Exercises in

Elementary Topics in Differential Geometry by J. A. Thorpe

1.10 $\text{graph}(f) = \{(x_1 \cdots x_{n+1}) \in R^{n+1} : (x_1 \cdots x_n) \in U,\ x_{n+1} = f(x_1 \cdots x_n)\}$
Then $\text{graph}(f)$ is a level set for $F(x_1 \cdots x_{n+1}) = 0$, where $F(x_1 \cdots x_{n+1}) = f(x_1 \cdots x_n) - x_{n+1}$

2.4 As integral curve, $\dot\alpha(t) = X(\alpha(t))$. If it crosses itself, then there exists t_1, t_2, s.t. $\alpha(t_1) = \alpha(t_2)$, $\dot\alpha(t_1) \neq \dot\alpha(t_2)$. But that isn't allowed.

2.7 (a) complete (b) incomplete say $p = (-1, 0)$ (c) complete
(d) $x_1 = tg(t+c)$, so $t \neq -c + \frac{\pi}{2}$. incomplete

2.8 Define $\tilde\beta(t) = \beta(t+t_0)$, $t \in \{x - t_0 \mid x \in \tilde I\}$, then $\tilde\beta(0) = p$, $\dot{\tilde\beta}(t) = \dot\beta(t+t_0) = X(\beta(t+t_0)) = X(\tilde\beta(t))$ $(t \in \tilde I - \underset{\text{offset}}{t_0})$
So $\tilde\beta(t)$ is an integral curve of X with $\tilde\beta(0) = p$. Since $\alpha(t)$ is the maximal of such curves, So for $\forall t \in \{x - t_0 \mid x \in \tilde I\}$. $\tilde\beta(t) = \alpha(t)$ i.e. $\beta(t) = \alpha(t - t_0)$ $\forall t \in \tilde I$

2.9 Define $\beta(t) \triangleq \alpha(t\ \cdot)$ $t \in I$, So $\beta(t_0) = \alpha(0)$. β is an integral curve of X on I
By Ex. 2.8 $\beta(t) = \alpha(t - t_0)$ i.e. $\alpha(t) = \alpha(t - t_0)$. i.e. α periodic.
(Don't worry about def. domain too much, only check & restrict in the last step)

~~2.10~~

2.10 (a) $\varphi_t(p) = p + (t, 0)$ translation. obviously one-to-one $R^2 \to R^2$
(b) $\varphi_0(p) = p + (0,0) = p$. $\varphi_{t_1+t_2}(p) = p + (t_1+t_2, 0) = (p + (t_2, 0)) + (t_1, 0)$
$\varphi_{-t}(p) = p + (-t, 0)$ $\varphi_t(\varphi_{-t}(p)) = p + (-t, 0) + (t, 0) = p$

2.11. (a) $\varphi_t(x_1, x_2) = (x_1 \cos t - x_2 \sin t,\ x_1 \sin t + x_2 \cos t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
rotation by t. one-to-one, additive group obviously
(b) $\varphi_t(x_1, x_2) = (x_1 e^t, x_2 e^t) = (x_1, x_2) \cdot e^t$ scaling bijection $e^{t_1+t_2} = e^{t_1} \cdot e^{t_2}$ so additive
(c). $\varphi_t(x_1, x_2) = \frac{1}{2} \begin{pmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\det = \frac{1}{2} \cdot 4$, invertible
$= \frac{1}{2}(e^t + e^{-t}) \begin{pmatrix} 1 & \tanh(t) \\ \tanh(t) & 1 \end{pmatrix}$ Use $\tanh(t_1 + t_2) = \dfrac{\tanh(t_1) + \tanh(t_2)}{1 - \tanh(t_1) \cdot \tanh(t_2)}$

2.12 Suppose $\beta(t)$ is the integral curve of X with $\beta(0) = \varphi_{t_2}(p)$. So $\alpha(0) = p$. $\alpha(t_2) = \beta(0)$

By using Ex. 2.8 (now the $\overset{\beta}{\alpha}$ here is the $\overset{\alpha}{\beta}$ is Ex. 2.8), $\beta(t) = \alpha(t + t_2)$
$\varphi_{t_1}(\varphi_{t_2}(p)) = \beta(t_1) = \alpha(t_1 + t_2) = \varphi_{t_1+t_2}(p)$, $\varphi_{-t_2}(\varphi_{t_2}(p)) = \beta(-t_2) = \alpha(0) = p$ $\forall t_2$