Stochastic Petri Net
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To study a formal model (personal view)

- Definition (and maybe history)
- Brief family tree: the branches and extensions
  - Advantages and disadvantages for each
  - Applications for each
- Relation to other models
Questions

- How came Stochastic PN (SPN)?
- SPN and Markov Chain (MC)?
- Application of SPN?
- Conclusion
History (in the 1980s or earlier)

- Performance Evaluation (PE) area
  - Modeling in the design phase

- Modeling requirement
  - integration of formal description, proof of correctness, and performance evaluation

Petri Net + Markov Chain = Stochastic Petri Net (SPN) //Markov-SPN
Petri Net (PN)

- A Petri Net is a directed bipartite graph \( N = \langle P, T, F \rangle \)
  - Elements: \( P \) for places, \( T \) for transitions; \( P \) and \( T \) are disjoint
  - Flow relations: \( F \subset (P \times T) \cup (T \times P) \) for arcs

- Execution
  - Token \( \rightarrow \) Transitions fire
  - Marking \( M \): a mapping which assigns tokens to each place
  - Reachability Graph (RG): illustrating marking transformation
Introduce temporal specifications in PN

- Concerns:
  - Associate timing with the PN elements
    - Places, or transitions
  - The semantics of the firing in the case of timed transitions
    - Atomic firing, or firing in three phases
  - The nature of the temporal specification
    - Deterministic, or probabilistic
Stochastic Petri Net (SPN)

- Transitions fire after a probabilistic delay
- Atomic firing

Formally, a SPN is a five-tuple \( <P, T, F, M_0, \Lambda> \):

- \( <P, T, F, M_0> \) is a PN with initial marking \( M_0 \).
- \( \Lambda \) = is the array of firing rates \( \lambda \)'s associated with the transitions in \( T \); each \( \lambda \) is a random variable, or a function \( \lambda(M) \) of current marking.
Different SPNs

- Different firing time probability density functions (pdf)

1) constant: \( X \sim \text{Const}(c), c \geq 0 \iff \Pr\{X \leq \theta\} = 0 \)
   if \( \theta < c \), 1 if \( \theta \geq c \).

2) geometric: \( X \sim \text{Geom}(p, \sigma), 0 < p \leq 1, \sigma \geq 0 \iff \Pr\{X \leq \theta\} = 1 - (1 - p)^{[\theta]} \),
   where \( \sigma \) is the length of the unit step. The constant distribution is a special case: \( \text{Const}(c) \) is equivalent to \( \text{Geom}(1, c) \).

3) discrete: \( X \sim \text{Disc} \iff \) the distribution function of \( X \) is obtained as a weighted sum of a (finite or countably infinite) number of constant distributions. The geometric distribution is a special case. It is possible to approximate any distribution arbitrarily well by using either a sufficiently large number of polynomials of small degree (e.g., constants, as for the discrete distributions) or by using a single polynomial of sufficiently large degree.

4) exponential: \( X \sim \text{Expo}(\lambda), \lambda > 0 \iff \Pr\{X \leq \theta\} = 1 - e^{-\lambda \theta} \). This distribution approaches \( \text{Const}(0) \) as \( \lambda \) increases.

5) uniform: \( X \sim \text{Unif}(a, b), b > a \geq 0 \iff \Pr\{X \leq \theta\} = 0 \) if \( \theta < a, (\theta - a)/(b - a) \) if \( a \leq \theta \leq b \), and 1 if \( \theta \geq b \). This distribution approaches \( \text{Const}(b) \) as \( a \) approaches \( b \).

6) polynomial: \( X \sim \text{Poly} \iff \) the distribution function of \( X \) is piecewise defined by polynomials in \( \theta \) (expressions of the form \( \sum_{i=0}^{n} a_i \theta^i, a_i \in \mathbb{R} \) and has finite support \([\theta_{\text{min}}, \theta_{\text{max}}]\). The finite discrete and uniform distributions are special cases. It is possible to approximate any distribution arbitrarily well by using either a sufficiently large number of polynomials of small degree (e.g., constants, as for the discrete distributions) or by using a single polynomial of sufficiently large degree.

7) expolynomial: \( X \sim \text{Expoly} \iff \) the distribution function of \( X \) is piecewise defined by expopolynomials in \( \theta \) (expressions of the form \( \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} \theta^i e^{-\lambda_i \theta}, a_{ij} \in \mathbb{R}, \lambda_{ij} \in [0, +\infty) \)). The polynomial and exponential distributions are special cases.
Markov SPN

- CTMC-SPN: negative exponential pdfs $e^{-\lambda_i \tau}$
- $\subseteq$ Markov SPN
- Assumption: CTMC (Continuous-time MC)
- Mature steady-state analysis for PE solution
The reachability graph (RG) of Markov SPN can be mapped directly to a Markov process.
Markov SPN - analysis

- Disadvantages:
  - Complexity
  - Structural analysis of underlying PN
  - Compute other performance measures

∴ extensions of SPNs introduced
Generalized SPN (GSPN) ⊂ SPN

- Allow “immediate” transitions (no firing delay)
  - Priority of firing: immediate transitions > timed transitions
  - Weight or Probabilities of immediate transitions
    - To determine the firing probability in case of conflicting immediate transitions.

- Advantage: better structural analysis of underlying PN
A small branch of the SPN family tree

CTMC-SPN
- Negative exponential pdfs

GSPN
- Immediate transitions allowed
- Constant timing = 0

DSPN
- Exponentially distributed and constant timing
Modern Application of SPN

- Stochastic Petri Net Identification for the Fault Detection and Isolation of Discrete Event System (DES) - 2011

- NSPN: SPN with normal and exponential transitions

- Methodology
  - Learn reference model from output sequences
  - Use reference model for fault detection and isolation
  - Consider NSPNs to represent faulty behaviors
Conclusion

A lot of ways to extend PN with time phase, SPN is one of them...

A lot of ways to define SPNs w.r.t. the firing delay, Markov SPN is one of them...

“Largeness problem”; active in DES study
Main Reference


Thank you!
Questions?
Continuous-time Markov Chain (CTMC)

A stochastic/random process \( \{X(t), t \in T\} \)
+ Markovian property (memoryless)
\[ \text{= Markov Process} \]
+ discrete state space
\[ \text{= Markov Chain (MC)} \]
+ continuous time parameter \( t \)
\[ \text{= Continuous-time MC (CTMC)} \]
A stochastic process \( \{X(t), t \in T\} \) is a family of random variables defined over the same probability space, taking values in the state space \( S \), and indexed by the parameter \( t \), which assumes values in the set \( T \); normally \( T = (0,\infty) \).
Markov Models

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<thead>
<tr>
<th>Markov Models</th>
<th>Do we have control over the state transitions?</th>
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<td></td>
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<td></td>
<td>YES</td>
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<tr>
<th>Markov Models</th>
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<tbody>
<tr>
<td>Markov Chain</td>
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<td>MDP</td>
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<tr>
<td>Hidden Markov Model</td>
<td>NO</td>
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<td>Partially Observable</td>
<td>NO</td>
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CTMC