Obviously, φ is continuous. $\varphi(t,c) = \alpha(t) - p$. $\varphi(t,d) = \alpha(t) - q$

$k(\varphi(t,c)) = k(\varphi(t,d)) \Rightarrow k_p(\alpha) = k_q(\alpha)$ (as $k_{\underset{q}{p}}(\alpha) = k(\alpha(t) - \underset{q}{p})$)

12.1 The matrix corresponding to L_p is $A = \|g\|^{-3}(g \cdot g^T - \|g\|^2 I)H$ ($\overset{g = \nabla f}{(H}$ is the Hessian)

$\cancel{k(v)=}$ $L_p(v) = -\|g\|^{-1} H \cdot V$, $k(v) = -\|g\|^{-1} V^T H V = \mathscr{S}_p(v)$ for $V \in S_p$.

12.2 $\nabla f = (1, 1, \cdots, 1)$, $V_i = \frac{1}{\sqrt{2}}(1, 0 \cdots 0, -1, 0 \cdots 0)$ where -1 is the i^{th} spot after 1. $i = 1 \cdots n$.

$\nabla f = \sqrt{n+1}$, $\frac{\partial^2 f}{\partial x_i \partial x_j} = 0$. So $\overset{k(v)}{=} \mathscr{S}_p(v) = 0$ $\forall V$ by Ex 12.1 So any $\cancel{orth}$ $V \in S_p$ $\|V\| = 1$ is a principle curvature direction, with principle curvature 0.

$K(p) = 0$ $H(p) = 0$.

12.3 $\nabla f = (2x_1, \cdots, 2x_{n+1})$ $\|\nabla f(p)\| = 2r$ $H = 2 \cdot I$. $k(v) = \cancel{2v} \frac{-1}{r} v \cdot v$.

Any $V \in S_p$ $\|V\| = 1$ is a principle curvature direction, with principle curvature $\frac{-1}{r}$.

$k(p) = (-r)^{-n}$, $H(p) = \frac{-1}{r}$

12.4 $\nabla f = \left(\frac{2x_1}{a^2}, \frac{2x_2}{b^2}, \frac{2x_3}{c^2}\right)$ $H = \begin{pmatrix} 2a^{-2} & 0 & 0 \\ 0 & 2b^{-2} & 0 \\ 0 & 0 & 2c^{-2} \end{pmatrix}$ $\cancel{k(v)=}$ $\|\nabla f(p)\| = \frac{2}{a}$ $\nabla f(p) = (\frac{2}{a}, 0, 0)$

$k(v) = -\frac{a}{2}\left(\frac{2}{a^2}V_1^2 + \frac{2}{b^2}V_2^2 + \frac{2}{c^2}V_3^2\right) = -a\left(\frac{V_1^2}{a^2} + \frac{V_2^2}{b^2} + \frac{V_3^2}{c^2}\right)$ $\forall V \in S_p$, $V = (0, V_2, V_3)$

$V_2^2 + V_3^2 = 1$, So $k(v)$ attains its extremum at $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$ i.e. principle curvature directions, corresponding to principal curvature $\frac{-a}{b^2}$ and $\frac{-a}{c^2}$ respectively.

$k(p) = a^2/b^2c^2$, $H(p) = \frac{-a}{2}\left(\frac{1}{b^2} + \frac{1}{c^2}\right)$

12.5 $\nabla f = \left(\frac{2x_1}{a^2}, \frac{2x_2}{b^2}, -\frac{2x_3}{c^2}\right)$ $H = 2\begin{pmatrix} a^{-2} & & \\ & b^{-2} & \\ & & -c^{-2} \end{pmatrix}$ $\|\nabla f(p)\| = \frac{2}{a}$ $\nabla f(p) = (\frac{2}{a}, 0, 0)$

$k(v) = -\frac{a}{2}\left(\frac{2}{a^2}V_1^2 + \frac{2}{b^2}V_2^2 - \frac{2}{c^2}V_3^2\right)$ $\forall V \in S_p$ $V = (0, V_2, V_3)$ $V_2^2 + V_3^2 = 1$ so

$k(v) = \frac{-a}{2}\left(\frac{2}{b^2}V_2^2 - \frac{2}{c^2}V_3^2\right) = -a\left(\frac{1}{b^2}(1 - V_3^2) - \frac{1}{c^2}V_3^2\right) = a\left[\left(\frac{1}{b^2} + \frac{1}{c^2}\right)V_3^2 - \frac{1}{b^2}\right]$ $V_3^2 \in [0, 1]$

$k(v)$ attains max when $V_3^2 = 1$, max $= \frac{a}{c^2}$, attains min when $V_3^2 = 0$ min $= \frac{-a}{b^2}$

So principal curvature and principle curvature directions are: $((0, 0, \pm 1), \frac{a}{c^2})$, $((0, \pm 1, 0), \frac{-a}{b^2})$

$k(p) = -\frac{a^2}{b^2c^2}$, $H(p) = \frac{a}{2}\left(\frac{1}{c^2} - \frac{1}{b^2}\right)$

12.6 $\nabla f = \left(2x_1, 2x_2\left(1 - 2(x_2^2 + x_3^2)^{\frac{-1}{2}}\right), 2x_3\left(1 - 2(x_2^2 + x_3^2)^{\frac{-1}{2}}\right)\right)$

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 - 4x_3^2(x_2^2 + x_3^2)^{-3/2} & 4x_2x_3(x_2^2 + x_3^2)^{-3/2} \\ 0 & 4x_2x_3(x_2^2 + x_3^2)^{-3/2} & 2 - 4x_2^2(x_2^2 + x_3^2)^{-3/2} \end{pmatrix}$$

For (a) $\nabla f(p) = (0, 2, 0)$ $\|\nabla f\| = 2$, $(p = (0, 3, 0))$

$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}$

(b) $p = (0, 1, 0)$, $\nabla f(p) = (0, -2, 0)$ $\|\nabla f\| = 2$

$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

(a) $k(v) = -V_1^2 - V_2^2 - \frac{1}{3}V_3^2$, $V = (V_1, 0, V_3)$ $V_1^2 + V_3^2 = 1$, $k(v) = -1 + \frac{2}{3}V_3^2$

min $= -1$ when $V_3 = 0$. max $= -\frac{1}{3}$ when $V_3 = \pm 1$. So $((\pm 1, 0, 0), -1)$ and $((0, 0, \pm 1), -\frac{1}{3})$

$H(p) = \frac{-2}{3}$, $K(p) = \frac{1}{3}$

(b) $k(V) = -V_1^2 - V_2^2 + V_3^2$ $V = (V_1, 0, V_3)$ $V_1^2 + V_3^2 = 1$ $k(V) = -1 + 2V_3^2$

min $= -1$ when $V_3 = 0$, max $= 1$ when $V_3 = \pm 1$

So $((\pm 1, 0, 0), -1)$, $((0, 0, \pm 1), 1)$ $H(p) = 0$. $K(p) = -1$

12.7 If (λ_i, V_i) are eigenvalue vector of L_p for S. For $\forall v \in S_p$ ~~then~~, $L_p(v) = -\nabla_v N = -(-\nabla_v(-N)) = -\tilde{L}_p(V)$ where $\tilde{L}_p$ stands for the Waingarten map for orientation $-N$. Thus specifically $L_p(V_i) = \lambda_i V_i \Leftrightarrow \tilde{L}_p(V_i) = -\lambda_i V_i$. So L_p's eigenvalue λ_i corresponds to $\tilde{L}_p$'s eigenvalue $-\lambda_i$. So $\tilde{K} = \prod_{i=1}^{n}(-\lambda_i) = (-1)^n \prod_{i=1}^{n} \lambda_i = (-1)^n K$

12.8 As $n = 2$, the Gaussian curvature is independent of orientation

Apply Thm 5. $Z = \frac{1}{2}\nabla f(p) = (p, x_1, x_2, -x_3)$ take $V_1 = (p, x_3, 0, x_1)$, $V_2 = (0, x_3, x_2)$

So $V_1 \perp Z$, $V_2 \perp Z$ $\det\begin{pmatrix} \nabla_{V_1} Z \\ \nabla_{V_2} Z \\ Z(p) \end{pmatrix} = \begin{vmatrix} x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \\ x_1 & x_2 & -x_3 \end{vmatrix} = (x_1^2 + x_2^2 - x_3^2) x_3$

$\det\begin{pmatrix} V_1 \\ V_2 \\ Z(p) \end{pmatrix} = \begin{vmatrix} x_3 & 0 & x_1 \\ 0 & x_3 & x_2 \\ x_1 & x_2 & -x_3 \end{vmatrix} = -x_3(x_1^2 + x_2^2 + x_3^2)$, $\|Z(p)\| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$

So $K(p) = x_3(x_1^2 + x_2^2 - x_3^2) / [(x_1^2 + x_2^2 + x_3^2) \cdot (-x_3)(x_1^2 + x_2^2 + x_3^2)] = 0$

This is because through each point p, there's a $\alpha(t)$ which lie completely in S, so S doesn't force any acceleration. Besides, if S is oriented outward, then S always bends away from N, so $k(v) \leq 0$. If oriented inward, then $k(v) \geq 0$. In whatever case, 0 is an extreme point of $k(v)$. So 0 is an eigenvalue of L_p. So $K(p) = 0$.

$p + (t - t_0)\frac{p}{\|p\|}$ $|\dot{\alpha}(t_0)| = 1$ $\ddot{\alpha} = 0$ $(k(\dot{\alpha}(t_0)) = 0)$

12.9 $Z = \frac{1}{2}\nabla f(p) = (p, x_1/a^2, x_2/b^2, -x_3/c^2)$ For $x_3 \neq 0$ we may take

$V_1 = (p, x_3/c^2, 0, x_1/a^2)$, $V_2 = (p, 0, x_3/c^2, x_2/b^2)$ $V_1, V_2 \perp Z$

$\det\begin{pmatrix} \nabla_{V_1} Z \\ \nabla_{V_2} Z \\ Z(p) \end{pmatrix} = \begin{vmatrix} x_3/a^2c^2 & 0 & -x_1/a^2c^2 \\ 0 & x_3/b^2c^2 & -x_2/b^2c^2 \\ x_1/a^2 & x_2/b^2 & -x_3/c^2 \end{vmatrix}$ $\det\begin{pmatrix} V_1 \\ V_2 \\ Z(p) \end{pmatrix} = \begin{vmatrix} x_3/c^2 & 0 & x_1/a^2 \\ 0 & x_3/c^2 & x_2/b^2 \\ x_1/a^2 & x_2/b^2 & -x_3/c^2 \end{vmatrix}$ $\|Z(p)\| = \left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)^{1/2}$

$= \frac{x_3}{a^2b^2c^4}(x_1^2/a^2 + x_2^2/b^2 - x_3^2/c^2) = \frac{x_3}{a^2b^2c^4}$ $= \frac{-x_3}{c^2}\left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)$

$K(p) = -\left[a^2b^2c^2\left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)^2\right]^{-1}$. negtitive At each point p, there are some directions bends towards N, some directions bending away from N. So the max of $k(v) > 0$, min $k(v) < 0$. As $K(p)$ = product of two extreme values, $K(p) < 0$

12.10 $Z = \frac{1}{2}\nabla f(p) = (p, \frac{2}{a^2}x_1, \frac{2}{b^2}x_2, -1)$, $V_1 = (p, 1, 0, \frac{2}{a^2}x_1)$ $V_2 = (p, 0, 1, \frac{2}{b^2}x_2)$

$$\det\begin{pmatrix}\nabla_{v_1}Z\\ \nabla_{v_2}Z\\ Z(p)\end{pmatrix} = \begin{vmatrix}2/a^2 & 0 & 0\\ 0 & 2/b^2 & 0\\ 2x_1/a^2 & 2x_2/b^2 & -1\end{vmatrix} = -4/a^2b^2 \qquad \det\begin{pmatrix}V_1\\ V_2\\ Z(p)\end{pmatrix} = \begin{vmatrix}1 & 0 & 2x_1/a^2\\ 0 & 1 & 2x_2/b^2\\ 2x_1/a^2 & 2x_2/b^2 & -1\end{vmatrix} = -1-4x_1^2/a^4-4x_2^2/b^4 \qquad \|Z(p)\|^2 = 1+\frac{4x_1^2}{a^4}+\frac{4x_2^2}{b^4}$$

~~$Z(p)$~~ $k(p) = 4/a^2b^2\left(1+\frac{4x_1^2}{a^4}+\frac{4x_2^2}{b^4}\right)^2$. $k(p)>0$ As can be seen from the fact that S bends towards N at all points in all directions in S_p if S is inward oriented. If outward, then always bend away from N in all directions. So the product $k(p)>0$.

12.11 $Z = \left(p, \frac{2x_1}{a^2}, \frac{-2x_2}{b^2}, -1\right)$, $V_1 = \left(p, 1, 0, \frac{2x_1}{a^2}\right)$, $V_2 = \left(p, 0, 1, \frac{-2x_2}{b^2}\right)$

$$\det\begin{pmatrix}\nabla_{v_1}Z\\ \nabla_{v_2}Z\\ Z(p)\end{pmatrix} = \begin{vmatrix}2/a^2 & 0 & 0\\ 0 & -2/b^2 & 0\\ 2x_1/a^2 & -2x_2/b^2 & -1\end{vmatrix} = 4/a^2b^2, \quad \det\begin{pmatrix}V_1\\ V_2\\ Z(p)\end{pmatrix} = \begin{vmatrix}1 & 0 & 2x_1/a^2\\ 0 & 1 & -2x_2/b^2\\ 2x_1/a^2 & -2x_2/b^2 & -1\end{vmatrix} = -1-4x_1^2/a^4-4x_2^2/b^4$$

$$\|Z(p)\|^2 = 1+\frac{4x_1^2}{a^4}+\frac{4x_2^2}{b^4}$$

$k(p) = -4/a^2b^2\left(1+\frac{4x_1^2}{a^4}+\frac{4x_2^2}{b^4}\right)^2<0$ hard to plot and analyze its shape but look at the graph at ~~http://here~~ http://users.rsise.anu.edu.au/~xzhang/reading/ex1211.jpg

12.12 (a) Cylinder C: $g(x_1,x_2,x_3) = \underbrace{f(x_1,x_2)}_{=f(x,y)}$, $Z = \nabla g(p) = (f'_x, f'_y, 0)$,

$V_1 = (0,0,1)$. $V_2 = (f'_y, f'_x, 0)$ $\nabla_{v_1}Z = (0,0,0)$, So ~~Z~~ $k(p)=0$ by Thm 5.

(b) $g(x_1\cdots x_{n+1}) = f(x_1\cdots x_n)$ $\quad Z = \nabla g(p) = (f'_{x_1}, \cdots, f'_{x_n}, 0)$

$V_1 = (0,\cdots,0,1)$ ~~V~~ and then decide $V_2\cdots V_n$. $\nabla_{V_1}Z = (0,\cdots,0)$ So $k(p)=0$.

12.13 ~~Z~~ $f = x_{n+1}-g(x_1\cdots x_n)$ $\quad Z = \nabla f(p) = (-g'_1,\cdots,-g'_n,1)$, (So $Z\cdot(0,0\cdots 0,1)>0$).

$V_1 = (1,0,\cdots,0,g'_1), \cdots, V_n = (0,\cdots,0,1,g'_n)$, $\nabla_{V_1}Z = (-g''_{11},\cdots,-g''_{n1},0), \cdots, \nabla_{V_n}Z = (-g''_{n1},\cdots,-g''_{nn},0)$

$$\det\begin{pmatrix}\nabla_{V_1}Z\\ \vdots\\ \nabla_{V_n}Z\\ Z(p)\end{pmatrix} = \begin{vmatrix}-g''_{11} & \cdots & -g''_{nn} & 0\\ \vdots & & \vdots & \vdots\\ -g''_{n1} & \cdots & -g''_{nn} & 0\\ -g'_1 & \cdots & -g'_n & 1\end{vmatrix} = (-1)^n\det\left(\frac{\partial^2 g}{\partial x_i\partial x_j}\right), \quad \det\begin{pmatrix}V_1\\ \vdots\\ V_n\\ Z(p)\end{pmatrix} = \begin{vmatrix}1 & 0 & \cdots & 0 & g'_1\\ & \ddots & & & \vdots\\ 0 & & \cdots & 1 & g'_n\\ -g'_1 & \cdots & & -g'_n & 1\end{vmatrix} = 1+\sum_{i=1}^{n}\left(\frac{\partial g}{\partial x_i}\right)^2 = \|Z_p\|^2$$

↑ easy proof by induction

$$k(p) = (-1)^n\cdot(-1)^n\det\left(\frac{\partial^2 g}{\partial x_i\partial x_j}\right)\Big/\left(1+\sum_{i=1}^{n}\left(\frac{\partial g}{\partial x_i}\right)^2\right)^{\frac{n}{2}+1} = \det\left(\frac{\partial^2 g}{\partial x_i\partial x_j}\right)\Big/\left(1+\sum_{i=1}^{n}\left(\frac{\partial g}{\partial x_i}\right)^2\right)^{1+n/2}$$

12.14. ~~If $v\times w=0$ then $\exists\lambda\in R$ $v=\lambda w$, $L_p(v)\times L_p(w) = \lambda L_p(w)\times L_p(w) = 0 = k(p)\, v\times w$~~

Both $L_p(v)\times L_p(w)$ and ~~Otherwise~~ $v\times w \in S_p^\perp$ (even if $v\times w=0$ i.e. $v//w$). So to prove the result, one only needs to prove that $N(p)\cdot L_p(v)\times L_p(w) = N(p)\cdot v\times w$, where $N(p)$ is Gauss map $\|N(p)\|=1$

By Thm 5, $k(p) = \begin{vmatrix}-L_p(v)\\ -L_p(w)\\ N(p)\end{vmatrix} \Big/ \|N(p)\|^2 \begin{vmatrix}v\\ w\\ N(p)\end{vmatrix}$ So

$N(p)\cdot L_p(v)\times L_p(w) = \begin{vmatrix}L_p(v)\\ L_p(w)\\ N(p)\end{vmatrix} = k(p)\cdot\begin{vmatrix}v\\ w\\ N(p)\end{vmatrix} = k(p)$ ~~$v\times w$~~ $N(p)\cdot v\times w$.

12.15. By Thm 5, $k(p) = \begin{vmatrix}\nabla_v Z\\ \nabla_w Z\\ Z(p)\end{vmatrix} \Big/ \|Z(p)\|^2 \begin{vmatrix}v\\ w\\ Z(p)\end{vmatrix} = Z(p)\cdot\nabla_v Z\times\nabla_w Z \,/\, \|Z(p)\|^4$

as $\begin{vmatrix}v\\ w\\ Z(p)\end{vmatrix} = Z(p)\cdot v\times w = Z(p)\cdot Z(p) = \|Z(p)\|^2$

$\begin{vmatrix}\nabla_v Z\\ \nabla_w Z\\ Z(p)\end{vmatrix} = Z(p)\cdot v\times w$

12.16 By Thm 2, the eigenvectors of L comprises an orthonormal basis for S_p. Let them be $(\alpha_1, \cdots \alpha_n)$ with corresponding eigenvalues $k_1(p) \cdots k_n(p)$. (Let $V = (V_1, \cdots V_n) = (\alpha_1 \cdots \alpha_n)T$ [A]. By Thm 3, ~~$k(V_i) = \sum_{j=1}^n k_j(p) t_i$~~ As $V_i = \sum_{j=1}^n \alpha_j t_{ji}$, so $k(V_i) = \sum_{j=1}^n k_j(p) t_{ji}^2$

So $\sum_{i=1}^n k(V_i) = \sum_{i,j=1}^n k_i(p) t_{ij}^2 = \sum_{i=1}^n k_i(p) \cdot \sum_{j=1}^n t_{ij}^2$, As Both V and A are orthonormal, $I = V^T V = T^T A^T A T = T^T T$, So T is also orthonormal. So $T T^T = I$ (I is identity)

So $\sum_{j=1}^n t_{ij}^2 = 1$ for all $i = 1 \cdots n$. So $\sum_{i=1}^n k(V_i) = \sum_{i=1}^n k_i(p)$, thus $H(p) = \frac{1}{n}\sum_{i=1}^n k_i(p) = \frac{1}{n}\sum_{i=1}^n k(V_i)$

12.17 (a) Obvious by Thm 3. Anyway $L(V(\theta)) = (\cos\theta) L(V_1) + (\sin\theta) L(V_2)$

$k(V(\theta)) = L(V(\theta)) \cdot V(\theta) = \cos^2\theta \cdot L(V_1) \cdot V_1 + \sin^2\theta \cdot L(V_2) V_2$

$+ \sin\theta\cos\theta \, (L(V_1) \cdot V_2 + L(V_2) \cdot V_1)$

$L(V_1) \cdot V_2 = k_1 \cdot V_1 \cdot V_2 = 0$. $L(V_2) \cdot V_1 = 0$.

So $k(V(\theta)) = \cos^2\theta \cdot k_1 + \sin^2\theta \cdot V_2$

(b) $H_p = \frac{1}{2}(k_1 + k_2)$, $\frac{1}{2\pi}\int_0^{2\pi} k(V(\theta))\,d\theta = \frac{1}{2\pi}\left[k_1 \int_0^{2\pi}\cos^2\theta\, d\theta + k_2 \int_0^{2\pi} \sin^2\theta\, d\theta\right] = \frac{1}{2}(k_1 + k_2) = H_p$.

12.18 $\operatorname{div} N = \operatorname{tr}(v \mapsto \nabla_v N) = \operatorname{tr}(-L_p)$ ~~$= -\operatorname{tr}(L_p)$~~

If $V_1 \cdots V_n$ ($\lambda_1 \cdots \lambda_n$) are eigenvalues of L_p, then $\pm V_1, \cdots \pm V_n$ ($-\lambda_1 \cdots -\lambda_n$) are eigenvalues of $-L_p$ because

$L_p(V_i) = \lambda_i V_i \Leftrightarrow -L_p(V_i) = -\lambda_i V_i$ So $\operatorname{tr}(-L_p) = \sum_{i=1}^n (-\lambda_i) = -\operatorname{tr}(L_p)$

So $H_p = \frac{1}{n}\operatorname{tr}(L_p) = \frac{-1}{n}\operatorname{div} N$

12.19 (a) $\tilde{S}$ is $g^{-1}(c)$. $\nabla g(p) = 0 \Leftrightarrow \frac{1}{a}\nabla f(p/a) = 0$

But S is n-surface, so $\nabla f(p/a) \neq 0$ for all p and thus $\nabla g(p) \neq 0 \ \forall p$, so $\tilde{S}$ is n-surface

$p \in S \Leftrightarrow f(p) = c \Leftrightarrow g(ap) = f(p) = c \Leftrightarrow ap \in \tilde{S}$

(b) $\forall N$ in the Gauss image of S, $\exists p$, s.t. $\nabla f / \|\nabla f\| \big|_p = N$. But $\nabla g(ap) = \frac{1}{a}\nabla f(p)$

$\nabla g / \|\nabla g\| \big|_{ap} = \frac{1}{a}\nabla f(p) / \|\frac{1}{a}\nabla f(p)\| = \nabla f / \|\nabla f\| = N$. So N is also in Gauss Image of $\tilde{S}$

$\forall N$ in Gauss image of $\tilde{S}$, $\exists q$ s.t. $\nabla g(q) / \|\nabla g(q)\| = N$. But $\nabla g(q) = \frac{1}{a}\nabla f(q/a)$

$\nabla f(q/a) / \|\nabla f(q/a)\| = a\nabla g(q) / \|a \nabla g(q)\| = \nabla g(q) / \|\nabla g(q)\| = N$.

So the spherical images of S and $\tilde{S}$ are the same

(c) $\forall v \in S_p$. $k(v) = -\nabla_v N \cdot v$, $\nabla_v N = (\nabla N_1(p) \cdot v, \cdots, \nabla N_{n+1}(p) \cdot v)^T$. v

$\nabla N_i(p) = \left(\frac{\partial N_i}{\partial x_1}(p), \cdots, \frac{\partial N_i}{\partial x_{n+1}}(p)\right)$. As short hand, denote $\nabla f = (f'_1, \cdots, f'_{n+1})$

So $\frac{\partial N_i}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{f'_i}{\|\nabla f\|} = \frac{1}{\|\nabla f\|^2}\left(f''_{ij}\|\nabla f\| - f'_i \cdot \frac{1}{\|\nabla f\|}\sum_{k=1}^n f'_k f''_{kj}\right) = \frac{1}{\|\nabla f\|^3}\left(f''_{ij}\|\nabla f\|^2 - f'_i \sum_{k=1}^n f'_k f''_{kj}\right)\Big|_p$ (1)

$\forall v \in \tilde{S}_{ap}$, $\tilde{k}(v) = -\nabla_v \tilde{N} \cdot v$. Using similar notation

$\frac{\partial \tilde{N}_i}{\partial x_j} = \frac{1}{\|\nabla g\|^3}\left(g''_{ij}\|\nabla g\|^2 - g'_i \sum_{k=1}^n g'_k f''_{kj}\right)\Big|_p$ (2) But $g(p) = f(p/a)$.

So $\nabla g(p) = \frac{1}{a}\nabla f(p/a)$, i.e. $\nabla g(ap) = \frac{1}{a}\nabla f(p)$, $g''_{ij}(p) = \frac{1}{a^2} f''_{ij}(p/a)$, i.e. $g''_{ij}(ap) = \frac{1}{a^2} f''_{ij}(ap)$

So $\frac{\partial \tilde{N}_i}{\partial x_j}\Big|_{ap} = \frac{1}{a}\frac{\partial \tilde{N}_i}{\partial x_j}\Big|_p$ (by plugging into (11, 12)). So $\tilde{k}(v) = k(v)/a$, which is ~~also~~ true at all the (shared) stationary points. on $\|v\|=1$

But mean curvature H is the average of k at stationary points, thus $H(ap) = \frac{1}{a}H(p)$.

(d) K (Gauss-Kronecker curvature) is the product of $k(v)$ at stationary points

So $\tilde{K}(ap) = a^{-n}K(p)$

Remark: Above argument based on stationary points is not strict enough, especially considering the multiplicity of L_p's eigenvalues. A better proof is: ~~$\forall v, w \in S_p$~~ As $\nabla g(ap) = \frac{1}{a}\nabla f(p)$

So $S_p = \tilde{S}_{ap}$. $\forall v, w \in S_p$ ($\tilde{S}_{ap}$). $L_p(v)\cdot w = \frac{1}{2}[k(v+w) - k(v) - k(w)]$

let $\tilde{L}_{ap}$ be Weingarten map on $\tilde{S}$ at ap $\rightarrow$ $\tilde{L}_{ap}(v)\cdot w = \frac{1}{2}[\tilde{k}(v+w) - \tilde{k}(v) - \tilde{k}(w)] = \frac{1}{a}L_p(v)\cdot w$ as $\tilde{k}(\cdot) = k(\cdot)/a$

Since w is arbitrary in S_p, so $\tilde{L}_{ap}(v) = \frac{1}{a}L_p(v)$ $\forall v$. So each eigenvalue λ_i of L_p corresponds to the eigenvalue λ_i/a of $\tilde{L}_{ap}$. As H and K are average/product of eigenvalues, we have

$H(ap) = \frac{1}{a}H(p)$ $\quad K(ap) = a^{-n}K(p)$

(Proof: set $w = \tilde{L}_{ap}(v) - \frac{1}{a}L_p(v) \in S_p$, then one has $(\tilde{L}_{ap}(v) - \frac{1}{a}L_p(v))\cdot(\tilde{L}_{ap}(v) - \frac{1}{a}L_p(v)) = 0$. So $\tilde{L}_{ap}(v) = \frac{1}{a}L_p(v)$)

13.1 If S is convex at p, then h_u ($u = N(p)$ Gauss map) attains local max/min at p. So $\mathcal{H}_p$ is semi-definite, so $\mathcal{L}_p = \pm\mathcal{H}_p$ is also semi-definite. But the Gauss-Kronecker curvature, which is equal to the product of all eigenvalues of $\mathcal{L}_p$, is negative. As S_p is even dimensional, there must be one positive and one negative eigenvalue, which contradicts the fact that $\mathcal{L}_p$ is semi-definite. So S is not convex at p.

13.2 $\forall v, w \in S_p$ $\quad \nabla_v(\text{grad } h)\, w = \nabla_v(\nabla h - (\nabla h\cdot N)N)w = \nabla_v(\nabla h)w - (\nabla h\cdot N)(\nabla_v N\cdot w)$

$\nabla_w(\text{grad } h)\, v = \nabla_w(\nabla h - (\nabla h\cdot N)N)v = \nabla_w(\nabla h)v - (\nabla h\cdot N)(\nabla_w N\cdot v)$

We know L_p is self-adjoint, i.e., $\nabla_v N\cdot w = \nabla_w N\cdot v$. Besides,

$\nabla_v(\nabla h)w = v^T H w = w^T H v = \nabla_w(\nabla h)w$ $\quad$ So $\nabla_v(\text{grad } h)w = \nabla_w(\text{grad } h)v$, so self-adjoint

13.3. (a) $\Rightarrow$ If $\mathcal{Q}$ is pos Def, then $\forall$ eigenvector v (value λ), $\mathcal{Q}(v) = \lambda v$, $\mathcal{Q}(v)\cdot v = \lambda > 0$ as $\mathcal{Q}$ is Pos Def

$\Leftarrow$ We know that the eigenvectors $v_1 \cdots v_n$ make up an orthonormal basis on S_p. $\forall v \in S_p$.

Let $v = \sum_{i=1}^n a_i v_i$ $\quad \mathcal{L}(v)\cdot v = \mathcal{Q}(\sum_{i=1}^n a_i v_i)\cdot \sum_{i=1}^n a_i v_i = (\sum_{i=1}^n a_i \mathcal{Q}(v_i))(\sum_{i=1}^n a_i v_i) = (\sum_{i=1}^n a_i\lambda_i v_i)(\sum_{i=1}^n a_i v_i)$

$= \sum_{i=1}^n a_i^2\lambda_i \geq 0$ because $\lambda_i > 0$.

It is equal to 0 iff ~~$\lambda_i = 0$~~ $a_i = 0$, i.e. $v = 0$

(b) $\Leftarrow$ Since L is self-adjoint linear transformation, its associated matrix L is symmetric so it has two real valued eigenvalues λ_1, λ_2. $\det L > 0 \Rightarrow \lambda_1\lambda_2 > 0$ But if $\lambda_1 < 0, \lambda_2 < 0$, then L is negative definition, i.e. there can't be any v: $\mathcal{Q}(v) > 0$.

thus $\lambda_1 > 0, \lambda_2 > 0$.