so $(p-q)\cdot\dot\alpha(0)=0$ so $f'(0)=2(\alpha(0)-q)\dot\alpha(0)=2(p-q)\cdot\dot\alpha(0)=0$.

By Thm 1 of chapter 9. $\ddot\alpha(t_0)\cdot N(p)=-\nabla_{\dot\alpha(0)}N(p)\cdot\dot\alpha(0)$ (N, N_1 are orientation of C and O resp.)

$N(p)=N_1(p)=\lambda(p-q)/r$ ($\lambda=\pm1$ which determines orientation) $\lambda=1$ outwards $\lambda=-1$ inwards

$\nabla_v N(p)=\nabla_v N_1(p)=\lambda\frac{1}{r}v$.

So $\ddot\alpha(t_0)\lambda(p-q)/r=\ddot\alpha(t_0)\cdot N(p)=-\nabla_{\dot\alpha(0)}N(p)\cdot\dot\alpha(0)=-\nabla_{\dot\alpha(0)}N_1(p)\cdot\dot\alpha(0)=-\lambda\frac{1}{r}\dot\alpha(0)\cdot\dot\alpha(0)=\frac{-\lambda}{r}$

So $\ddot\alpha(p-q)=-1$. So $f''(0)=2+2(p-q)\ddot\alpha(0)=0$

"If part" $f(0)=0 \Rightarrow \|p-q\|=r^2$ So $p\in O$. ①

$f'(0)=0\Rightarrow(p-q)\cdot\dot\alpha(0)=0$ As we are in 2D, and $p-q\in O_p^\perp$. So $\dot\alpha(0)\in O_p$.

But $\dot\alpha(0)\in C_p$ as well and O_p and O_p are both one dimensional, So $O_p=C_p$ ②; then we can easily choose an orientation of O such that its orientation at p is the same as C's ③

$f''(0)\Rightarrow(p-q)\cdot\ddot\alpha(0)=-1$ ~~$\forall v\in C_p$, i.e. $v=\mu\dot\alpha(0)$, N~~

Since $\nabla_v N\cdot N=0$ and $N\perp\dot\alpha(0)$ So $\nabla_{\dot\alpha(0)}N=a\cdot\dot\alpha(0)$ $a\in\mathbb{R}$ as we are in 2D

$a=\nabla_{\dot\alpha(0)}N\cdot\dot\alpha(0)=-\ddot\alpha(0)N(p)=-\ddot\alpha(0)N_1(p)=-\lambda(p-q)/r\cdot\ddot\alpha(0)=\frac{\lambda}{r}$,

So $\nabla_{\dot\alpha(0)}N=\frac{\lambda}{r}\dot\alpha(0)$. But $\nabla_{\dot\alpha(0)}N_1=\frac{\lambda}{r}\dot\alpha(0)$ By Example in chapter 9 or page 56

So $\nabla_{\dot\alpha(0)}N=\nabla_{\dot\alpha(0)}N_1$. Furthermore, $\forall v\in C_p$, v must be $v=\mu\dot\alpha(0)$ $\mu\in\mathbb{R}$.

But $\nabla_v N=\nabla_{\mu\dot\alpha(0)}N=\mu\cdot\nabla_{\dot\alpha(0)}N=\mu\nabla_{\dot\alpha(0)}N_1=\nabla_{\mu\dot\alpha(0)}N_1=\nabla_v N_1$ ④

Combining ①-④. O is circle of curvature of C at p.

10.10 $\alpha(t)=(\cos\theta(t),\sin\theta(t))$ As $\alpha(t)$ is local parametrization of C

$N(\alpha(t))=(-\sin\theta(t),\cos\theta(t))$. $\ddot\alpha(t)=(-\sin\theta(t)\cdot\dot\theta(t),\cos\theta(t)\dot\theta(t))$. As α is unit speed, $k\circ\alpha=\ddot\alpha(t)\cdot N(\alpha(t))=\dot\theta(t)\stackrel{\Delta}{=}d\theta/dt$.

11.1 $l(\alpha)=\int_0^2\|(2t,3t^2)\|dt=\int_0^2\sqrt{4+9t^2}\,t\,dt\stackrel{u=t^2}{=}\frac{1}{2}\int_0^4\sqrt{4+9u}\,du$

$=\frac{1}{18}\int_0^4\sqrt{4+9u}\,d(4+9u)=\frac{1}{18}\frac{2}{3}(4+9u)^{\frac{3}{2}}\Big|_0^4=\frac{1}{27}(10\sqrt{10}-1)$

11.2 $l(\alpha)=\int_{-1}^1\|(-3\sin3t,3\cos3t,4)\|dt=10$

11.3 $l(\alpha)=\int_0^{2\pi}\|-2\sqrt2\sin2t,2\cos2t,2\cos2t\|dt=\int_0^{2\pi}2\sqrt2\,dt=4\pi\sqrt2$

11.4 $l(\alpha)=\int_0^{2\pi}\|(-\sin t,\cos t,-\sin t,\cos t)\|dt=2\sqrt2\pi$

11.5. $\alpha(t)=(12t,-5t)$ $t\in(-1,1)$ $l(C)=l(\alpha)=\int_{-1}^1 13\,dt=26$ Ref Ex 11.9 changes sign

Actually, don't bother with orientation and $\dot\alpha$ compliance, because $l(C)\geq0$.

11.6 $\alpha(t)=(2\sin t, 1+2\cos t)$ $l(C)=l(\alpha) = \int_0^{2\pi}\|\dot\alpha(t)\|dt = \int_0^{2\pi}\|(2\cos t, -2\sin t)\|dt = 4\pi$

11.7 $\alpha(t)=(\sqrt{1+t^2}, t)$, $t\in(-\sqrt3,\sqrt3)$, $l(C)=l(\alpha)= \int_{-\sqrt3}^{\sqrt3}\|(t(1+t^2)^{-1/2},1)\|dt$
$= 2\int_0^{\sqrt3}\sqrt{1+t^2/(1+t^2)}\,dt$

11.8 $\alpha(t)=(\frac23 t^{\frac32}, t)$ $t\in(0,3)$ $l(C)=l(\alpha)=\int_0^3\|(t^{\frac12},1)\|dt = \int_0^3\sqrt{t+1}\,d(t+1) = 14/3$

11.9 If $\alpha(t)$ $t\in(a,b)$ is consistent with N, then $\beta(t)\triangleq\alpha(-t)$ $t\in(-b,-a)$ is consistent with $-N$
$(\dot\alpha_1(t),\dot\alpha_2(t))^T = R_{-\pi/2}(N_1(\alpha(t)),N_2(\alpha(t)))^T$. for $\forall\, t\in(a,b)$
$(\dot\beta_1(t),\dot\beta_2(t))^T=(-\dot\alpha_1(-t),-\dot\alpha_2(-t))^T = R_{-\pi/2}(-N_1(\alpha(-t)),-N_2(\alpha(-t)))^T$
$\int_{-b}^{-a}\alpha(-t)dt = \int_a^b\alpha(t)dt$ So $l(C)=l(\tilde C)$

11.10 (a) $\int_a^b|k\circ\alpha(t)|dt = \int_a^b|\ddot\alpha\cdot N(\alpha(t))|dt = \int_a^b\|\ddot\alpha(t)\|dt$. If β is reparametrization of α. $\beta:(c,d)\to C$
$\beta=\alpha\circ h$. (Since α,β are both one-to-one, such h must exist, $h(t)\triangleq\alpha^{-1}(\beta(t))$) since both α and β are ~~smooth~~ regular. ($\dot\alpha\neq0$, $\dot\beta\neq0$) h must be differentiable
$\dot\beta(t)=\dot\alpha(h(t))\cdot h'(t)$. But $\|\dot\alpha\|=\|\dot\beta\|=1$, so $\|h'(t)\|\equiv1$. But h' is continuous.
So $h'\equiv1$ or $h'\equiv-1$. In whichever case $\ddot\beta(t)=\ddot\alpha(h(t))(h'(t))^2=\ddot\alpha(h(t))$
so $\int_c^d|k\circ\beta(t)|dt = \int_c^d\|\ddot\beta(t)\|dt = \begin{cases}\int_c^d\|\ddot\alpha(h(t))\|\,h'(t)\,dt & \text{if } h'\equiv1\\ \int_d^c\|\ddot\alpha(h(t))\|\cdot h'(t)\,dt & \text{if } h'\equiv-1\end{cases}$
$=\int_a^b\ddot\alpha(u)\,du$, $u\triangleq h(t)$.
(b) By Ex 10.6. $l(N\circ\alpha)=\int_a^b\|\dot N\circ\alpha\|dt = \int_a^b\|-k\circ\alpha\cdot\dot\alpha\|dt = \int_a^b|k\circ\alpha|\,dt$.

11.11 (a) $d(f+g)(v)=\nabla(f+g)\cdot v = \nabla f\cdot v+\nabla g\cdot v = df(v)+dg(v)$ $\forall\, v\triangleq(p,v)\in R_p^{n+1}$, $p\in U$
(b) $d(fg)(v)=\nabla(fg)\cdot v = \nabla f(p)\cdot g(p)\cdot v+\nabla g(p)\cdot f(p)\cdot v$
So $dfg = g\,df+f\,dg$
(c) $d(h\circ f)(v)=\nabla(h\circ f)\cdot v = h'(f(p))\nabla f(p)\cdot v$, So $d(h\circ f)=(h'\circ f)\,df$

11.12 (a) $\int_\alpha(x_2dx_1-x_1dx_2) = \int_0^{2\pi}[2\sin t(-2\sin t)-2\cos t(2\cos t)]dt = -8\pi$
(b) $\int_C(-x_2dx_1+x_1dx_2) = \int_0^{2\pi}[(-b\sin t)(-a\sin t)+(a\cos t)(b\cos t)]dt = 2\pi ab$
(c) $\int_\alpha\sum_{i=1}^{n+1}x_idx_i = f(\alpha(1))-f(\alpha(0)) = \frac12(n+1)$, where $f(x)=\frac12\sum_{i=1}^{n+1}x_i^2$: $df(v)=\nabla f(p)\cdot v=\sum_{i=1}^{n+1}p_iv_i$
$df(X_j)=p_j$, So $df=\sum_{i=1}^{n+1}x_idx_i$

11.13 $w(\dot\alpha(t))=\sum_{i=1}^{n+1}f_i(\alpha(t))\dot\alpha_i(t) = \sum_{i=1}^{n+1}f_i(\alpha(t))\cdot\frac{dx_i}{dt}$ So $\int_\alpha w = \int_a^b\sum_{i=1}^{n+1}(f_i\circ\alpha)\frac{dx_i}{dt}dt$

11.14 If C is connected, then there is a one-to-one, global parametrization $\alpha(t): [a,b] \to C$.
$\int_C w_X = \int_a^b X(\alpha(t))\cdot\dot{\alpha}(t)\,dt = \int_a^b \|\dot{\alpha}(t)\|\,dt = \ell(C)$ (as X is rotating $\nabla f/\|\nabla f\|$ by $-\pi/2$)
If C is not connected, then the above is true for each segment, so globally holds too

11.15 Treat $\dot{\alpha}$ as α, then $\dot{\alpha}(t) = (\cos\theta(t), \sin\theta(t))$ by proof in Thm 3 ($\theta(t) \triangleq \theta_0 + \int_{t_0}^{t}\eta(\beta(t))\,dt$)
As for uniqueness: If $\theta_1(t)$ and $\theta_2(t)$ satisfy: $\cos\theta_1(t) \equiv \cos\theta_2(t)$, $\sin\theta_1(t) \equiv \sin\theta_2(t)$
$\theta_1(t_0) = \theta_2(t_0) = \theta_0$, then $\theta_1(t) = \theta_2(t)$ for all $t \in I$. Proof: By first two equations
$\sin(\theta_1(t) - \theta_2(t)) = 0$ so $\cos(\theta_1(t) - \theta_2(t))\cdot(\dot{\theta}_1(t) - \dot{\theta}_2(t)) = 0$.
But $\sin(\theta_1 - \theta_2) = 0 \Rightarrow \cos(\theta_1 - \theta_2) \neq 0$, so $\dot{\theta}_1(t) - \dot{\theta}_2(t) = 0$ So $\theta_1(t) \equiv \theta_2(t)$ as it holds for $t=0$.

11.16 Let $\beta(t) = f(t)\cdot\alpha(t)$. Define $\varphi_1(t) = \varphi_1(a) + \int_{\alpha_t}\eta$, $\varphi_2(t) = \varphi_2(a) + \int_{\beta_t}\eta$
$\varphi_1(a)$ is chosen so that $\alpha(a)/\|\alpha(a)\| = (\cos\varphi_1(a), \sin\varphi_1(a))$ and $\varphi_1(a) \in [0, 2\pi)$
$\varphi_2(a)$ - - - - - - $\beta(a)/\|\beta(a)\| = (\cos\varphi_2(a), \sin\varphi_2(a))$ and $\varphi_2(a) \in [0, 2\pi)$
As $\beta(a)/\|\beta(a)\| = \alpha(a)/\|\alpha(a)\|$ (since $f>0$) and such choice of $\varphi_1(a), \varphi_2(a)$ is unique, we have
$\varphi_1(a) = \varphi_2(a)$. Furthermore, by proof in Thm 3,
$\alpha(t)/\|\alpha(t)\| = (\cos\varphi_1(t), \sin\varphi_1(t))$. $\beta(t)/\|\beta(t)\| = (\cos\varphi_2(t), \sin\varphi_2(t))$
As $\alpha(t)/\|\alpha(t)\| \equiv \beta(t)/\|\beta(t)\|$, $\cos\varphi_1(t) \equiv \cos\varphi_2(t)$, $\sin\varphi_1(t) \equiv \sin\varphi_2(t)$
and $\varphi_1(a) = \varphi_2(a)$. Same as the proof of uniqueness in Ex 11.15 (now may need piece-wise, but still true) we have
$\varphi_1(t) \equiv \varphi_2(t)$, $k(\alpha) = \frac{1}{2\pi}(\varphi_1(b) - \varphi_1(a)) = \frac{1}{2\pi}(\varphi_2(b) - \varphi_2(a)) = k(\beta)$.
Let $f = \|\alpha\|^{-1}$ (As $\|\alpha\| \neq 0$), then $k(\alpha) = k(\alpha/\|\alpha\|)$.
Actually no need of α being closed and $f(a) = f(b)$. $\int_\alpha \eta \equiv \int_\beta \eta$.

11.17 Since by Ex 11.16, α and $\alpha/\|\alpha\|$ have the same winding number, it is now equivalent to proving that with $\varphi(t,u)$ redefined as $\hat{\varphi}(t,u) = \varphi(t,u)/\|\varphi(t,u)\|$, the result holds. Now $\|\hat{\varphi}_u(t)\| = 1$ for all u, and t, and $\varphi(t,u)/\|\varphi(t,u)\|$ is continuous as $\|\varphi(t,u)\|$ is continuous and $\hat{\varphi}_u(t)$ is smooth on each $[t_i, t_{i+1}]$, $\hat{\varphi}_u(a) = \hat{\varphi}_u(b)$.
As $[a,b]\times[0,1]$ is compact, and $\hat{\varphi}$ is continuous, $\hat{\varphi}$ must be uniform continuous, i.e.
$\forall \varepsilon_1 > 0\ \exists \delta_1 > 0\ \forall \|(t_1, u_1) - (t_2, u_2)\| < \delta_1$ $\|\hat{\varphi}(t_1, u_1) - \hat{\varphi}(t_2, u_2)\| < \varepsilon_1$. Specifically, let $t_1 = t_2$
$\|\hat{\varphi}(t, u_1) - \hat{\varphi}(t, u_2)\| < \varepsilon_1$ i.e. $\hat{\varphi}(t, u_1)\cdot\hat{\varphi}(t, u_2) \geq 1 - \frac{\varepsilon_1^2}{2} = 1 - \varepsilon_2$ ($\varepsilon_2 \triangleq \frac{\varepsilon_1^2}{2}$) $\forall |u_1 - u_2| < \varepsilon_1$ (*)
Define $\theta_u(t) = \theta_u(a) + \int_{\hat{\varphi}_u}\eta$, $\theta_x(t) = \theta_x(a) + \int_{\hat{\varphi}_x}\eta$. $\forall u \in [0,1]$, $x \in (u - \varepsilon_1, u + \varepsilon_1) \cap [0,1]$
$\theta_u(a)$ is chosen so that $\hat{\varphi}_u(a) = (\cos\theta_u(a), \sin\theta_u(a))$. Likewise, $\hat{\varphi}_x(a) = (\cos\theta_x(a), \sin\theta_x(a))$
and $\theta_u(a), \theta_x(a) \in [0, 2\pi)$ unique! For $\forall t$, by proof in Thm 3,
$\hat{\varphi}_u(t) = (\cos\theta_u(t), \sin\theta_u(t))$, $\hat{\varphi}_x(t) = (\cos\theta_x(t), \sin\theta_x(t))$

So $\hat{\varphi}_u(t)\cdot\hat{\varphi}_x(t) = \cos(\theta_u(t)-\theta_x(t))$. By (*) $\cos(\theta_u(t)-\theta_x(t)) > 1-\varepsilon_2$ (**)

Let $\theta_0 = \arccos(1-\varepsilon_2)$. ~~As $\theta_u(a), \theta_x(a) \in [0, 2\pi)$, $|\theta_u(a)-\theta_x(a)| < \theta_0$~~

So for $\forall t$, $\exists k_t \in \mathbb{Z}$, s.t. $|\theta_u(t)-\theta_x(t)| \in (2k_t\pi-\theta_0, 2k_t\pi+\theta_0)$

If $\exists t_1, t_2$, $t_1 < t_2$, $k_{t_1} \neq k_{t_2}$, $|\theta_u(t_1)-\theta_x(t_1)| \in (2k_{t_1}\pi-\theta_0, 2k_{t_1}\pi+\theta_0)$

$|\theta_u(t_2)-\theta_x(t_2)| \in (2k_{t_2}\pi-\theta_0, 2k_{t_2}\pi+\theta_0)$ Note $\theta_0 \in (0, \pi/2]$

As $\theta_u(t)-\theta_x(t)$ is continuous with respect to t, there must exist $t_3 \in (t_1, t_2)$

s.t. $\theta_u(t_3)-\theta_x(t_3) = 2\pi\cdot\min(k_{t_1}, k_{t_2})+\pi$, ~~if ε_2 is small enough and thus θ_0 is small enough~~. But $\cos(\theta_u(t_3)-\theta_x(t_3)) = -1$ violating (**). Thus there is a $k \in \mathbb{Z}$,

s.t. $\forall t \in [a,b]$, $|\theta_u(t)-\theta_x(t)| \in (2k\pi-\theta_0, 2k\pi+\theta_0)$

But $|k(\varphi_u)-k(\varphi_x)| = \frac{1}{2\pi}|(\theta_u(b)-\theta_u(a))-(\theta_x(b)-\theta_x(a))|$

~~$\leq \frac{1}{2\pi}(|\theta_u(b)-\theta_x(b)| + |\theta_u(a)-\theta_x(a)|)$~~

$= \frac{1}{2\pi}|(\theta_u(b)-\theta_x(b))-(\theta_u(a)-\theta_x(a))|$

$< \frac{1}{2\pi}2\theta_0 = \frac{\theta_0}{\pi}$ ~~$\pi/3$ is to make θ_0 sufficiently small to ensure $\exists t_3$~~

So $\forall u \in [a,b]$, $\varepsilon \in (0, 1/2)$, $\exists \varepsilon_2 = 1-\cos(\varepsilon\pi)$, $\varepsilon_1 = \sqrt{2\varepsilon_2}$, s.t. $\forall x \in (u-\varepsilon_1, u+\varepsilon_1)\cap[0,1]$,

$|k(\varphi_x)-k(\varphi_u)| < \varepsilon$. So $k(\varphi_u)$ is a continuous function of u

Finally, as $k(\varphi_u)$ can only assume integer value, $k(\varphi_0) = k(\varphi_1)$

Note: φ can be on any $[a,b]$ $(a, b \neq \infty)$ and $k(\varphi_a) = k(\varphi_b)$

11.18 (a) $\forall n$. define $\alpha(t) = (\cos nt, \sin nt)$, i.e. $\alpha'(t) = (\frac{1}{n}\sin nt, \frac{-1}{n}\cos nt)$

Then following Example 2 on Pg 75, $\int_\alpha \eta = n\int_0^{2\pi} \frac{1\cdot 1}{\cos^2 t+\sin^2 t}\,dt = 2n\pi$

i.e. the rotation index of α is n.

(b) We follow the definitions of φ, ψ, ϕ as in the hint, but define t_0 more formally.

Let $u \in \mathbb{R}^2$, $u \neq 0$ define $h(t) = \alpha(t)\cdot u$. Since α is compact, h must attain its minimum, say, at t_0. ~~By Lagrange multiplier. Thus, $u = \lambda$~~

So $h'(t_0) = \dot\alpha(t_0)\cdot u = 0$, i.e. $\dot\alpha(t_0) \perp u$. By definition, ϕ is continuous.

$k(\varphi_0)$ is the rotation index of α, because $\varphi_0(t) = \psi(t,t) = \dot\alpha(t)/\|\dot\alpha(t)\|$.

As for $k(\varphi_1)$, when $t \in (t_0, t_0+\tau/2]$ $\varphi_1(t) = \psi(t_0, -t_0+2t) = (\alpha(-t_0+2t)-\alpha(t_0))/\|\alpha(-t_0+2t)-\alpha(t_0)\|$ ($t_0 \to t_0+\tau$)

Now the η is exact because $\varphi_1(t)\cdot u \geq 0$ and We can set $v = -u$, and have ~~$\theta_v = \theta_u$~~

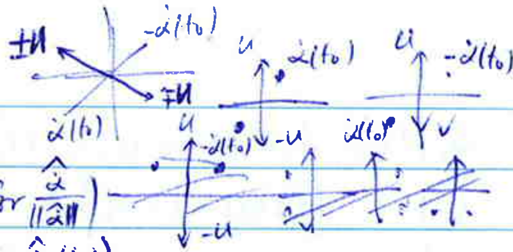
$V = \mathbb{R}^2 - \{rv \mid r>0\}$, η is exact on V. Here $\varphi_1(t)\cdot u \geq 0$, because $\forall t$

$h(t) \geq h(t_0) \Rightarrow (\alpha(t)-\alpha(t_0))\cdot u \geq 0 \Rightarrow \varphi_1(t)\cdot u \geq 0$ $(t \in (t_0, t_0+\tau/2))$

So $\int_{\varphi_1|(t_0, t_0+\tau/2]}\eta = \theta_v(\varphi_1(t_0+\frac{\tau}{2}))-\theta_v(\varphi_1(t_0)) = \theta_v(-\dot\alpha(t_0)/\|\dot\alpha(t_0)\|)-\theta_v(\dot\alpha(t_0)/\|\dot\alpha(t_0)\|)$

$= \pm\pi$

For $t \in (t_0+\frac{\tau}{2}, t_0+\tau]$, $\varphi_1(t) = \psi(2t-t_0-\tau, t_0+\tau) = (\alpha(t_0)-\alpha(2t-t_0-\tau))/\|\cdot\|$ ($t_0 \to t_0+\tau$)



Now we set $V'=u$, $V'=\mathbb{R}^2-\{rv \mid r>0\}$. η is exact on V'

$\int_{\varphi_1|(t_0+\pi/2,\, t_0+\tau]}\eta = \theta_{V'}(\hat{\dot\alpha}(t_0)) - \theta_{V'}(-\hat{\dot\alpha}(t_0))$ ($\hat{\dot\alpha}$ stands for $\frac{\dot\alpha}{\|\dot\alpha\|}$)

But for $\forall u$, $\theta_u(-\hat{\dot\alpha}(t_0)) - \theta_u(\hat{\dot\alpha}(t_0)) + \theta_{-u}(\hat{\dot\alpha}(t_0)) - \theta_{-u}(-\hat{\dot\alpha}(t_0))$

$= \pm 2\pi$ (if $-u \searrow +u$, then 2π, if $-u \nwarrow +u$, then -2π)

Thus. $k(\varphi_0) = k(\varphi_1) = \pm 1$ i.e. rotation index is ± 1.

11.19 (a) $h'(t_0) = 0 \iff \dot\alpha(t_0)\cdot u = 0 \iff \dot\alpha(t_0) \perp u$. Since $\dot\alpha(t_0) \perp N_{(\alpha(t_0))}$ $\iff N(\alpha(t_0)) = \pm u$. Let $N(\alpha(t_0)) = \delta u$, $\delta = \pm 1$.

$h''(t_0) = \ddot\alpha(t_0)\cdot u = \ddot\alpha \cdot \delta N = k\cdot\delta = k(\alpha(t_0))\cdot N(\alpha(t_0))\cdot u$.

(b) construct $\theta(t)$ as in Ex 11.15. By Ex 10.10 $\frac{d\theta}{dt} = k\circ\alpha$. Then rotation index is

$\frac{1}{2\pi}\int_{\dot\alpha}\eta = \frac{1}{2\pi}\left(\theta(t_0+\tau) - \theta(t_0)\right) = \frac{1}{2\pi}\int_{t_0}^{t_0+\tau}\frac{d\theta}{dt}dt = \int_{t_0}^{t_0+\tau}(k\circ\alpha)(t)dt$

Gauss map N of C is onto because: $\forall u \in S^1$, $h(t_1) = h(t_1+\tau)$ $\forall t_1, t_1+\tau \in \mathbb{R}$, τ is period

so there must be $t_0 \in (t_1, t_1+\tau)$ s.t. $h'(t_0) = 0$. So $N(\alpha(t_0)) = \pm u$

Since $\alpha(t)$ is periodic. $h(t)$ must have both minimum and maximum say, t_0, t_0' resp.

$h'(t_0) = h'(t_0') = 0$, $h''(t_0) \geq 0$, $h''(t_0') \leq 0$. But since $N = \pm u$, $N\cdot u \neq 0$. So $h''(t_0) \neq 0$

So $h''(t_0) > 0$. Likewise, $h''(t_0') < 0$. $h''(t_0) > 0 \Rightarrow u\cdot N(\alpha(t_0)) > 0 \Rightarrow N(\alpha(t_0)) = u$

$h''(t_0') < 0 \Rightarrow u\cdot N(\alpha(t_0')) < 0 \Rightarrow N(\alpha(t_0')) = -u$. So N is onto

(c) As $k>0$ $\int_{t_0}^{t}(k\circ\alpha)(t)dt$ monotonically increasing wrt t. Set $\theta(t) = \theta_0 + \int_{t_0}^{t}\eta(\dot\alpha(\tau))d\tau$

then $\dot\alpha = (\alpha \cdot \cos\theta(t), \sin\theta(t))$ As $N(c) = N(t_0)$, so $(\cos\theta(c), \sin\theta(c)) = (\cos\theta_0, \sin\theta_0)$

So $\theta(c) = 2n\pi + \theta_0$. But by Ex 10.10, $\int_{t_0}^{c}(k\circ\alpha)(t)dt = \int_{t_0}^{c}\frac{d\theta}{dt}dt = \theta(c) - \theta_0 = 2n\pi$

As $k>0$, $c > t_0$. so $n>0$. If $n=2$ then there is a $t_1 \in (t_0, c)$: $\theta(t_1) = \theta_0 + 2\pi$

because $\theta(t)$ is continuous. But $(\cos\theta(t_1), \sin\theta(t_1)) = (\cos\theta_0, \sin\theta_0)$, So $N(\alpha(t_1)) = N(\alpha(t_0))$

But that contradicts with the assumption that $N(t) \neq N(t_0)$ $\forall t \in (t_0, c)$

So $n=1$, i.e. $\int_{t_0}^{c}(k\circ\alpha)(t)dt = 2\pi$.

By (b) $\int_{t_0}^{t_0+\tau}(k\circ\alpha)(t)dt$ = rotation index of $\alpha * 2\pi = \pm 2\pi$.

As $k>0$. It equals 2π, which in turn equals $\int_{t_0}^{c}(k\circ\alpha)(t)dt$.

As $k>0$ $c = t_0+\tau$. (a) has shown the Gaussian map is onto.

Now we've proven that $N(t) = N(t_0)$ iff $t = t_0 + \tau\cdot n$. But τ is period of α.

So Gaussian map is injection. In sum, it is one-to-one.

11.20 (a) α_f is just one point a_0, so obviously $k(f) = 0$ (construct $v = -a_0$, θ_v)

(b) $\alpha_f(t) = (a_n\cos nt, a_n\sin nt)$ Similar to Example 2 on Pg. 75, $k(f) = n$.

(c) Construct $\varphi: [0,2\pi]\times[0,1] \to \mathbb{R}^2 - \{0\}$. $\varphi(t,u) = f(u(\cos t + i\sin t))$

↑ by $f(z) \neq 0$ $\forall z: |z| \leq 1$ — obviously continuous

$\varphi_0(t) = f(0) \neq 0$, So $k(\varphi_0(t)) = 0$.

$\varphi_1(t) = f(\cos t + i\sin t)$ $\quad k(f) = k(\varphi_1(t)) \overset{*}{=} k(\varphi_0(t)) = 0$ (*by Ex 11.17)

(d) Construct $\varphi(t,u) = \begin{cases} u^n f\left(\frac{1}{u}(\cos t + i\sin t)\right) & \text{if } u \neq 0 \\ a_n(\cos t + i\sin t)^n & \text{if } u = 0 \end{cases}$ on $[0,2\pi]\times[0,1]$

By def of φ and $f(z)\neq 0$ $\forall z\in\mathbb{C}, |z|\geq 1$, we have $\varphi(t,u)\not\equiv 0$.

As $\lim_{u\to 0} u^n f\left(\frac{1}{u}(\cos t + i\sin t)\right) = \lim_{u\to 0} u^n \sum_{k=0}^{n} a_n(\cos kt + i\sin kt)\frac{1}{u^k} = a_n(\cos nt + i\sin nt)$
$= \varphi(t,0)$ So φ is continuous

$\varphi(t,0) = a_n(\cos nt + i\sin nt)$. By Example 2 on Pg 75. $k(\varphi(t,0)) = n$

By Ex 11.17. $k(f) = k(\varphi(t,1)) = k(\varphi(t,0)) = n$.

(e). Combining (c),(d), (c) says $k(f)=0$, (d) says $k(f) = n$. So ~~either~~ $n=0$

11.21 If no point of $\alpha(t)$ lies on positive x_1-axis, then choose $v=(1,0)$, by θ_v, we have $k(\alpha)=0$, correct.

Else Let $a < t_1 < t_2 < \cdots < t_m < b$ be the set of all $t\in(a,b)$ such that $\alpha(t)$ lies on the positive x_1-axis. we will discuss $t=a,b$ specially. ~~Note as $\alpha(a)=\alpha(b)$, even if $\alpha(a)$ is not on positive x_1-axis, we can still reparametrize $\alpha(t)$ into $\beta(t)=\alpha(t_1+t-a)$, then $\beta(a)=\alpha(t_1)$ which is on x_1-axis.~~ Denote $t_0=a$, $t_{m+1}=b$. For all $i=1,2,\ldots,m$, if $\alpha(t_i)$ crosses positive x_1-axis upward, define $\delta_i=1$. If crosses downward, define $\delta_i=-1$. If $\alpha(a)=\alpha(b)$ is on positive x_1-axis, define $\delta_0=\delta_{m+1}$ likewise in $\{\pm1\}$. ~~If $\alpha(a)$ is not on positive x_1-axis, define $\delta_0=\delta_{m+1}=0$~~.

$k(\alpha) = \frac{1}{2\pi}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\eta(\dot\alpha(t))\,dt = \frac{1}{2\pi}\sum_{i=0}^{m}\lim_{\varepsilon\to0^+}\int_{t_i+\varepsilon}^{t_{i+1}-\varepsilon}\eta(\alpha(t))\,dt = \frac{1}{2\pi}\sum_{i=0}^{m}\lim_{\varepsilon\to0^+}\int_{t_i+\varepsilon}^{t_{i+1}-\varepsilon}d\theta_v(\alpha(t))\,dt$

where $v=(1,0)$ and θ_v is defined as in proof of Thm 3. We check two consecutive crossings of positive x_1-axis: $(i=1,\ldots,m-1)$:

i	$i+1$	δ_i	δ_{i+1}	angle	formula
↗	↗	1	+1	2π	$(\delta_i+\delta_{i+1})\cdot\frac{2\pi}{2}$
↗	↘	1	-1	0	
↘	↘	-1	-1	-2π	
↘	↗	-1	1	0	

angle means $\lim_{\varepsilon\to0^+}[\theta_v(\alpha(t_{i+1}-\varepsilon))-\theta_v(\alpha(t_i+\varepsilon))]$.

So $k(\alpha) = \frac{1}{2}\sum_{i=1}^{m-1}(\delta_i+\delta_{i+1})$
$+\frac{1}{2\pi}\left(\theta_v(\alpha(b)) - 2\pi\cdot\frac{1-\delta_m}{2}\right)$
$+\frac{1}{2\pi}\left(2\pi\frac{1+\delta_1}{2} - \theta_v(\alpha(b))\right)$ (if $\alpha(a)$ is not on pos x_1-axis)
$= \sum_{i=1}^{m}\delta_i$

If $\alpha(a)$ is on pos x_1-axis, then $k(\alpha) = \frac{1}{2}\sum_{i=0}^{m}(\delta_i+\delta_{i+1}) = \sum_{i=0}^{m}\delta_i$ (as $\delta_{m+1}=\delta_0$)

So the conclusion is correct in both cases.

11.22 (a) let $\beta(t)=\alpha(t)-p$

$\eta(\dot\beta) = -\frac{\beta_2}{\beta_1^2+\beta_2^2}dx_1 + \frac{\beta_1}{\beta_1^2+\beta_2^2}dx_2 = -\frac{(\alpha_2(t)-b)\,dx_1}{(\alpha_1(t)-a)^2+(\alpha_2(t)-b)^2} + \frac{(\alpha_1(t)-a)\,dx_2}{(\alpha_1(t)-a)^2+(\alpha_2(t)-b)^2}$

So $k(\beta) = \int_\beta \eta(\beta) = \int_\alpha k_p(\alpha)$. We know $k(\beta)=2k\pi$ $k\in\mathbb{Z}$, so $\frac{1}{2\pi}\int_\alpha k_p(\alpha)$ is integer

(b) Suppose p and q are joined by continuous $\beta:[c,d]\to\mathbb{R}^2-\text{Image }\alpha$ s.t. $\beta(c)=p$. $\beta(d)=q$

Define $\varphi(t,u)=\alpha(t)-\beta(u)$ on $[a,b]\times[c,d]\to\mathbb{R}^2-\{0\}$ (Since $\beta\to\mathbb{R}^2-\text{Image }\alpha$ so $\varphi\not\equiv0$)

Obviously, φ is continuous. $\varphi(t,c) = \alpha(t) - p$. $\varphi(t,d) = \alpha(t) - q$

$k(\varphi(t,c)) = k(\varphi(t,d)) \Rightarrow k_p(\alpha) = k_q(\alpha)$ (as $k_{\underset{q}{p}}(\alpha) = k(\alpha(t) - \underset{q}{p})$)

12.1 The matrix corresponding to L_p is $A = \|g\|^{-3}(g \cdot g^T - \|g\|^2 I)H$ ($g = \nabla f$, H is the Hessian)

~~$k(v)$~~ $L_p(v) = -\|g\|^{-1} H \cdot V$, $k(v) = -\|g\|^{-1} V^T H V = \mathcal{L}_p(v)$ for $V \in S_p$.

12.2 $\nabla f = (1, 1, \dots, 1)$, $V_i = \frac{1}{\sqrt{2}}(1, 0 \dots 0, -1, 0 \dots 0)$ where -1 is the i^{th} spot after 1, $i = 1 \dots n$.

$\nabla f = \sqrt{n+1}$, $\frac{\partial^2 f}{\partial x_i \partial x_j} = 0$. So $k(v) = \mathcal{L}_p(v) = 0$ $\forall V$ by Ex 12.1 So any ~~orth~~ $V \in S_p$ $\|V\| = 1$ is a principle curvature direction, with principle curvature 0.

$K(p) = 0$ $H(p) = 0$.

12.3 $\nabla f = (2x_1, \dots, 2x_{n+1})$ $\|\nabla f(p)\| = 2r$ $H = 2 \cdot I$. $k(v) = \frac{-1}{r} V \cdot V$.

Any $V \in S_p$ $\|V\| = 1$ is a principle curvature direction, with principle curvature $\frac{-1}{r}$.

$k(p) = (-r)^{-n}$, $H(p) = \frac{-1}{r}$

12.4 $\nabla f = \left(\frac{2x_1}{a^2}, \frac{2x_2}{b^2}, \frac{2x_3}{c^2}\right)$ $H = \begin{pmatrix} 2a^{-2} & 0 & 0 \\ 0 & 2b^{-2} & 0 \\ 0 & 0 & 2c^{-2} \end{pmatrix}$ ~~$k(v)=$~~ $\|\nabla f(p)\| = \frac{2}{a}$ $\nabla f(p) = \left(\frac{2}{a}, 0, 0\right)$

$k(v) = -\frac{a}{2}\left(\frac{2}{a^2}V_1^2 + \frac{2}{b^2}V_2^2 + \frac{2}{c^2}V_3^2\right) = -a\left(\frac{V_1^2}{a^2} + \frac{V_2^2}{b^2} + \frac{V_3^2}{c^2}\right)$ $\forall V \in S_p$, $V = (0, V_2, V_3)$

$V_2^2 + V_3^2 = 1$, So $k(v)$ attains its extremum at $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$ i.e. principle curvature directions, corresponding to principal curvature $\frac{-a}{b^2}$ and $\frac{-a}{c^2}$ respectively.

$k(p) = a^2/b^2c^2$, $H(p) = \frac{-a}{2}\left(\frac{1}{b^2} + \frac{1}{c^2}\right)$

12.5 $\nabla f = \left(\frac{2x_1}{a^2}, \frac{2x_2}{b^2}, -\frac{2x_3}{c^2}\right)$ $H = 2\begin{pmatrix} a^{-2} & & \\ & b^{-2} & \\ & & -c^{-2} \end{pmatrix}$ $\|\nabla f(p)\| = \frac{2}{a}$ $\nabla f(p) = \left(\frac{2}{a}, 0, 0\right)$

$k(v) = -\frac{a}{2}\left(\frac{2}{a^2}V_1^2 + \frac{2}{b^2}V_2^2 - \frac{2}{c^2}V_3^2\right)$ $\forall V \in S_p$. $V = (0, V_2, V_3)$ $V_2^2 + V_3^2 = 1$. so

$k(v) = \frac{-a}{2}\left(\frac{2}{b^2}V_2^2 - \frac{2}{c^2}V_3^2\right) = -a\left(\frac{1}{b^2}(1 - V_3^2) - \frac{1}{c^2}V_3^2\right) = a\left[\left(\frac{1}{b^2} + \frac{1}{c^2}\right)V_3^2 - \frac{1}{b^2}\right]$ $V_3^2 \in [0, 1]$

$k(v)$ attains max when $V_3^2 = 1$, $\max = \frac{a}{c^2}$, attains min when $V_3^2 = 0$ $\min = \frac{-a}{b^2}$

So principal curvature and principle curvature directions are: $\left((0,0,\pm 1), \frac{a}{c^2}\right)$, $\left((0,\pm 1,0), \frac{-a}{b^2}\right)$

$k(p) = -\frac{a^2}{b^2c^2}$, $H(p) = \frac{a}{2}\left(\frac{1}{c^2} - \frac{1}{b^2}\right)$

12.6 $\nabla f = \left(2x_1, 2x_2\left(1 - 2(x_2^2 + x_3^2)^{\frac{-1}{2}}\right), 2x_3\left(1 - 2(x_2^2 + x_3^2)^{\frac{-1}{2}}\right)\right)$

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 - 4x_3^2(x_2^2 + x_3^2)^{-3/2} & 4x_2x_3(x_2^2 + x_3^2)^{-3/2} \\ 0 & 4x_2x_3(x_2^2 + x_3^2)^{-3/2} & 2 - 4x_2^2(x_2^2 + x_3^2)^{-3/2} \end{pmatrix}$$

For (a) $\nabla f(p) = (0, 2, 0)$ $\|\nabla f\| = 2$, ($p = (0, 3, 0)$)

$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}$

(b) $p = (0, 1, 0)$, $\nabla f(p) = (0, -2, 0)$ $\|\nabla f\| = 2$

$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$