by plugging into (11,12)

So $\frac{\partial \tilde{N}_i}{\partial x_j}\Big|_{ap} = \frac{1}{a}\frac{\partial \tilde{N}_i}{\partial x_j}\Big|_p$ So $\tilde{k}(v) = k(v)/a$, which is ~~also~~ true at all the (shared) stationary points. on $\|v\|=1$

But mean curvature H is the average of k at stationary points, thus $H(ap) = \frac{1}{a}H(p)$.

(d) K (Gauss-Kronecker curvature) is the product of $k(v)$ at stationary points

So $\tilde{K}(ap) = a^{-n}K(p)$

Remark Above argument based on stationary points is not strict enough, especially considering the multiplicity of L_p's eigenvalues. A better proof is: ~~$\forall v, w \in S_p$~~ As $\nabla g(ap) = \frac{1}{a}\nabla f(p)$

So $S_p = \tilde{S}_{ap}$. $\forall v, w \in S_p$ ($\tilde{S}_{ap}$). $L_p(v)\cdot w = \frac{1}{2}[k(v+w) - k(v) - k(w)]$ as $\tilde{k}(\cdot) = k(\cdot)/a$

let $\tilde{L}_{ap}$ be Weingarten map on $\tilde{S}$ at ap $\rightarrow \tilde{L}_{ap}(v)\cdot w = \frac{1}{2}[\tilde{k}(v+w) - \tilde{k}(v) - \tilde{k}(w)] = \frac{1}{a}L_p(v)\cdot w$

Since w is arbitrary in S_p, so $\tilde{L}_{ap}(v) = \frac{1}{a}L_p(v)$ $\forall v$. So each eigenvalue λ_i of L_p corresponds to the eigenvalue λ_i/a of $\tilde{L}_{ap}$. As H and K are average / product of eigenvalues, we have

$H(ap) = \frac{1}{a}H(p)$ $\quad K(ap) = a^{-n}K(p)$

Proof: set $w = \tilde{L}_{ap}(v) - \frac{1}{a}L_p(v) \in S_p$

then one has $(\tilde{L}_{ap}(v) - \frac{1}{a}L_p(v))\cdot(\tilde{L}_{ap}(v) - \frac{1}{a}L_p(v)) = 0$. So $\tilde{L}_{ap}(v) = \frac{1}{a}L_p(v)$

13.1 If S is convex at p, then h_u ($u = N(p)$ Gauss map) attains local max/min at p. So $\mathcal{H}_p$ is semi-definite, so $\mathcal{L}_p = \pm\mathcal{H}_p$ is also semi-definite. But the Gauss-Kronecker curvature, which is equal to the product of all eigenvalues of $\mathcal{L}_p$, is negative. As S_p is even dimensional, there must be one positive and one negative eigenvalue, which contradicts the fact that $\mathcal{L}_p$ is semi-definite. So S is not convex at p.

13.2 $\forall v, w \in S_p$ $\quad \nabla_v(\text{grad}\, h)\, w = \nabla_v(\nabla h - (\nabla h\cdot N)N)w = \nabla_v(\nabla h)w - (\nabla h\cdot N)(\nabla_v N\cdot w)$

$\nabla_w(\text{grad}\, h)\, v = \nabla_w(\nabla h - (\nabla h\cdot N)N)v = \nabla_w(\nabla h)v - (\nabla h\cdot N)(\nabla_w N\cdot v)$

We know L_p is self-adjoint, i.e., $\nabla_v N\cdot w = \nabla_w N\cdot v$. Besides,

$\nabla_v(\nabla h)w = v^T H w = w^T H v = \nabla_w(\nabla h)w$ $\quad$ So $\nabla_v(\text{grad}\, h)w = \nabla_w(\text{grad}\, h)v$, so self-adjoint

13.3. (a) $\Rightarrow$ If $\mathcal{Q}$ is pos Def, then $\forall$ eigenvector v (value λ), $\mathcal{Q}(v) = \lambda v$, $\mathcal{Q}(v)\cdot v = \lambda > 0$ as $\mathcal{Q}$ is Pos Def

$\Leftarrow$ We know that the eigenvectors $v_1 \cdots v_n$ make up an orthonormal basis on S_p. $\forall v \in S_p$.

Let $v = \sum_{i=1}^n a_i v_i$ $\quad \mathcal{L}(v)\cdot v = \mathcal{Q}(\sum_{i=1}^n a_i v_i)\cdot\sum_{i=1}^n a_i v_i = \left(\sum_{i=1}^n a_i\mathcal{Q}(v_i)\right)\left(\sum_{i=1}^n a_i v_i\right) = \left(\sum_{i=1}^n a_i\lambda_i v_i\right)\left(\sum_{i=1}^n a_i v_i\right)$

$= \sum_{i=1}^n a_i^2\lambda_i \geq 0$ because $\lambda_i > 0$.

It is equal to 0 iff ~~$\lambda_i = 0$~~ $a_i = 0$, i.e. $v = 0$

(b) $\Leftarrow$ Since L is self-adjoint linear transformation, its associated matrix L is symmetric so it has two real valued eigenvalues λ_1, λ_2. $\det L > 0 \Rightarrow \lambda_1\lambda_2 > 0$ But if $\lambda_1 < 0, \lambda_2 < 0$, then L is negative definition, i.e. there can't be any v: $\mathcal{Q}(v) > 0$.

thus $\lambda_1 > 0, \lambda_2 > 0$.

⇒ by definition $Q(v)>0$ for all $v\neq 0$. As Q is Pos Def. both eigenvalues are positive, thus $\det L = \lambda_1\lambda_2 > 0$

(c) L is non-singular ⟺ $\det L = \prod_{i=1}^{n}\lambda_i \neq 0 \iff \lambda_i \neq 0$ (λ_i are eigenvalues)

Q is non-degenerate ⟺

i.e. p is non-degenerate

$\mathcal{H}_p$ is non-degenerate ⟺ $L: v \mapsto \nabla_v(\operatorname{grad} h)$ is non-singular ⟺ $\lambda_i \neq 0 \iff \forall v\ \nabla_v(\operatorname{grad} h)\neq 0$

13.4 If h is height function or any function which has constant $\|\nabla h\|$, then

$(h\circ\beta)'(t) = \nabla h(\beta(t))\cdot\dot\beta(t) \leq \|\nabla h(\beta(t))\|\cdot\|\dot\beta(t)\| = \|\nabla h(\alpha(t))\|\cdot\|\dot\alpha(t)\|$

$= \nabla h(\alpha(t))\cdot\dot\alpha(t) = (h\circ\alpha)'(t)$.

So $h(\alpha(b)) = h(\alpha(a)) + \int_a^b (h\circ\alpha)'(t) \geq h(\beta(a)) + \int_a^b (h\circ\beta)'(t)\,dt = h(\beta(b))$

Equality holds iff $\nabla h(\beta(t)) = \lambda\dot\beta(t)$ $\lambda\geq 0$. But $\nabla h(\alpha(t)) = \nabla h(\beta(t))$.

So $\|\nabla h(\alpha(t))\| = \lambda\|\dot\beta(t)\| = \lambda\|\dot\alpha(t)\|$ But $\dot\alpha(t) = \nabla h(\alpha(t))$. So $\lambda = 1$

So $\nabla h(\beta(t)) = \dot\beta(t)$, i.e. β is also a gradient line passing thru $\alpha(a)$, but such a line is unique, so $\beta = \alpha$.

this $\tilde h$ is actually the h in the question

If $\|\nabla h\| = \text{const}$ is not guaranteed, WE FEEL that this proposition may not hold. The following is a counter-example. let $\tilde h(x_1,x_2) = h(x_1)$ $f(x_1,x_2) = x_2$

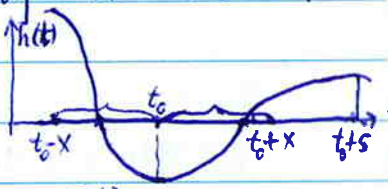
then $\nabla\tilde h = (h'(x_1), 0)$ $\nabla f = (0,1)$ so $S = f^{-1}(0)$ is n-surface. $\nabla\tilde h \perp \nabla f \Rightarrow \underset{=\nabla\tilde h}{\operatorname{grad}\tilde h}$

$\alpha(t), \beta(t) \in S$. So we can write in brief $\alpha(t) = (\alpha(t), 0)$. $\beta(t) = (\beta(t), 0)$

here $R\to R$ $\quad R\to R$

So now $\dot\alpha(t) = h'(\alpha(t))$ $|\dot\beta(t)| = |\dot\alpha(t)|$

As $\beta(t)$ appears in the conclusion only inside $h(\beta(t))$, the only constraint on β is actually $\ell(\beta) = \int_a^b \|\dot\beta(t)\|dt = \int_a^b \|\dot\alpha(t)\|dt = \ell(\alpha)$. Now we check function $h(t) = \frac{1}{t}\sin\frac{1}{t}$ $(t>0)$ so $\tilde h$ and f are defined on (R^+, R) which is open. let $\alpha(a) = t_0$, s.t. $\sin\frac{1}{t_0} = -1$



the first peak to the left of t_0 is $t_0 - X$, where $\frac{1}{t_0-X} = \frac{1}{t_0}+\pi$, $X = \frac{\pi t_0^2}{1+\pi t_0}$.

$h(t_0-X) = \frac{1}{t_0}+\pi$, $h(t_0+X) = \frac{1+\pi t_0}{t_0+2\pi t_0^2}\sin\frac{1+\pi t_0}{t_0+2\pi t_0^2} < \frac{1+\pi t_0}{t_0+2\pi t_0^2} < \frac{1+\pi t_0}{t_0} = h(t_0-X)$.

Besides, the first peak to the right of t_0 is t_0+S, s.t. $\frac{1}{t_0+S} = \frac{1}{t_0}-\pi$, $S = \frac{\pi t_0^2}{1-\pi t_0} > X$

in (t_0, t_0+S) $\dot\alpha > 0$. Now suppose $\alpha(a) = t_0+\varepsilon$ where $\varepsilon \overset{>0}{}$ is sufficiently small, . for $t>a$, $\alpha(t)$ monotonically increases. and $\beta(t)$ is forced to decrease monotonically.

As ε can be arbitrarily small, by above discussion, $\beta(t)$ first reaches t_0-X, while $\alpha(t)$ hasn't reached t_0+S, i.e. $\dot\alpha = h\circ\alpha > h'(t_0+\varepsilon)$ guarantees that α has enough impetus to go right and meanwhile, β goes left. β reaches t_0-X while α only reaches $t_0+X+2\varepsilon$.

Suppose b is chosen (stop) at such a moment, then we have $h(\alpha(b)) < h(\beta(b))$ which contradicts the exercise assertion.

13.5 $\left.\begin{array}{l} h(\beta(t)) = c \Rightarrow \nabla h(\beta(t_1)) \cdot \dot\beta(t_1) = 0 \\ \alpha(t_0) = \beta(t_1) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \nabla h(\alpha(t_0)) \cdot \dot\beta(t_1) = 0 \\ \dot\alpha(t_0) = (\operatorname{grad} h)(\alpha(t_0)) \end{array}\right\}$

$\cancel{\Rightarrow (\operatorname{grad} h)(\alpha(t_0)) \cdot \dot\beta(t_1) = 0 \Rightarrow \nabla h}$

$\Rightarrow \dot\alpha(t_0) \cdot \dot\beta(t_1) = (\operatorname{grad} h)(\alpha(t_0)) \cdot \dot\beta(t_1) = \left(\nabla h(\alpha(t_0)) - (\nabla h(\alpha(t_1)) \cdot N(\alpha(t_0)))\, N(\alpha(t_0)) \right) \cdot \dot\beta(t_1) = 0$

$= \nabla h(\alpha(t_0)) \cdot \dot\beta(t_1) = 0$ As $N(\alpha(t_0)) \cdot \dot\beta(t_1) = N(\beta(t_1)) \cdot \dot\beta(t_1) = 0$.