derive the equation (G) in terms of β_1. (G) itself guarantees β_1 is on S as shown by the proof in Thm of chapter 7, given that $\beta_1(t_0) = p \in S$, $\dot\beta_1(t_0) = \dot\beta_2(t_0) = v \in S_p$.

10.1 $\alpha = (x, y)$ $\dot\alpha = (x', y')$, $\ddot\alpha = (x'', y'')$ $N = (-y', x')$ (due to consistency).

So $k \circ \alpha = \ddot\alpha \cdot N \circ \alpha / \|\dot\alpha\|^2 = (-x''y' + y''x') / (x'^2 + y'^2)^{3/2}$

10.2 $f: x_2 - g(x_1)$, ~~$f' = (-g'(x_1), 1)$~~ $f^{-1}(0)$ can be viewed as $\alpha(t) = \begin{cases} t \\ g(t) \end{cases}$ $t \in I$

By Ex 10.1. curvature of C at point $(t, g(t)) = k \circ \alpha = g''(t) / [1 + (g'(t))^2]^{3/2}$

10.3 (a) $\nabla = (a, b)$ $X = (+b, -a)$ $\dot\alpha(t) = X(\alpha(t)) \Rightarrow$ $\alpha(t) = \begin{pmatrix} +bt + c_1 \\ -at + c_2 \end{pmatrix}$, ~~let $\alpha(0) = \frac{c}{2}\begin{pmatrix} 1/a \\ 1/b \end{pmatrix} \Rightarrow \alpha(t) = \begin{pmatrix} -bt + \frac{c}{2a} \\ at + \frac{c}{2b} \end{pmatrix}$~~

Since $(a, b) \neq (0, 0)$ let $a \neq 0$. let $\alpha(0) = \begin{pmatrix} c/a \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 = c/a \\ c_2 = 0 \end{pmatrix} \Rightarrow \alpha(t) = \begin{pmatrix} +bt + c/a \\ -at \end{pmatrix}$ $t \in R$

(b) $\nabla = \left(\frac{2x_1}{a^2}, \frac{2x_2}{b^2}\right)$ $X = \left(\frac{2x_2}{b^2}, \frac{-2x_1}{a^2}\right)$ $\dot\alpha(t) = X(\alpha(t))$, $\frac{1}{a^2}\alpha_1^2(t) + \frac{1}{b^2}\alpha_2^2(t) = 1$ $\Rightarrow \begin{cases} \alpha_1(t) = a \sin \frac{2}{ab} t \\ \alpha_2(t) = b \cos \frac{2}{ab} t \end{cases}$ $t \in R$

(c) $\nabla = (-2ax_1, 1)$, $X = (1, 2ax_1)$, $\dot\alpha(t) = X(\alpha(t)) \Rightarrow \begin{cases} \alpha_1(t) = t + c_1 \\ \alpha_2(t) = at^2 + 2ac_1t + c_2 \end{cases}$

$\alpha_2(t) - a(\alpha_1(t))^2 = c \Rightarrow c_2 = c + a c_1^2$. let $c_1 = 0$, $c_2 = c$. So $\begin{cases} \alpha_1(t) = t \\ \alpha_2(t) = at^2 + c \end{cases}$ $t \in R$

(d) $\nabla = (+2x_1, -2x_2)$ $X = (-2x_2, -2x_1)$ $\dot\alpha(t) = X(\alpha(t))$, $\alpha_1^2 - \alpha_2^2 = 1$ $\Rightarrow$ ~~$\alpha_1(t) = \cos t$, $\alpha_2(t) = \sin t$~~ ~~$t \in [0, 2\pi) \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}$~~

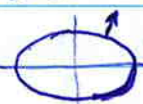
$\alpha_1(t) = \frac{c_1e^{2t} + c_2e^{-2t}}{2}$ ~~$\alpha_1(t) = \frac{1}{2}(e^{2t} + e^{-2t})$, $\alpha_2(t) = \frac{c_2}{2}(e^{-2t} - e^{2t})$~~

$\alpha_2(t) = \frac{c_2e^{-2t} - c_1e^{2t}}{2}$ $\forall$ ~~$c_1, c_2 \in ...$~~ $c_1c_2 = 1$ if ignore $x_1 > 0$

10.4 (a) $k = 0$ as $\ddot\alpha = 0$. (b) $\alpha = \begin{pmatrix} a \sin 2t/ab \\ b \cos 2t/ab \end{pmatrix}$, $\dot\alpha = \begin{pmatrix} 2/b \cos(2t/ab) \\ -2/a \sin(2t/ab) \end{pmatrix}$, $\ddot\alpha = \begin{pmatrix} -4/ab^2 \sin(2t/ab) \\ -4/a^2b \cos(2t/ab) \end{pmatrix}$

$N = \lambda \begin{pmatrix} 2/a \sin(2t/ab) \\ 2/b \cos(2t/ab) \end{pmatrix}$ ~~$= \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} b \sin(2t/ab) \\ a \cos(2t/ab) \end{pmatrix}$~~, $k(p) = \frac{\ddot\alpha \cdot N \circ \alpha}{\|\dot\alpha\|^2}$ ~~$= \frac{-4/ab}{4(a^2+b^2)/a^2b^2} = \frac{-ab}{a^2+b^2}$~~ ~~$\ddot\alpha \cdot N\alpha = \frac{-4}{ab\sqrt{a^2+b^2}}$~~

~~So $k(p) =$~~ $\|\dot\alpha\|^2 = \frac{4}{a^2b^2}\left(a^2\cos^2\frac{2t}{ab} + b^2\sin^2\frac{2t}{ab}\right)$ ~~So $k(p) =$~~

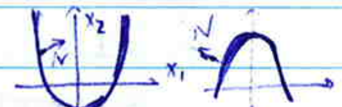


$\ddot\alpha \cdot N \circ \alpha = \frac{-4}{a^2b^2}\begin{pmatrix} a \sin\frac{2t}{ab} \\ b \cos\frac{2t}{ab} \end{pmatrix} \cdot \frac{2}{ab}\begin{pmatrix} b \sin\frac{2t}{ab} \\ a \cos\frac{2t}{ab} \end{pmatrix} \Big/ \frac{2}{ab}\sqrt{a^2\cos^2\frac{2t}{ab} + b^2\sin^2\frac{2t}{ab}}$

So $k(p) = \frac{\ddot\alpha \cdot N \circ \alpha}{\|\dot\alpha\|^2} = -ab\left(a^2\cos^2\frac{2t}{ab} + b^2\sin^2\frac{2t}{ab}\right)^{-3/2}$ If $a = b = r$. then $k(p) = -\frac{1}{r}$.

$= -ab\left(\frac{a^2}{b^2}x_2^2 + \frac{b^2}{a^2}x_1^2\right)^{-3/2}$

(c) Use Ex 10.2. ~~$k \circ \alpha =$~~ $g(t) = at^2 + c$, $g'(t) = 2at$. $g''(t) = 2a$

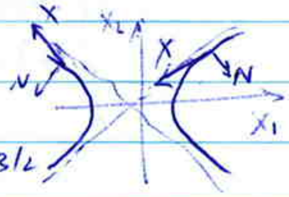


$k \circ \alpha = 2a / (1 + 4a^2t^2)^{3/2} = 2a / (1 + 4a^2x_1^2)^{3/2}$

(d) Use Ex 10.1 ~~$\alpha(t) = \begin{pmatrix} \cos t \\ tg t \end{pmatrix}$ $\dot\alpha(t) = \begin{pmatrix} \sin t / \cos^2 t \\ 1 / \cos^2 t \end{pmatrix}$ $\ddot\alpha(t) = \frac{1}{\cos^3 t}\begin{pmatrix} 1 + \sin^2 t \\ 2 \sin t \end{pmatrix}$~~

~~$k \circ \alpha = -\cos^3 t / (1 + \sin^2 t)^{3/2} = -(x_1^2 + x_2^2)^{3/2} \cdot sgn(x_1)$~~

In general for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $k = -ab / (a^2 tg^2 t + b^2 sec^2 t)^{3/2}$



$\alpha(t) = \frac{1}{2}(c_1e^{2t} + c_2e^{-2t}, c_2e^{-2t} - c_1e^{2t})^T$, $\dot\alpha(t) = (c_1e^{2t} - c_2e^{-2t}, -c_2e^{-2t} - c_1e^{2t})^T$

$\ddot\alpha(t) = 2(c_1e^{2t} + c_2e^{-2t}, c_2e^{-2t} - c_1e^{2t})$ So $k \circ \alpha = 8 / [2(e^{4t} + e^{-4t})]^{3/2}$

$k = 1 / (x_1^2 + x_2^2)^{3/2}$, So curve is always turning (according to X) towards N

10.5 $h(t_0) = (\alpha(t_0) - p)\cdot N(p) = (p-p)\cdot N(p) = 0$ $h'(t_0) = (\dot\alpha(t_0))\cdot N_p = 0$.

$h''(t_0) = \ddot\alpha(t_0)\cdot N(p) = k(p)$ because $\|\dot\alpha(t_0)\| = 1$

10.6 (a) As $\|\dot\alpha\| = \text{const}$ $\dot\alpha\cdot\ddot\alpha = 0$ But $\dot\alpha\cdot N\circ\alpha = 0$ and $\{v \mid v\cdot\dot\alpha = 0\}$ is one dimensional (as C is in 2D plane) So $\ddot\alpha = \lambda N\circ\alpha$, $\lambda = \ddot\alpha\cdot N\circ\alpha = k\circ\alpha$, So $\dot T = \ddot\alpha = (k\circ\alpha)\cdot(N\circ\alpha)$

(b) $\|N\| = 1$. So $(N\circ\alpha)\cdot(\dot N\circ\alpha) = 0$. But $(N\circ\alpha)\cdot\dot\alpha = 0$. and we are in 2-D plane

so $\dot N\circ\alpha = \lambda\dot\alpha$ $\lambda = \dot N\circ\alpha\cdot\dot\alpha$ Besides, as $\dot\alpha\cdot N\circ\alpha = 0$ we have

$\ddot\alpha\cdot N\circ\alpha + \dot\alpha\cdot\dot N\circ\alpha = 0$ So $\lambda = -\ddot\alpha\cdot N\circ\alpha = -k\circ\alpha$.

Thus. $\dot N\circ\alpha = -(k\circ\alpha)\cdot\dot\alpha = -(k\circ\alpha)\cdot T$.

10.7 (a) $\|\dot\alpha\| = 1 \Rightarrow \dot\alpha\ddot\alpha = 0 \Rightarrow T\perp N$. $B\perp N$ and $B\perp T$ are by definition of B (cross product)

(b) $\dot T = \ddot\alpha \,/\!/\, N(t) = \ddot\alpha/\|\ddot\alpha\|$ So $\dot T = \|\ddot\alpha\|\cdot N$ so $k \triangleq \|\ddot\alpha\|$

$\dot B = \dot T\times N + T\times\dot N = T\times\dot N$ So $\dot B\perp T$. $\dot B\perp\dot N$ But we know $N\perp T$

and $\|N\| = 1 \Rightarrow \dot N\perp N$. As we are in 3-D space $\dot B = -\tau\cdot N$ wher $\tau: I\to R$

$\tau(t) = -\dot B(t)\cdot N(t)$ so $\tau(t)$ is smooth.

$\dot N\perp N$. We know $B\perp N$, $T\perp N$ and $B\perp T$. So there exist $\lambda_1, \lambda_2 : I\to R$

$\dot N = \lambda_1 B + \lambda_2 T$ $\lambda_1 = \dot N\cdot B = -N\cdot\dot B = \tau$ (Since $N\cdot B = 0 \Rightarrow \dot N B + N\dot B = 0$)

$\lambda_2 = \dot N\cdot T = -N\cdot\dot T = -k$ (since $N\cdot T = 0 \Rightarrow \dot N\cdot T + N\dot T = 0$)

So $\dot N = \tau B - kT$.

10.8 By definition of circle of curvature, $C_p = O_p$. $\dot\alpha(0)\in C_p$ $\dot\beta(0)\in O_p$ C_p and O_p are one dimensional, $\|\dot\alpha(0)\| = \|\dot\beta(0)\| = 1$ So $\dot\alpha(0) = \pm\dot\beta(0)$. Furthermore $\dot\alpha(0), \dot\beta(0)$ are both consistent with $N(p)$ and $N_1(p)$ resp. ($N(p)$ and $N_1(p)$ are orientation norms of C and O). But $N_1(p) = N(p)$

Thus $\dot\alpha(0) = \dot\beta(0)$

As $\ddot\alpha\cdot N(p) = -\nabla_{\dot\alpha(0)}N\cdot\dot\alpha(0)$ by Thm 1 of chapter 9

$\ddot\beta\cdot N_1(p) = -\nabla_{\dot\beta(0)}N_1\cdot\dot\beta(0)$

But $\dot\alpha(0) = \dot\beta(0)$ and by definition of circle of curvature, $\nabla_{\dot\alpha(0)}N = \nabla_{\dot\beta(0)}N_1$

So $\ddot\alpha(0)N(p) = \ddot\beta(0)\cdot N_1(p)$ (*) As $\ddot\alpha /\!/ N(p)$, suppose

$\ddot\alpha(0) = \lambda_1 N(p)$, suppose $\ddot\beta(0) = \lambda_2 N_1(p)$ similarly as $\ddot\beta /\!/ N_1(p)$

So $\lambda_1 = \ddot\alpha(0)\cdot N(p) \overset{\text{by }(*)}{=} \ddot\beta(0)\cdot N_1(p) = \lambda_2$, $\ddot\alpha(0) = \lambda_1 N(p) = \lambda_2 N_1(p) = \ddot\beta(0)$ (as $N_1(p) = N(p)$)

10.9 "only if": O: $\|x - q\|^2 = r^2$. $p\in O \Rightarrow \|p-q\|^2 = r^2 \Rightarrow f(0) = \|p-q\|^2 - r^2 = 0$

$C_p = O_p \Rightarrow$ the normal vector of O at $p = 2(p-q)\perp O_p = C_p = \{\lambda\dot\alpha(0) \mid \lambda\in R\}$

so $(p-q)\cdot\dot\alpha(0)=0$ so $f'(0)=2(\alpha(0)-q)\dot\alpha(0)=2(p-q)\cdot\dot\alpha(0)=0$.

By Thm 1 of chapter 9. $\ddot\alpha(t_0)\cdot N(p) = -\nabla_{\dot\alpha(0)}N(p)\cdot\dot\alpha(0)$ (N, N_1 are orientation of C and O resp.)

$N(p)=N_1(p)=\lambda(p-q)/r$ ($\lambda=\pm1$ which determines orientation) $\lambda=1$ outwards $\lambda=-1$ inwards

$\nabla_v N(p)=\nabla_v N_1(p)=\lambda\frac{1}{r}v$.

So $\ddot\alpha(t_0)\lambda(p-q)/r=\ddot\alpha(t_0)\cdot N(p) = -\nabla_{\dot\alpha(0)}N(p)\cdot\dot\alpha(0) = -\nabla_{\dot\alpha(0)}N_1(p)\cdot\dot\alpha(0) = -\lambda\frac{1}{r}\dot\alpha(0)\cdot\dot\alpha(0) = \frac{-\lambda}{r}$

So $\ddot\alpha(p-q)=-1$. So $f''(0)=2+2(p-q)\ddot\alpha(0)=0$

"If part" $f(0)=0 \Rightarrow \|p-q\|=r^2$ So $p\in O$. ①

$f'(0)=0\Rightarrow(p-q)\cdot\dot\alpha(0)=0$ As we are in 2D, and $p-q\in O_p^\perp$. So $\dot\alpha(0)\in O_p$.

But $\dot\alpha(0)\in C_p$ as well and O_p and O_p are both one dimensional, So $O_p=C_p$ ②; then we can easily choose an orientation of O such that its orientation at p is the same as C's ③

$f''(0)\Rightarrow(p-q)\cdot\ddot\alpha(0)=-1$ ~~$\forall v\in C_p$, i.e. $v=\mu\dot\alpha(0)$, N~~

Since $\nabla_v N\cdot N=0$ and $N\perp\dot\alpha(0)$ So $\nabla_{\dot\alpha(0)}N=a\cdot\dot\alpha(0)$ $a\in\mathbb{R}$ as we are in 2D

$a=\nabla_{\dot\alpha(0)}N\cdot\dot\alpha(0)=-\ddot\alpha(0)N(p)=-\ddot\alpha(0)N_1(p)=-\lambda(p-q)/r\,\ddot\alpha(0)=\frac{\lambda}{r}$,

So $\nabla_{\dot\alpha(0)}N=\frac{\lambda}{r}\dot\alpha(0)$. But $\nabla_{\dot\alpha(0)}N_1=\frac{\lambda}{r}\dot\alpha(0)$ By Example in chapter 9 on page 56

So $\nabla_{\dot\alpha(0)}N=\nabla_{\dot\alpha(0)}N_1$. Furthermore, $\forall v\in C_p$, v must be $v=\mu\dot\alpha(0)$ $\mu\in\mathbb{R}$.

But $\nabla_v N=\nabla_{\mu\dot\alpha(0)}N=\mu\cdot\nabla_{\dot\alpha(0)}N=\mu\nabla_{\dot\alpha(0)}N_1=\nabla_{\mu\dot\alpha(0)}N_1=\nabla_v N_1$ ④

Combining ①-④. O is circle of curvature of C at p.

10.10 $\dot\alpha(t)=(\cos\theta(t),\sin\theta(t))$ As $\alpha(t)$ is local parametrization of C

$N(\alpha(t))=(-\sin\theta(t),\cos\theta(t))$. $\ddot\alpha(t)=(-\sin\theta(t)\cdot\dot\theta(t),\cos\theta(t)\dot\theta(t))$. As α is unit speed, $k\circ\alpha=\ddot\alpha(t)\cdot N(\alpha(t))=\dot\theta(t)\stackrel{\triangle}{=}d\theta/dt$.

11.1 $l(\alpha)=\int_0^2\|(2t,3t^2)\|dt=\int_0^2\sqrt{4+9t^2}\,t\,dt\overset{u=t^2}{=}\frac{1}{2}\int_0^4\sqrt{4+9u}\,du$

$=\frac{1}{18}\int_0^4\sqrt{4+9u}\,d(4+9u)=\frac{1}{18}\frac{2}{3}(4+9u)^{\frac{3}{2}}\Big|_0^4=\frac{1}{27}(10\sqrt{10}-1)$

11.2 $l(\alpha)=\int_{-1}^1\|(-3\sin3t,3\cos3t,4)\|dt=10$

11.3 $l(\alpha)=\int_0^{2\pi}\|-2\sqrt2\sin2t,2\cos2t,2\cos2t\|dt=\int_0^{2\pi}2\sqrt2\,dt=4\pi\sqrt2$

11.4 $l(\alpha)=\int_0^{2\pi}\|(-\sin t,\cos t,-\sin t,\cos t)\|dt=2\sqrt2\pi$

11.5. $\alpha(t)=(12t,-5t)$ $t\in(-1,1)$ $l(C)=l(\alpha)=\int_{-1}^1 13\,dt=26$ ~~...~~ Ref Ex 11.9 changes sign

Actually, don't bother with orientation and α compliance, because $l(C)\geq0$. and ~~orientation only~~