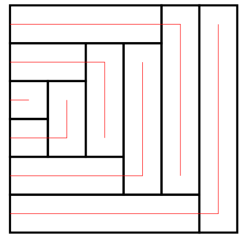

Figure 2.27: Two-dimensional discrete spiral limited to half-space

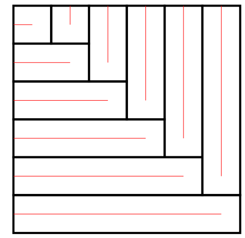

Figure 2.28: Two-dimensional discrete spiral limited to quarter-space

of it each time, alternating between the upper and lower halves. Figure 2.27 shows this construction.

We can limit the spiral even farther to quarter space. Now the same one-dimensional slats wrap around quarter of the space filled so far, that is one side of it, alternating between the two limiting lines. Figure 2.28 shows this limited spiral.

Notice, that the discrete spiral limited to quarter-space is nothing more than the two-dimensional Herringbone arrangement. However, there are several generalizations possible to this idea of a limited discrete spiral since there are several parameters involved. The quarter-space two-dimensional spiral is made by wrapping one-dimensional slats in two dimensions. In k dimensions we consider a "spiral" limited to $1/2^k$th orthant of the space. If the slats making up the k-dimensional spiral are $(k-1)$–dimensional then the resulting object is precisely k-dimensional Herringbone arrangement. However, one can consider the generalization of the two-dimensional spiral made by wrapping one-dimensional slats in k-dimensions. The object created by this process is a discrete ball in k dimensions limited to a k-dimensional hypercube. That is, all points of distance d from the first within the limiting orthant. In general, for any $1 \le l \le k-1$ we can consider wrapping l–dimensional slats in k dimensions. The resulting objects are cartesian products of l–dimensional hypercubes of Herringbone arrangements and $(k-l)$–dimensional hyperballs.

- **Monotonic.** As defined earlier, an arrangement is *monotonic* if the values in any line do not descend with the increase of the changing coordinate. That is, an arrangement A is monotonic if for all $0 \le i_t, j_t \le n_t - 1,\ 1 \le s \le k$

$$((t \neq s \rightarrow i_t = j_t) \wedge i_s < j_s) \rightarrow A(i_1, ..., i_s, ..., i_k) \le A(j_1, ..., j_s, ..., j_k).$$

By construction, Herringbone arrangements are monotonic.

- **Nested.** We will say that an arrangement is *nested* if for $l < m$ the k-dimensional arrangement of l numbers is the k-dimensional arrangement of m numbers, limited to the first l. Obviously, Herringbone arrangements are nested since they are constructed in an iterative way.

- **Compact.** For any arrangement of m numbers limited to a k-dimensional orthant we can define the *surface area* as the number of cells that have neighbors not in the arrangement (that is, the arrangement does not assign any value to those cells).

  Herringbone arrangement of m numbers in an k-dimensional orthant is the smallest surface area arrangement, since at any point it fills the next cell with the maximum number of neighbors in the arrangement, or, to put it the other way, the minimum number of neighbors outside the arrangement.

- **Recursive.** By definition, a Herringbone arrangement is recursive. But there is an even stronger recursive property, which is that for $l \le k$ any l-dimensional slice (a complete l-dimensional submatrix) of a Herringbone arrangement is an l-dimensional Herringbone arrangement of the numbers that are in the slice.

These various properties of the Herringbone arrangement make it a useful combinatorial structure for different problems.

2.4.2 Applications of the Herringbone Arrangement

Multichannel communication. While the optimal solution for the Hamming Graph bandwidth, which is a graph theoretical model we use for one of the settings of the multichannel communication problem, does not use the Herringbone arrangement, some of the properties of the arrangement are still useful.

As we have shown in Section 2.2, monotonic arrangements do not increase the maximum spread of an arrangement. Fishburn, Tetali, and Winkler [19] have shown that monotonic arrangements do not increase the average spread of an arrangement. This holds true also for an arrangement of any set of numbers, not necessarily distinct, which is an extension of the graph labeling problem.

Consider an infinite matrix and an arrangement of an infinite set of numbers in it. What do the arrangements that minimize *partial* maximum and average spread look like? In the limit, going from discrete to continuous equivalent of the problem. Maximum spread becomes maximum diameter and the average spread is equivalent to the surface area to volume ratio. The continuous object that minimizes both the maximum diameter and the surface area to volume ratio in two dimension was known to the ancient Greeks. It is, of course, the circle. In higher dimensions a ball is the natural and correct generalization[1]. So what is the correct discrete analogue of the ball for this problem? We would like to have a spiral object that minimizes the surface area.

From the above, Herringbone arrangements seem to be good candidates if we want to minimize some combination of the maximum and average spread simultaneously. This is often the case in multichannel communication, where we want to bound both the maximum and average error.

Another property of the Herringbone arrangement that is particularly useful in communication setting is its recursive structure. Both maximum and average spread of an arrangement are defined over the spreads in a line, which is equivalent to one channel failing. However, we need to design an encoding that minimizes the error also in case of two or more channels failing. Vaishampayan [50] has stated that there is no encoding that is optimal both for one and two channels failing. That is, if an encoding minimizes the error for one channel failing then it cannot minimize the error for two channels failing and vice versa. Thus, if we want to have an encoding that gives a relatively small error for any number of channels failing, we must agree to non-optimality. The recursive property of Herringbone arrangements ensures that there is graceful degradation as the number of the failed channels goes up.

Thus, overall, the monotonic, compact, spiral, and recursive properties of Herringbone arrange-

[1]Problems of this sort are known as *isoperimetric problems*. For more on the subject see [43]

$i_3 = 0$

-1	0	0	1
-2	-1	-1	0
-2	-1	-1	0
-3	-2	-2	-1

$i_3 = 1$

0	1	1	2
-1	0	0	1
-1	0	0	1
-2	-1	-1	0

$i_3 = 2$

0	1	1	2
-1	0	0	1
-1	0	0	1
-2	-1	-1	0

$i_3 = 3$

1	2	2	3
0	1	1	2
0	1	1	2
-1	0	0	1

Figure 2.29: Coefficients in the wirelength expression of each cell in a $4 \times 4 \times 4$ arrangement.

ments make it a good encoding scheme for channel communication problem.

Grid graph wirelength. As we said, Fishburn, Tetali, and Winkler [19] have shown that monotonic labelings do not increase the wirelength of a two-dimensional grid graph. The statement easily extends to arbitrary dimensions. Wirelength of a graph is the sum over all the edges of the difference of the labels on the edge ends. For a grid graph in two dimensions this becomes a telescoping sum with just the surface values left. In general, in k dimensions, only the values in cells of the form $(..., 0, ...)$ or $(..., n_i - 1, ...)$ appear in the wirelength expression. The coefficient with which any value appears in the expression is (# of $n_i - 1$'s minus # of 0s in its coordinate). Figure 2.29 shows the coefficients for each cell in a $4 \times 4 \times 4$ arrangement. Thus to minimize the wirelength, we want to maximize the minimum surface values and minimize the maximum surface values. A Herringbone arrangement is monotonic and minimizes the surface area, which means it maximizes the values on the surface. Thus a combination of a Herringbone and a reverse Herringbone arrangements is a good candidate for minimizing grid graph wirelength and is, indeed, used in two dimensions. Figure 2.30 shows the two-dimensional arrangement and the proposed extension of it to higher dimensions.

Edge isoperimetric set. For a simple connected graph an *edge isoperimetric set* is a set of vertices that has either the minimum or the maximum number of either interior edges (both endpoints are set vertices) or boundary edges (one endpoint is a set vertex). A survey of the current state of edge isoperimetric problems on graphs has been done by Bezrukov [9]. Generally, explicit construction of edge isoperimetric sets for a given graph is possible if the problem has the *nested solutions property.* That is, there is a sequence of isoperimetric sets $A_i \subseteq V$ with $|A_i = i$ for $1 \leq i \leq |V|$ such that

$$A_1 \subset A_2 \subset \cdots \subset A_{|V|}.$$

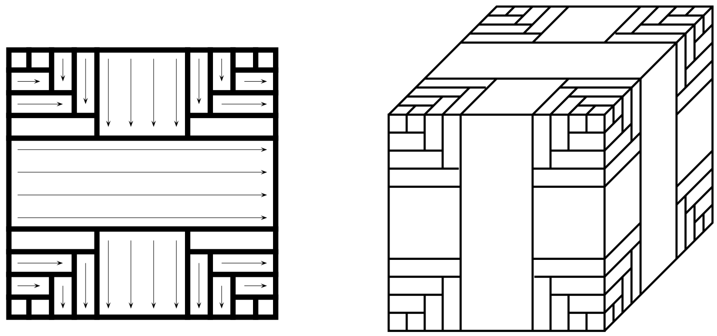

Figure 2.30: A two-dimensional labeling for grid graph wirelength by Fishburn, Tetali, and Winkler and a proposed generalization to higher dimensions.

For regular graphs, obviously, a set that minimizes or maximizes the number of internal edges also maximizes or minimizes the number of boundary edges.

Harary and Harboth [23] showed that for an infinite two-dimensional gird graph the maximum number of internal vertices for any set of m vertices is

$$\lfloor 2(m - \sqrt{m}) \rfloor$$

and the sets themselves grow spirally as m grows. The spiral limited to one orthant of the grid is still an edge isoperimetric set and is exactly the a two-dimensional Herringbone arrangement. Since Herringbone arrangements minimize surface area, that is the number of boundary edges, and an infinite grid is a regular graph, Herringbone arrangements maximize the number of the interior edges and thus are edge isoperimetric sets for infinite grids in any dimension. The maximum number of interior edges, I, for any set of m vertices in an infinite grid in k dimensions is, therefore,

$$
\begin{aligned}
I &= \frac{\text{degree sum over the vertices} - \text{boundary edges}}{2} \\
&= \frac{2km - \text{boundary edges}}{2} \\
&= km - \left\{ \begin{array}{ll}
k\lceil \sqrt[k]{m} \rceil - (k-1) & \text{if } (\lceil \sqrt[k]{m} \rceil - 1)^k < m \leq (\lceil \sqrt[k]{m} \rceil - 1)^{k-1} \lceil \sqrt[k]{m} \rceil \\
k\lceil \sqrt[k]{m} \rceil - (k-2) & \text{if } (\lceil \sqrt[k]{m} \rceil - 1)^{k-1} \lceil \sqrt[k]{m} \rceil < m \leq (\lceil \sqrt[k]{m} \rceil - 1)^{k-2} \lceil \sqrt[k]{m} \rceil^2 \\
\ldots & \\
k\lceil \sqrt[k]{m} \rceil - 1 & \text{if } (\lceil \sqrt[k]{m} \rceil - 1)^2 \lceil \sqrt[k]{m} \rceil^{k-2} < m \leq (\lceil \sqrt[k]{m} \rceil - 1) \lceil \sqrt[k]{m} \rceil^{k-1} \\
k\lceil \sqrt[k]{m} \rceil & \text{if } (\lceil \sqrt[k]{m} \rceil - 1) \lceil \sqrt[k]{m} \rceil^{k-1} < m \leq \lceil \sqrt[k]{m} \rceil^k
\end{array} \right\} \\
&= \lfloor k(m - \sqrt[k]{m}) \rfloor
\end{aligned}
$$

In this section we have defined the Herringbone arrangement and have looked at its various properties and applications. We believe that it merits further research and is a good construction to consider in various graph labeling and isoperimetric problems.

Chapter 3

Conclusions, Extensions, and Open Problems

The channel coding problem of designing an encoding scheme to minimize the error of sending a number in a system of multiple unreliable channels is closely related the the problem of minimizing bandwidth and wirelength of cartesian products of graphs. In fact, the formulation of the graph problems by Harper [25], [26] in 1960's was originally motivated by the channel coding problem. Harper also provided a fundamental proof technique called isoperimetric problem. Most of the progress in bandwidth and wirelength of cartesian product of Hamming and grid graphs in the last 35 years has been based on this technique. However, this technique turned out to be not strong enough to obtain results for bandwidth of cartesian product of any number of cliques of arbitrary size, or optimal results for wirelength of general grid graphs.

For the Hamming graph, we propose two different approaches. One is based on creating labelings that bound the smallest and labeling that bound the largest numbers in each cliques. The difference between those bounds is the lower bound on the bandwidth. If one labeling exists that bounds both numbers simultaneously, it must be an optimal labeling, which is the case for 2-dimensional Hamming graphs. This is a generic way to provide lower bounds for the bandwidth of Hamming graphs and their induced subgraphs. The method is also constructive, thus giving an immediate upper bound. However, this approach does not give an optimal labeling. We propose a different technique taking advantage of monotonic arrangements that gives a much better lower bound and nearly matching and, we believe, optimal upper bound for the problem of Hamming graph bandwidth. As we have mentioned, this problem is equivalent to the channel communication problem of creating encodings that minimize maximum single channel failure error in case of complete information loss failure, with no redundancy in the system.

The labeling that bounds the minima or the maxima for each clique we call Herringbone arrangement. It appears in several places in the research of graph isoperimetric problems. However, its pattern has not been recognized, and thus the arrangement has not been generalized to dimensions higher than 2 (that is, cartesian products of 2 graphs). One of such cases is the wirelength problem of a grid graph. Fishburn, Tetali, and Winkler [19] solve the problem for the 2-dimensional grid. Using Herringbone arrangement, we propose a generalization of this result to higher dimensions. In channel communication setting, this problem is equivalent to creating encodings that minimize average single channel failure error in case of distance-1 failure, with no redundancy in the system.

Until now, only bandwidth of an entire graph has been considered. In this work we introduce a new graph isoperimetric problem: given a graph G, find an induced subgraph of order m with the smallest bandwidth/wirelength. We investigate this problem for the bandwidth of Hamming graph. A surprising negative result is that a diagonal subgraph does *not* have the smallest bandwidth. We presented the best known subgraph structure and its labeling. Considering subgraphs instead of entire graphs corresponds to allowing redundancy in the channel communication problem.

3.1 Maximum Error of Complete Channel Failure with Redundancy - Hamming Subgraph Bandwidth

In this section we return to the the problem of designing encodings that minimize the maximum error in case of the complete information loss failure. As we have mentioned before, when we assume no redundancy in the system and only single channel failure, the problem is equivalent to the bandwidth minimization problem of a Hamming graph. In this section we focus on this problem *with* redundancy allowed in the encoding. In the graph domain, this problem is equivalent to finding bandwidth optimal induced subgraphs of a Hamming graph. Both formulations are open. We present some results and outline the challenges.

First we state the problem formally. Given k channels with channel i having capacity $\log n_i$ and loosing all the transmitted information in case of failure, the goal is to design encoding to

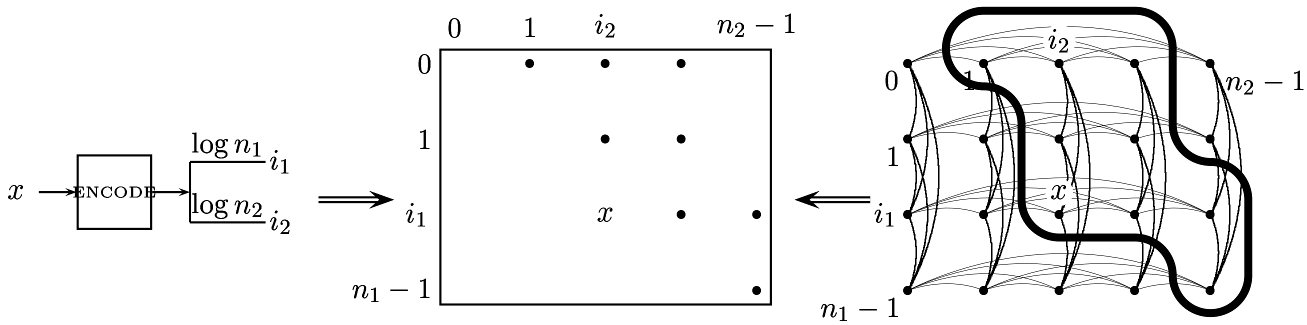

Figure 3.1: Correspondence between the channel communication with two channels with redundancy, the bandwidth minimization of induced subgraphs of a product of two cliques, and the incomplete number arrangements in a two-dimensional matrix problems.

transmit numbers 0 through $m-1$ to minimize the maximum error in case of failure. In graph theory, given a k-dimensional Hamming graph $K_{n_1} \times \cdots \times K_{n_k}$ the problem is to find minimum bandwidth induced subgraphs of order m for $1 \leq m \leq \prod_i n_i$. The problem can be viewed as two subproblems:

1. Find the *shape* or the *topology* of the minimal bandwidth induced subgraphs; and
2. find the minimal bandwidth *labeling* of a given subgraph.

Again, we visualize the problem as the one of filling a $n_1 \times n_2 \times \cdots \times n_k$ matrix with m numbers so that the maximum difference between any two numbers in any line – complete one-dimensional submatrix – is minimized. Figure 3.1 shows the correspondence between the three settings.

For $m = \prod_i n_i$ we presented a solution in Section 2.2. For small m the solution is obvious. For $m \leq \min_i\{n_i\}$ the induced subgraph is any maximal independent set and the bandwidth of this subgraph is 0 since no two vertices share an edge. (In the matrix representation, this is at most one number in any row or column, or, without loss of generality, a diagonal.) We now focus on $\min_i\{n_i\} < m \leq (k - \sum_{t=0}^{k-1} \binom{t}{\lfloor t/2 \rfloor}) \prod_i \lceil n_i/2 \rceil$. Vaishampayan [48], [47], [49] designed a solution for this case in the setting of the channel communication problem. The solutions is to arrange the numbers into a uniform diagonal. Below we examine this solution. However, to avoid the boundary effects of a finite matrix, we first consider the infinite diagonal. We consider an arrangement of numbers in a infinite k-dimensional diagonal of thickness l; that is, any line in the diagonal has exactly l elements in it. This case is also equivalent to deriving the achievable rate-distortion tuples for an unbounded discrete information source with k channels of rate l.

We derive the lower bound on the spread in this case using the same technique we employed in

Section 2.1 for the case of no redundancy complete failure. We use the herringbone arrangement to maximize the smalls sequence and to minimize the bigs sequence. Since the smalls and the bigs sequence arrangements can start at any point in the diagonal, we consider the difference between the sequences relative to the starting points. Recall that the lower bound on the spread is the maximum over all lines of the difference between smalls and the bigs sequences. However, in this case this difference turns out to be constant.

Definition 4

A *herringbone arrangement* of a k-dimensional infinite l-diagonal is defined recursively as follows. Assign an arbitrary order to the coordinates of the system $\langle i_1, i_2, ..., i_k \rangle$.
A herringbone arrangement of k-dimensional 0-diagonal is empty.
A herringbone arrangement of a 0-dimensional diagonal is any arrangement of one number in the cell.

Given a herringbone arrangement of the k-dimensional diagonal up to coordinate t_i in dimension i, we define a larger herringbone arrangement recursively:

- project the existing arrangement onto the $(k-1)$-dimensional hyperplanes $(*, ..., *, t_i, *, ..., *)$, limited to the diagonal,
- calculate the volume of each projection,
- recursively fill the largest volume projection (using coordinate order to break ties) with the herringbone arrangement for $k-1$ dimensions.

We denote the element in the cell $(i_1, ..., i_k)$ of the k-dimensional herringbone arrangement by $HB_k(i_1, ..., i_k)$. Examples of a herringbone arrangement of a diagonal are shown in Figure 3.2.

Lemma 6 *A herringbone arrangement of values in a k-dimensional diagonal maximizes the smalls sequence (the ascending list of the smallest numbers in a line) for that matrix.*

The proof is similar to the proof of Lemma 1.

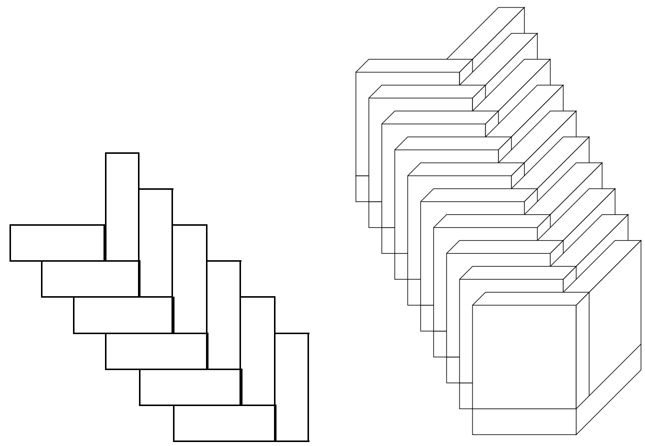

Figure 3.2: Uniform infinite diagonal herringbone arrangements in two and three dimensions.

Corollary 4 *If the smallest number in line* $(i_1, ..., *, ..., i_k)$ *of a herringbone arrangement is* a_j*, then the smallest number in line* $(i_1+1, ..., *, ..., i_k+1)$ *is*

$$a_j + \sum_{i=0}^{k-1} \lfloor l/2 \rfloor^i \lceil l/2 \rceil^{k-1-i} = \begin{cases} a_j + k(l/2)^{k-1}, & \textit{if } l \textit{ is even,} \\ \\ a_j + \frac{(l-1)^k+(l+1)^k}{2^k}, & \textit{if } l \textit{ is odd.} \end{cases}$$

Corollary 5 *If the largest number in line* $(i_1, ..., *, ..., i_k)$ *of the herringbone arrangement is* b_j*, then the largest number in line* $(i_1+1, ..., *, ..., i_k+1)$ *is*

$$b_j + \sum_{i=0}^{k-1} \lfloor l/2 \rfloor^i \lceil l/2 \rceil^{k-1-i} = \begin{cases} b_j + k(l/2)^{k-1}, & \textit{if } l \textit{ is even,} \\ \\ b_j + \frac{(l-1)^k+(l+1)^k}{2^k}, & \textit{if } l \textit{ is odd.} \end{cases}$$

Thus, combining the results of Corollary 4 and Corollary 5, the difference $b_j - a_j$ remains constant along any diagonal. That is, the difference between the largest and the smallest numbers in line $(i_1, i_2, ..., *, ..., i_k)$ equals that of line $(i_1+s, i_2+s, ..., *, ..., i_k+s)$ for some integer s. To see this, notice that while we start the smalls arrangement from 0 at some point, since the diagonal is infinite, we can continue the arrangement in the opposite direction using negative numbers.

Similarly with the bigs arrangement, we can continue the arrangement in the direction of the increasing of coordinates using larger numbers. Thus, the smalls and the bigs arrangements are the same arrangements, offset by a fixed value. These arrangements are also "facing" in opposite directions: the herringbone arrangement can be viewed as cones stacked into each other, and in case of the smalls sequence the "cones" face the direction of the coordinate decrease, while in the bigs sequence the "cones" face in the direction of the coordinate increase. However, the brims of these cones from both sequences coincide. Since that is where the smallest and the largest numbers in each line occur, the difference along any diagonal remains constant.

A consequence of the structure of the herringbone arrangement is the fact that the difference is maximized over the central diagonal. That is, if $(t, t, ..., t)$ is a cell in the center of the diagonal, then the maximum difference is achieved for any line $(t+s, t+s, ..., *, ..., t+s)$, where s is some integer, and this difference equals

$$HB(t + \lfloor l/2 \rfloor, t, ..., t) - HB(t - \lceil l/2 \rceil, t, ..., t) =$$

$$(\lceil l/2 \rceil - 1) \sum_{i=0}^{k-1} \lfloor l/2 \rfloor^i \lceil l/2 \rceil^{k-1-i} + \lceil l/2 \rceil^{k-1}.$$

We now restrict the diagonal to a finite matrix and investigate how the boundary effects change the spread.

Consider a diagonal arrangement limited to the n^k cube. This arrangement is a restriction of an infinite diagonal, therefore it has the same bound on the spread in the central part of the diagonal. However, the boundary parts of the arrangement are complete cubes of size $\lfloor l/2 \rfloor^k$, so the spread in this case is the spread of completely-filled cube derived in Section 2.1. By comparing the boundary cubic and the central diagonal spreads, we show that the spread in the diagonal part is always greater. Thus, the spread is dominated by the diagonal part, as long as a true diagonal part exists. That is, if $l \leq n$, then there is at least one complete line in each dimension which belongs entirely to the diagonal, and the spread in this line dominates the overall spread in the matrix. But this means that we can increase the size of the initial cubic part and decrease the width of the diagonal part, thus decreasing the overall spread. Moreover, we can decrease the spread even more by introducing non-overlapping cubic parts along the diagonal. By balancing the entire structure,

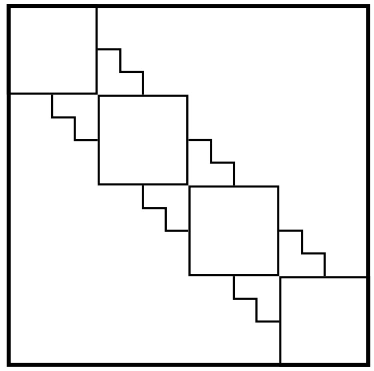

Figure 3.3: Possibly an optimal arrangement for for an incompletely filled cube.

we can make all of the cubic parts smaller, further reducing the overall spread. This construction is demonstrated in Figure 3.3. The spread in this arrangement is less than the uniform diagonal. However, currently we do not know whether this arrangement is optimal.

Now, if $l > n$, then the initial cubic parts of the diagonal overlap. Remembering from Section 2.1, the maximum spread of the entirely filled cube is the spread in line $(\lceil (n-1)/2 \rceil, ..., *, ..., \lfloor (n-1)/2 \rfloor)$, a line in the intersection of the first-orthant cube and the last-orthant cube. Those are precisely the cubic parts of the diagonally-filled cube. The spread of a cube optimally filled with $m < n^k$ numbers is at most the spread of the cube filled with n^k numbers; thus, for an l-diagonal with $l > n$, the spread is the the same as in a completely-filled cube.

The above arrangement gives the best known upper bound on the distortion in a multiple description system if redundancy is allowed and gives an upper bound on the bandwidth of induced subgraphs of order m of a Hamming graph. However, the question of we whether it is optimal remains open.

3.2 Open Problems

There are many different related open problems, in very different areas of research. We have mentioned a few of them in previous chapters. Following are some other possible directions of extending the results.

- **Different channel failure types - Product of different graphs.**

  Currently graph bandwidth is known for products of paths, cycles, paths and cycles, paths and cliques, and now cliques. Only bounds are known for products of graphs of other types. Wirelength is know for products of paths, cycles, and cliques. However, it seems that at least for some graphs with relatively regular structure, such as complete bipartite graphs, regular trees, path degrees, the techniques presented in this work may apply. This is especially important in the context of multichannel communications since different graphs correspond to different types of failure.

- **Different type of product of graphs.**

  The graph product used in this work is Cartesian product. There are two other types of graph products: tensor and strong. Bandwidth is known for both products of paths or cycles with cliques, products of paths, products of cycles, and paths with cycles. Bandwidth and wirelength of tensor product of paths with bicliques is also known. Again, it seems that the approach presented in this work may be useful in tackling tensor and strong products of other graphs.

- **Probability of failure - Weighted graphs.**

  We have not taken into consideration the probability of channel failure. One possible way to address it is to assign weights to the graphs corresponding to channels which would give weights to edges of that graph. We may then consider weighted bandwidth or wirelength.

- **Probability distribution for input - Labeling with repetition.**

  Throughout the work we have assumed uniform probability distribution over the input. However, in practice, this is not always the case. A possible way to take this into account is to duplicate the more frequent input proportionally to its probability.

- **Dynamic (on-line) encoding algorithm/heuristic.**

  In a dynamic setting the number of channels, their capacities, the types of failure, and the probability of failure can change, thus the encoding has to be done online. There is a tradeoff between the channel capacity and the error size. A possible heuristic could choose and encoding for the available channels slightly below their capacity and break up the information further into smaller pieces or recombine it into larger pieces if the number of channels and their capacities change. The herringbone approach seems particularly useful in this context due to its strong recursive structure.

- **Error correction codes.**

  The specific data encoding technique investigated in this thesis encodes information into several pieces in a way that allows graceful degradation in case of the corruption or loss of some of the pieces. That is, the error is monotonically decreasing function of the number of uncorrupted pieces. A classical way to counter the effects of unreliable media is to use error correcting codes. While they do not provide graceful degradation, error correcting codes guarantee zero error up to a certain number of corrupted bits. It would be interesting to explore a way to combine the two techniques that would result in better error correction overall. One way, for example, is to use error correction up to a fixed number of errors and use the Multiple Description approach if more errors result. An interesting question is to find the tradeoff between the higher redundancy of using codes with greater error correcting radius versus the lower error that would achieve.

- **Geometric data decomposition.**

  Another application of our approach is in geometric data decomposition. Currently, any such decomposition uses hierarchical approach, breaking the data into levels, each subsequent level being a refinement of the previous one. This is a fast and compact way to transmit the information if the order of the pieces can be guaranteed. However, since each layer relies on the previous one, if even one layer is lost or delayed, the information in all the following layers cannot be reconstructed. The question is, given a mesh, how to break it up into (possibly overlapping) pieces, so that each piece is a coarse representation of the data, and the more pieces are combined the better representation results, with all the pieces together giving the full data. We want this iterative quality improvement to be independent of which specific pieces are combined, rather just how many of them there are. This problem is trivial for infinite regular meshes. The obvious solutions is to decompose the original mesh into non-overlapping pieces by taking the shifted subsets of points and connecting each point to its nearest neighbors. When two meshes are combined, the edges are recomputed for the new set of points. The first, although not a difficult, problem is the boundary. However, even if the boundary problem is solved, this approach may loose important features in some layers and exaggerate them in others if the mesh is not regular. One possible approach is to use the existing mesh coarsening techniques and combine several coarse meshes obtained in this way.

- **Data storage.**

  To address the rising demand of high availability data storage systems, an approach similar to that of bandwidth-constrained communication has been proposed by Kirkpatrick, Wilcke, Garner, and Huels and Morris [32, 38]. The suggestion is to break the storage medium into many autonomous pieces, "bricks", assembled in a cube. The stored information is spread and replicated in many pieces. This creates a highly reliable storage device based on unreliable components. This is strongly reminiscent of transmitting data reliably in unreliable pieces. Here the main goal is to find a mapping of a number to a location in the array that minimizes a given function over the mapping. In the data storage problem, the mapping has to be created of the particular combination of the information in each brick so that all the data is accessible even if some of the bricks fail. The in [32] some of the heuristics of data replication location under the model of percolation have been studied. The other two aspects, spreading and combining of data, have not been investigated deeply. It would be interesting to approach this problem from the combinatorial point of view. First, we need to find the number of copies that should be created, given the failure rate of the bricks, the data to storage space ratio, and the overhead on mirroring the data. Currently, this number assumed to be two or three [32]. Next, it would be interesting to investigate arrangements of numbers in a 3-dimensional array (or labeling of lattice vertices) where the numbers can be repeated, several numbers can be mapped to the same location, and numbers can be erased. We believe there are a lot of questions about these arrangements that can be answered either experimentally or theoretically.

  - How should the data be spread? In how many pieces? What modification of Rabin's information dispersal algorithm [44] is appropriate in this case?
  - What are the effects of the data connectivity graph on these arrangements? Is it better to combine related data or not? The preliminary answer to these questions can be obtained experimentally. If it seems preferable to preserve connectivity of data in storing (for access speed or integrity), then a distance [37] or volume [16] preserving embedding of a

graph in a normed space can be used to achieve it. If the preference is not to preserve the connectivity, then the same approach can be used on a complement graph. Of course, a random heuristic is always an option if there is no preference.

  - Is it better to choose the mirror locations for the data regularly or randomly? Again, preliminary results could be obtained experimentally. However, the work in [32] points out that some lattice structures seem to appear even if the failure pattern is random. It would be interesting to investigate further what substructures appear as a result of failures and the effect of those on mirror location of data.

- **Bandwidth/wirelength preserving permutations.**

  We showed that sorting values within each line does not increase bandwidth of Hamming graphs or wirelength of grid graphs. It is obvious that switching lines within the same coordinate does not change either quantity. An interesting question then is, what other operations preserve or do not increase/decrease bandwidth or wirelength of a graph? This question also comes up some areas of coding theory, when designing families of codes with a given code distance.

- **Monotonicity of bandwidth.**

  Recently Fishburn, Tanenbaum, and Trenk [18] addressed the problem of *linear discrepancy*, that is labelings of the vertices of a poset that are monotone (consistent with the poset) and minimize the maximum difference between incomparable pairs. It turns out that the optimal labelings are the one that minimize the bandwidth of the complimentary graph. Thus for the bandwidth of the complimentary graph, to find the optimal labeling, it is sufficient to consider only monotone assignments. There seems to be a possibility of generalization of the idea of monotone or monotonic assignments that do not increase the bandwidth of a labeling.

These are the directions in which the research in this thesis can lead.

Appendix A

Definitions

A.1 Graph Theory

- *Graph* $G(V, E)$ is defined by a set V of vertices and a set of E of vertex pairs. For the purpose of current work, no *loops* (edges of type (v, v)) or multiple edges are allowed. We also assume the graphs to be undirected (that is, the edges have no direction) unless otherwise mentioned.

- *Subgraph* of a graph $G(V, E)$ is a graph $H(V', E')$ such that $V' \subseteq V$, $E' \subseteq E'$.
  Induced Subgraph of a graph $G(V, E)$ is a subgraph of G, $H(V', E')$ such that $V' \subseteq V$ and for any $u, v \in V'$ if $(u, v) \in E$ then $(u, v) \in E'$.

- *Clique* K_n is a complete graph on n vertices, that is for any $u, v \in V \ (u, v) \in E$.

- *Path* P_n is a graph $G(\{v_1, v_2, ..., v_n\}, \{(v_i, v_{i+1})\})$.

- *Cartesian Product of Graphs* $G \times H$ is the graph defined as follows:

$$V(G \times H) := V(G) \times V(H)$$

$$E(G \times H) := \{\{(x, y), (x', y')\} : (x = x' \wedge (y, y') \in E(H)) \vee (y = y' \wedge (x, x') \in E(G))\}.$$

- *Hamming Graph* is the cartesian product of two or more cliques.

- *Grid Graph* is the cartesian product of two or more paths.

- *Hypercube* is the cartesian product of two or more two cliques.

- *Vertex labeling* of a graph $G(V, E)$ is a one-to-one function

$$f : V(G) \to \{0, ..., |V(G)| - 1\}.$$

- *Bandwidth* $B(G)$ of a graph $G(V, E)$ is

$$\min_{\text{labeling } f} \max_{(u,v) \in E(G)} |f(u) - f(v)|.$$

- *Wirelength* $L(G)$ of a graph $G(V, E)$ is

$$\min_{\text{labeling } f} \sum_{(u,v) \in E(G)} |f(u) - f(v)|.$$

A.1.1 Isoperimetric Problems

- *Edge isoperimetric problems* are, given a graph $G(V, E)$, find

$$\min_{A \subseteq V, |A|=m} (\# \text{ boundary edges of } A),$$

$$\max_{A \subseteq V, |A|=m} (\# \text{ interior edges of } A).$$

- *Vertex isoperimetric problems* are, given a graph $G(V, E)$, find

$$\min_{A \subseteq V, |A|=m} (\# \text{ boundary vertices of } A),$$

$$\max_{A \subseteq V, |A|=m} (\# \text{ interior vertices of } A),$$

- *Isoperimetric set* is a subset of vertices that achieves the extremal value in an isoperimetric problem.

- *Nested Solution Structure* is a characteristic of an isoperimetric problem whose solutions are isoperimetric subsets $A_i \subseteq V(G)$, $|A_i| = i$, s.t.

$$A_1 \subset A_2 \subset \cdots \subset A_{|V(G)|}.$$

A.2 Coding Theory

- *Source coding* is the part of coding theory that studies information measurement and its conversion into different forms. [2]

- *Channel coding* is the part of coding theory that studies the amount of information a channel can transfer without error. [2]

- *Vector Quantizer* is a code that, given an input vector $X \in \mathcal{R}^t$, maps it into a finite set of channel vectors $U \in \mathcal{Z}^k$.

  Scalar Quantizer is a Vector Quantizer with $t = 1$. [22]

A.3 Number Arrangements

- *Arrangement* is the inverse of encoding, that is a function from the cells of the matrix to the numbers to be put in those cells:

$$A : I \to \{1, ..., m\},$$

  where I is a subset of the product of the sets of indices $I_1 \times \cdots \times I_k$.

- *Slice* is a full submatrix. An l-dimensional slice is a subset of all cells with $k - l$ coordinates fixed:

$$(*, ..., *, i_{j_1}, *, ..., *, i_{j_2}, *, ..., *, i_{j_{k-l}}, *, ..., *).$$

- *Line* is a one-dimensional slice:

$$(i_1, i_2, ..., *, ..., i_k).$$

- *Spread* is the difference between the largest and the smallest number in a slice.

- *Maximum spread* of an arrangement, $spread(A)$, is the maximum over all the spreads in slices of the same fixed dimension.

- *Smalls* is a plural form of "the smallest number", a set of the smallest numbers in a set of slices.

- *Bigs* is similar to smalls, a plural form of "the largest number".

A.4 Miscellaneous

- *Floor* of an integer x, $\lfloor x \rfloor$, is the largest integer not greater than x.

- *Ceiling* of an integer x, $\lceil x \rceil$, is the smallest integer not less than x.

- *Whole part* of an integer x, $[x]$, is the integer closest to x.

References

[1] Ahlswede, R., "The rate-distortion region for multiple descriptions without excess rate", *IEEE Transactions on Information Theory*, **31** (1985), 721–726.

[2] Anderson, J. B. and S. Mohan, *Source and Channel Coding: An Algorithmic Approach*, Kluwer Academic Publishers, 1991.

[3] Batllo, J- C. and V. A. Vaishampayan, "Multiple description transform codes with an application to packetized speech", *IEEE International Symposium on Information Theory - Proceedings*, 1994, IEEE, Piscataway, NJ, USA.

[4] Bennet, W. R., "Spectra of quantized signals", *Bell Systems Technical Journal*, **27** (1948), 446–472.

[5] Berlekamp, E. R., R. J. McEliece and H.C.A. van Tilborg, "On the in the inherent intractability if certain coding problems", *IEEE Transactions on Information Theory*, **24** (1978), 384–386.

[6] Berger, T. and Z. Zhang, "Minimum breakdown degradation in binary source encoding", *IEEE Transactions on Information Theory*, **29** (1983), 807–814.

[7] Berger-Wolf, T. Y and E. M. Reingold, "Optimal multichannel communication under failure", *ACM-SIAM Symposium on Discrete Algorithms - Proceedings*, 1999, Baltimore, MD, S858–S859.

[8] Berger-Wolf, T. Y. and E. M. Reingold. "Index assignment for multichannel communication under failure." *IEEE Transactions on Information Theory, to appear*

[9] Bezrukov, S. L., "Edge isoperimetric problem on graphs", *Graph Theory and Combinatorial Biology*, Bolyai Society Mathematical Studies **7**, L. Lovasz, A. Gyarfas, G.O.H. Katona, A. Recski, L. Szekely eds., Budapest 1999, 157–197.

[10] Bollobas, B. and I. Leader, "Compressions and isoperimetric inequalities", *Journal of Combinatorial Theory Series A*, **56** (1991), 47–62.

[11] Bresenham, J., "Algorithm for computer control of digital plotter", IBM Systems Journal **4** (1965), 25–30.

[12] Buzi, L., V. A. Vaishampayan and R. Laroia, "Design and asymptotic performance of a structured multiple description vector quantizer", *IEEE International Symposium on Information Theory - Proceedings*, 1994, IEEE, Piscataway, NJ, USA.

[13] Chinn, P. Z., J. Chvátalová, A. K. Dewdney, and N. E. Gibbs, "The bandwidth problem for graphs and matrices - a survey", *Journal of Graph Theory*, **6** (1982), 223–254.

[14] Chvátalová, J., "Optimal labeling of product of two paths", *Discrete Mathematics*, **11** (1975), 249–253.

[15] El Gamal, A. A. and T. M. Cover, "Achievable rates for multiple descriptions",*IEEE Transactions on Information Theory*, **28** (1982), 851–857.

[16] Feige, U., "Approximating the bandwidth via volume respecting embedding", *13th Annual ACM Symposium on Theory of Computing-Proceedings*, 1998, ACM, Dallas, TX, USA.

[17] Fishburn, P. and P. Wright, "Bandwidth edge counts for linear arrangements of rectangular grids", *Journal of Graph Theory*, **26** (1997), 195–202.

[18] Fishburn, P., P. Tanenbaum and A. Trenk, "Linear discrepancy and bandwidth", *preprint*

[19] Fishburn, P., P. Tetali and P. Winkler, "Optimal linear arrangement of a rectangular grid", *Discrete Math*, to appear.

[20] Gale, D. and R. Karp, "A phenomenon in the Theory of Sorting", *Journal of Computer and System Sciences* **6**, 103-115 (1972)

[21] Graham, R. L., D. E. Knuth and O. Patashnik, *Concrete Mathematics: A Foundation For Computer Science*, 2nd ed., Addison-Wesley, 1994.

[22] Gray, R. M., *Source Coding Theory*, Kluwer Academic Publishers, 1990.

[23] Harary, F and H. Harboth, "Extremal animals", *Journal of Combinatorics, Information and System Sciences*, **1** (1976), 1–8.

[24] Hardy, G. H. and E. M. Wright, *An Introduction to the Theory of the Numbers*, 5th ed., Oxford Science Publications, 1995.

[25] Harper, L. H., "Optimal assignment of numbers to vertices", *Journal of SIAM*, **12** (1964), 131–135.

[26] Harper, L. H., "Optimal numberings and isoperimetric problems on graphs", *Journal of Combinatorial Theory*, **1** (1966), 385–393.

[27] Harper, L. H., "On an isoperimetric problem for Hamming graphs", *Discrete Applied Mathematics*, **95** (1999), 285–309.

[28] Hendrich, U. and M. Stiebitz, "On the bandwidth of graph products", *Journal of Information Processing and Cybernetics*, **EIK 28** (1992), 113–125.

[29] Jayant, N. S., "Subsampling of a DPCM speech channel to provide two self-contained half-rate channels", *The Bell System Technical Journal*, **60** (1981), 501–501.

[30] Jayant, N. S. and S. W. Christensen, "Effect of packet losses in waveform coded speech and improvements due to odd-even sample interpolation procedure", *IEEE Transactions on Communications*, **29** (1981), 101–109.

[31] Karpinski, M., J. Wirtgen and A. Zelikovsky, "An approximation algorithm for the bandwidth problem on dense graphs", *ECCC TR97-017*.

[32] Kirkpatrick, S., W. Wilcke, R. Garner, and H. Huels. "Percolation in dense storage arrays", *Physica A, to appear*, 2002.

[33] Kloks, T., D. Kratsch and H. Muller, "Approximating the bandwidth for asteroidal triple-free graphs", *Algorithms-ESA '95, Paul Spirakis (Ed.), Lecture Notes in computer Science 979*, 434–447.

[34] Lai, Y. and K. Williams, "A survey of solved problems and applications on bandwidth, edge-sum, and profile of graphs", *Journal of Graph Theory*, **31** (1999), 75–94.

[35] Lawler, E., *Combinatorial Optimization Networks and Matroids*,

[36] Lindsey, J. H., "Assignment of numbers to vertices", *American Mathematics Monthly*, **7** (1964), 508–516.

[37] Linial. N., E. London, and Y. Rabinovich. "The geometry of graphs and some of its algorithmic applications", *Combinatorica* **15** (2), 1995, 215–245.

[38] Morris, R., "Storage: From Atoms to People", *Keynote address at Conference on File and Storage Technologies (FAST)*, 2002, Monterey, CA

[39] Moghadam, H. S., "Compression operators and a solutions to the bandwidth problem of the product of n paths", *Ph. D. Thesis*, University of California, Riverside, 1983.

[40] Nakano, K., "Linear layouts of generalized hypercubes", *Proceedings of WG'93, Lecture Notes in Computer Science*, **790**, Springer Verlag, 1994, 34–375.

[41] Ozarow, L., "On a source coding problem with two channels and three receivers", *The Bell System Technical Journal*, **59** (1980), 1909–1921.

[42] Papadimitriou, C. H., "The NP-completeness of the bandwidth minimization problem", *Computing*, **16** (1976), 263–270.

[43] Pólya, G. and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics. Annals of Mathematical Studies*, **27**, Princeton University Press, 1951.

[44] Rabin, M. "Efficient dispersal of information for security, load balancing, and fault tolerance", *Journal of the ACM* **36** (2), 1989, 335–348.

[45] Shannon, C. E., “A mathematical theory of communication”, *Bell Systems Technical Journal*, **27** (1948), 379–423, 623–656.

[46] Shannon, C. E., “Coding theorems for discrete source with fidelity criterion”, *IRE National Convention Record*, **4** (1959), 142–163.

[47] Vaishampayan, V. A., “Vector quantizer design for diversity systems”, in *Proceedings of 25 Annual Conference on Inform. Sci. Syst.*, Johns Hopkins University, Mar. 20–22, 1991, 564–569.

[48] Vaishampayan, V. A.,“Design of multiple description scalar quantizers,” *IEEE Transactions on Information Theory,* **93** (1993), 821–834.

[49] Vaishampayan, V. A. and J. Domaszewicz, “Design of entropy-constrained multiple-description scalar quantizers”, *IEEE Transactions on Information Theory*, **40** (1994), 245–250.

[50] Vaishampayan, V. A., personal communication, 2001.

[51] West, D. B., *Introduction to Graph Theory*, Prentice Hall, 1996.

[52] West, D. B., personal communication.

[53] Witsenhausen, H. S. and A. D. Wyner, “Source coding for multiple descriptions II: A binary source”, *The Bell System Technical Journal*, **60** (1981), 2281–2292.

[54] Wolf, J. K., A. D. Wyner and J. Ziv, “Source coding for multiple descriptions”, *The Bell System Technical Journal*, **59** (1980), 1417–1426.

[55] Yang, S.- M. and V. A. Vaishampayan, “Low-delay communication for Rayleigh fading channels: an application of the multiple description quantizer”, *IEEE Transactions on Communications*, **43** (1995), 2771–2783.

[56] Zhang, Z. and T. Berger, “New results in binary multiple description”, *IEEE Transactions on Information Theory*, **33** (1987), 502–521.

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Yonit Berger-Wolf (also known as Tanya Berger-Wolf) was born in 1972 in Vilnius, Lithuania. She graduated from the Specialized Physics and Mathematics High School attached to the Leningrad State University in 1989. She received her B.Sc. in mathematics and computer science (double major) from the Hebrew University in Jerusalem, Israel, in 1995.

Yonit (Tanya) has held various jobs, including an instructor of extracurricular mathematics courses for middle school gifted children (1992, Hebrew University Youth Science Center) and Geographic Information Systems programmer (1993-1995, various places)

Yonit has started her PhD at the University of Illinois at Urbana-Champaign in 1996. At the University of Illinois, she was an NSF Graduate Fellow from 1998 to 2001. In the summer of 1999, she was a research intern at the Sandia National Laboratory. Throughout her graduate studies Yonit was a teaching assistant in the computer science department at the University of Illinois, and in 2001 she was an instructor.

During her graduate studies, Yonit has been involved in numerous volunteer and outreach activities, including science summer camp for middle school girls and grade school science fairs. She has co-founded and was the first president of the Women in Computer Science organization at the University of Illinois.

Yonit has received the C. W. Gear Outstanding Graduate Student Award in 2000 and the departmental Outstanding Graduate Student Service Award in 2001. She was awarded the Mavis Memorial Fund Scholarship from the College of Engineering in 2000, based on academic performance, interest in engineering education, and published research. Her research interests include design and analysis of algorithms with emphasis on combinatorial optimization, approximation, and related complexity theory. Recently, Yonit has become interested in the emerging field of computational ecology.