13.5 $\left.\begin{array}{r} h(\beta(t)) = c \Rightarrow \nabla h(\beta(t_1)) \cdot \dot\beta(t_1) = 0 \\ \alpha(t_0) = \beta(t_1) \end{array}\right\} \Rightarrow \left.\begin{array}{l} \nabla h(\alpha(t_0)) \cdot \dot\beta(t_1) = 0 \\ \dot\alpha(t_0) = (\mathrm{grad}\, h)(\alpha(t_0)) \end{array}\right\}$

$\cancel{\Rightarrow (\mathrm{grad}\, h)(\alpha(t_0)) \cdot \dot\beta(t_1) = 0 \Rightarrow \nabla h}$

$\Rightarrow \dot\alpha(t_0) \cdot \dot\beta(t_1) = (\mathrm{grad}\, h)(\alpha(t_0)) \cdot \dot\beta(t_1) = \left( \nabla h(\alpha(t_0)) - (\nabla h(\alpha(t_0)) \cdot N(\alpha(t_0)))\, N(\alpha(t_0)) \right) \cdot \dot\beta(t_1) = 0$

$= \nabla h(\alpha(t_0)) \cdot \dot\beta(t_1) = 0$ As $N(\alpha(t_0)) \cdot \dot\beta(t_1) = N(\beta(t_1)) \cdot \dot\beta(t_1) = 0$.

14.1 Let $S_1 = f^{-1}(c)$. $S_2 = g^{-1}(d)$, $\alpha(t): I \to S_1$, $\alpha(t_0) = p$. $\dot\alpha(t_0) = v$. As $\varphi(S_1) \subseteq S_2$, $\overset{g(\varphi(\alpha(t)))}{\cancel{g(\alpha(t))}} = d$

So $\nabla g(\varphi(\alpha(t))) \cdot \dot{\overline{\varphi \circ \alpha}}(t) = 0$. But $d\varphi(p,v) = \dot{\overline{\varphi\circ\alpha}}(t_0)$, So $d\varphi(p,v) \perp \nabla g(\varphi(p))$, i.e.

$d\varphi(p,v) \in S_{2\,\varphi(p)}$ So $d\varphi: T(S_1) \to T(S_2)$

14.2 For $\forall\, p \in U_1$, $v \in \mathbb{R}^n$, $d(\psi\circ\varphi)(p,v) = (\psi(\varphi(p)), \nabla f_1(p)\cdot v, \cdots, \nabla f_k(p)\cdot v)$, $f_i(p) = \psi_i(\varphi(p))$

$d\varphi(p,v) = (\varphi(p), \nabla\varphi_1(p)\cdot v, \cdots \nabla\varphi_m(p)\cdot v)$. Let $u = (\nabla\varphi_1(p)\cdot v, \cdots, \nabla\varphi_m(p)\cdot v)$

$d\psi \circ d\varphi(p,v) = (\psi(\varphi(p)), \nabla\psi_1(\varphi(p))\cdot u, \cdots \nabla\psi_k(\varphi(p))\cdot u)$

But $\nabla\psi_i(\varphi(p))\cdot u = \sum_{j=1}^m \frac{\partial\psi_i}{\partial x_j}(\varphi(p))(\nabla\varphi_j(p)\cdot v) = \left(\sum_{j=1}^m \frac{\partial\psi_i}{\partial x_j}(\varphi(p))\cdot\nabla\varphi_j(p)\right)\cdot v$ and

$$\nabla f_i(p) = \left(\frac{\partial\psi_i}{\partial x_1} \cdots \frac{\partial\psi_i}{\partial x_m}\right)\begin{pmatrix} \frac{\partial\varphi_1}{\partial x_1} & \cdots & \frac{\partial\varphi_1}{\partial x_n} \\ \frac{\partial\varphi_m}{\partial x_1} & & \frac{\partial\varphi_m}{\partial x_n}\end{pmatrix} = \left(\sum_{j=1}^m \frac{\partial\psi_i}{\partial x_j}\frac{\partial\varphi_j}{\partial x_1}, \cdots, \sum_{j=1}^m \frac{\partial\psi_i}{\partial x_j}\frac{\partial\varphi_j}{\partial x_1}\right) = \sum_{j=1}^m \left(\frac{\partial\psi_i}{\partial x_j}\frac{\partial\varphi_j}{\partial x_1}, \cdots, \frac{\partial\psi_i}{\partial x_j}\frac{\partial\varphi_j}{\partial x_n}\right)$$

$$= \sum_{j=1}^m \frac{\partial\psi_i}{\partial x_j}(\varphi(p))\cdot\left(\frac{\partial\varphi_j}{\partial x_1}(p), \cdots, \frac{\partial\varphi_j}{\partial x_n}(p)\right) = \sum_{j=1}^m \frac{\partial\psi_i}{\partial x_j}(\varphi(p))\cdot\nabla\varphi_j(p)$$

So $d(\psi\circ\varphi) = d\psi \circ d\varphi$.

14.3. Example 9. $J^T = \begin{pmatrix} -\sin\theta & \cos\theta & \overset{0}{\cancel{\sin\phi}} & \overset{0}{\cancel{\cos\phi}} \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix}$ rank $J = 2$.

Example 10. $J^T = \begin{pmatrix} \cos\frac{\theta}{2}\cos\theta & \cos\frac{\theta}{2}\sin\theta & \sin\frac{\theta}{2} \\ -\sin\theta - \frac{t}{2}\sin\frac{\theta}{2}\cos\theta - t\cos\frac{\theta}{2}\sin\theta, & \cos\theta - \frac{t}{2}\sin\frac{\theta}{2}\sin\theta + t\cos\frac{\theta}{2}\cos\theta, & \frac{t}{2}\cos\frac{\theta}{2} \end{pmatrix} \triangleq \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

$A \triangleq \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = \frac{t}{2}(\cos\theta + \sin^2\theta) + \sin\frac{\theta}{2}\sin\theta$. $B \triangleq \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = \frac{t}{2}\sin\theta(1-\cos\theta) - \sin\frac{\theta}{2}\cos\theta$

If $A = B = 0$ then $(\cos\theta + \sin^2\theta)\frac{-t}{2} = \sin\frac{\theta}{2}\sin\theta$ cross multiply $\times$ we have

$(\cos\theta - 1)\sin\theta \cdot \frac{-t}{2} = \sin\frac{\theta}{2}\cos\theta$

$\frac{t}{2}\sin\frac{\theta}{2}\sin^2\theta(\cos\theta - 1) = \frac{t}{2}\sin\frac{\theta}{2}\cos\theta(\cos\theta + \sin^2\theta)$. i.e. $\frac{t}{2}\sin\frac{\theta}{2} = 0$. So $t = 0$ or $\theta = 2k\pi$ $k \in \mathbb{Z}$

If $t = 0$. $J^T = \begin{pmatrix} \cos\frac{\theta}{2}\cos\theta & \cos\frac{\theta}{2}\sin\theta & \sin\frac{\theta}{2} \\ -\sin\theta & \cos\theta & 0 \end{pmatrix}$, $A^2 + B^2 + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^2 = 1$ So rank $J = 2$.

If $\overset{t\neq 0}{\theta} = 2k\pi$, $J^T = \begin{pmatrix} \cos k\pi & 0 & 0 \\ 0 & 1 + t\cos k\pi & \frac{t}{2}\cos k\pi \end{pmatrix}$, $A^2 + B^2 + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^2 = (1+t)^2 + \frac{1}{4}t^2 > 0$ So rank $J = 2$.

In all, rank $J = 2$ for all $t, \theta$.

14.4 Let $\alpha: I \to \mathbb{R}^2$ be a parametrized curve $\alpha(t) = \begin{pmatrix} \alpha_1(t) \\ \alpha_3(t) \end{pmatrix}\begin{matrix} x_1 \\ x_3 \end{matrix}$, then the parametrized surface obtained by rotating about $x_3$-axis is $(\alpha_1(t)\cos\theta, \alpha_1(t)\sin\theta, \alpha_3(t))$. In Example 4 $\alpha(\phi) = \begin{pmatrix} r\sin\phi \\ r\cos\phi \end{pmatrix}$.

Example 8 $\alpha(\phi) = \begin{pmatrix} a + b\cos\phi \\ b\sin\phi \end{pmatrix}$

14.5 (a) $J^T = \begin{pmatrix} \cos\phi\sin\theta\sin\psi & -\sin\phi\sin\theta\sin\psi & 0 & 0 \\ \sin\phi\cos\theta\sin\psi & \cos\phi\cos\theta\sin\psi & -\sin\theta\sin\psi & 0 \\ \sin\phi\sin\theta\cos\psi & \cos\phi\sin\theta\cos\psi & \cos\theta\cos\psi & -\sin\psi \end{pmatrix} \triangleq (A_1, A_2, A_3, A_4)$

$|A_1\,A_2\,A_3|^2 + |A_1A_2A_4|^2 + |A_1A_3A_4|^2 + |A_2A_3A_4|^2 = 1 + \sin^2\theta\sin^6\psi > 0$, So rank $J = 3$

(b) $(\sin\phi\sin\theta\sin\psi)^2 + (\cos\phi\sin\theta\sin\psi)^2 + (\cos\theta\sin\psi)^2 + \cos^2\psi = 1$

14.6 $J_\psi = \begin{pmatrix} t_{n+1} J_\varphi & \begin{matrix} -a_1 \\ \vdots \\ -a_{n+1} \end{matrix} \\ 0 & -a_{n+2} \end{pmatrix}$ $|J_\psi| = -a_{n+2}\, t_{n+1} |J_\varphi| \neq 0$ (as $t \neq 0$, $a_{n+2} \neq 0$, $|J_\varphi| \neq 0$ by assumption)

14.7 Let $d\varphi(v) = (\varphi(v), u) = (\varphi(v), \nabla\varphi_1 \cdot v, \cdots, \nabla\varphi_{n+k}\cdot v)$ Let $Y = X\circ\varphi$

$\nabla_v(X\circ\varphi) = (\nabla Y_1 \cdot v, \cdots \nabla Y_{n+k}\cdot v)$ $\quad \nabla_{d\varphi(v)} X = (\nabla X_1 \cdot u, \cdots, \nabla X_{n+k}\cdot u)$

But $\nabla X_i \cdot u = \sum_{j=1}^{n+k} \frac{\partial X_i}{\partial x_j} \nabla\varphi_j \cdot v = \left(\sum_{j=1}^{n+k} \frac{\partial X_i}{\partial x_j}\nabla\varphi_j\right)\cdot v$

$\nabla Y_i = \left(\frac{\partial X_i}{\partial x_1} \cdots \frac{\partial X_i}{\partial x_{n+k}}\right)\begin{pmatrix} \frac{\partial\varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_{n+k}} \\ \vdots & & \vdots \\ \frac{\partial\varphi_{n+k}}{\partial x_1} & \cdots & \frac{\partial\varphi_{n+k}}{\partial x_{n+k}} \end{pmatrix} = \sum_{j=1}^{n+k}\frac{\partial X_i}{\partial x_j}\nabla\varphi_j =$ So $\nabla X_i \cdot u = \nabla Y_i \cdot v$

i.e. $\nabla_v(X\circ\varphi) = \nabla_{d\varphi(v)} X$

14.8 (a) $\|N\| = 1$. $\det\begin{pmatrix} E_1 \\ E_2 \\ N(p)\end{pmatrix} = \|E_1 \times E_2\| > 0$ as $E_1, E_2$ are linearly independent for parametrized 2-surface.

$N \perp E_1$, $N \perp E_2$, $E_1, E_2$ form a basis for $d\varphi_p$. So $N \perp \operatorname{Image} d\varphi_p$. So $N$ is orientation vector field

As for uniqueness. $N \perp E_1$, $N \perp E_2 \Rightarrow N = \lambda \cdot E_1 \times E_2$, then $\|N\| = 1 \Rightarrow \lambda = \pm E_1 \times E_2$.

then $\det\begin{vmatrix} E_1 \\ E_2 \\ N \end{vmatrix} > 0 \Rightarrow N = E_1 \times E_2 / \|E_1 \times E_2\|$.

(b) $E_1, E_2$ are smooth wrt $p$ as they are just the $i$-th column of ~~Jabo~~ Jacobian, So $N$ is smooth

14.9 (a) Look at matrix $A = \begin{pmatrix} E_1 \\ \vdots \\ E_n \\ X \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n+1} \\ E_{n1} & E_{n2} & \cdots & E_{nn+1} \\ X_1 & X_2 & \cdots & X_{n+1} \end{pmatrix}$ Let $A_i = \begin{pmatrix} E_{11} & \cdots & E_{1,i-1} & E_{1,i+1} & \cdots & E_{1,n+1} \\ \vdots \\ E_{n1} & \cdots & E_{n,i-1} & E_{n,i+1} & \cdots & E_{n,n+1} \end{pmatrix}$ $i = 1, 2, \cdots, n+1$ ~~So det A =~~

So $\det A = \sum_{i=1}^{n+1} (-1)^{n+1+i} \det A_i X_i = \sum_{i=1}^{n} (\det A_i)^2$ as $X_i = (-1)^{n+1+i}\det A_i$.

So $\det A \geq 0$ and $\det A = 0$ iff $\det A_i = 0$ for all $i = 1 \cdots n+1$. But that contradicts the fact that $\varphi$ is a parametrized $n$-surface, i.e. Jacobian is non-singular. So $\det A > 0$

If $X(p) = 0$, then $|A| = 0$ which is impossible. Hence $X(p) \neq 0$ for all $p \in U$.

(b) For $i = 1 \cdots n$, $E_i \cdot X = \det\begin{pmatrix} E_1 \\ \vdots \\ E_i \end{pmatrix} = 0$ So $X \perp E_i$ So $X \perp \operatorname{Image} d\varphi_p$. So $X$ is normal vector field along $\varphi$

(c) Combinging (b), $\det A > 0$, and $\|N\| = 1$. We have $N$ is orientation vector field along $\varphi$.

(d) $X_i$ is ~~continuous~~ smooth and $X(p) \neq 0$. So $N$ is smooth.

14.10 $E_i(p) = (\varphi(p), 0, \cdots, 1, \cdots 0, \frac{\partial g}{\partial u_i}(p))$. So $E_i(p)\cdot N(p) = 0$ Let $A = \begin{pmatrix} E_1 \\ E_n \\ N \end{pmatrix}$ ~~then~~ before normalization let $a_i = \frac{\partial g}{\partial u_i}(p)$, then

$\det A = \begin{bmatrix} 1 & & a_1 \\ & \ddots & a_n \\ -a_1 & \cdots -a_n & 1 \end{bmatrix} = 1 + \sum_{i=1}^{n} a_i^2 > 0$. $\|N\| = 1$ So $N$ is orientation vector field along $\varphi$

14.11 Proof is essentially similar to proving Thm2 in Chapter 9. Let $v, w \in \mathbb{R}^n_p$ and orientation $N$

We need to prove $L_p(d\varphi_p(v))\cdot d\varphi_p(w) = L_p(d\varphi_p(w))\cdot d\varphi_p(v)$ i.e. $+\nabla_v N\cdot d\varphi_p(w) = \nabla_w N d\varphi_p(v)$

Let $X$ be the one defined in Ex 14.9. then

$\nabla_v N\cdot d\varphi_p(w) = \nabla_v\left(\frac{X}{\|X\|}\right)\cdot d\varphi_p(w) = \left[(\nabla_v X)\frac{1}{\|X\|} + \left(\nabla_v \frac{1}{\|X\|}\right)X\right]d\varphi_p(w) = \frac{1}{\|X\|}\nabla_v X\cdot d\varphi_p(w) = v^T J_X^T(p) J_{\varphi}^{(p)} w / \|X(p)\|$

Similarly $\nabla_w N\cdot d\varphi(v) = w^T J_N^T(p) J_\varphi(p) v / \|X(p)\|$ So we only need to prove that $J = J_X^T J_\varphi$ is symmetric

But $J_{ij} = \sum_{k=1}^{n+1} \frac{\partial X_k}{\partial x_i}\frac{\partial \varphi_k}{\partial x_j}$ $J_{ji} = \sum_{k=1}^{n+1}\frac{\partial X_k}{\partial x_j}\frac{\partial\varphi_k}{\partial x_i}$. Let $J_\varphi = (J_{\varphi_1}, J_{\varphi_2}, \dots, J_{\varphi_n})$. by def Ex 14.9.

$X \perp J_{\varphi i}$ $(i=1\cdots n)$ So $X\cdot J_{\varphi i} = 0$ Taking derivative $J_X^T J_{\varphi i} + H_{\varphi i}^T X = 0$ where

$H_{\varphi i} = \begin{pmatrix} \frac{\partial^2\varphi_1}{\partial x_i\partial x_1} & \cdots & \frac{\partial^2\varphi_1}{\partial x_i \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2\varphi_{n+1}}{\partial x_i\partial x_1} & \cdots & \frac{\partial^2\varphi_{n+1}}{\partial x_i\partial x_n}\end{pmatrix}$ So we have for $j=1\cdots n$. $\sum_{k=1}^{n+1}\frac{\partial X_k}{\partial x_j}\frac{\partial\varphi_k}{\partial x_i} + \sum_{k=1}^{n+1}\frac{\partial^2\varphi_k}{\partial x_i\partial x_j}X_k = 0$

Similarly: $\sum_{k=1}^{n+1}\frac{\partial X_k}{\partial x_i}\frac{\partial \varphi_k}{\partial x_j} + \sum_{k=1}^{n+1}\frac{\partial^2\varphi_k}{\partial x_j\partial x_i}X_k = 0$.

As $\frac{\partial^2\varphi_k}{\partial x_i\partial x_j} = \frac{\partial^2\varphi_k}{\partial x_i \partial x_j}$ So $\sum_{k=1}^{n+1}\frac{\partial X_k}{\partial x_j}\frac{\partial\varphi_k}{\partial x_i} = \sum_{k=1}^{n+1}\frac{\partial X_k}{\partial x_i}\frac{\partial\varphi_k}{\partial x_j}$, i.e. $J_{ij} = J_{ji}$.

14.12 $d\varphi(p,v) = (\varphi(p), \nabla_v\varphi_1, \cdots \nabla_v\varphi_{n+k})$, $J_{d\varphi} = \begin{pmatrix} J_\varphi(p) & 0 \\ \text{sth.} & \begin{matrix}\nabla\varphi_1 \\ \vdots \\ \nabla\varphi_{n+k}\end{matrix}\end{pmatrix} = \begin{pmatrix} J_\varphi(p) & 0 \\ \text{sth} & J_\varphi(p)\end{pmatrix}$ As $J_\varphi(p)$ is full ranked, $J_{d\varphi(p,v)}$ must be full ranked as well

14.13 $\nabla_{e_i}E_j = \nabla_{e_i}\left(\varphi(p), \frac{\partial\varphi_1}{\partial x_j}, \dots \frac{\partial\varphi_{n+k}}{\partial x_j}\right) = \left(\frac{\partial^2\varphi_1}{\partial x_i\partial x_j}, \frac{\partial^2\varphi_2}{\partial x_i\partial x_j}, \dots, \frac{\partial^2\varphi_{n+k}}{\partial x_i\partial x_j}\right) = \nabla_{e_j}E_i$

14.14 (a) Let $\begin{pmatrix} L_p(E_1(p)) \\ \vdots \\ L_p(E_n(p)) \\ N(p)\end{pmatrix} = A^T \begin{pmatrix} E_1(p) \\ \vdots \\ E_n(p)\end{pmatrix}$, $\begin{pmatrix} A^T & \\ & 1\end{pmatrix}\begin{pmatrix}E_1(p) \\ \vdots \\ E_n(p) \\ N(p)\end{pmatrix}$ By definition of $N(p)$, $\begin{pmatrix}E_1(p) \\ E_n(p) \\ N(p)\end{pmatrix} \neq 0$, so

$\det A = \det\begin{pmatrix} L_p(E_1(p)) \\ L_p(E_n(p)) \\ N(p)\end{pmatrix} / \det\begin{pmatrix}E_1(p) \\ E_n(p) \\ N(p)\end{pmatrix}$, as $L_p(E_i(p)) = -\nabla_{e_i}N = -\left(\varphi(p), \frac{\partial N_1}{\partial x_i}(p), \dots, \frac{\partial N_{n+1}}{\partial x_i}(p)\right)$

$= (-1)^n \det\begin{pmatrix}J_N^T \\ N\end{pmatrix} / \det\begin{pmatrix}J_\varphi^T \\ N\end{pmatrix}$ (1)

On the other hand $L_p[E_i(p)]\cdot E_j(p) = -\nabla_{e_i}N\cdot E_j(p) = -\left(\frac{\partial N_1}{\partial x_i}\cdots\frac{\partial N_{n+1}}{\partial x_i}\right)\cdot E_j(p) = -\sum_{k=1}^{n+1}\frac{\partial N_k}{\partial x_i}\frac{\partial\varphi_k}{\partial x_j}$ (*)

$E_i(p)\cdot E_j(p) = \sum_{k=1}^{n+1}\frac{\partial\varphi_k}{\partial x_i}\frac{\partial\varphi_k}{\partial x_j}$

So $\det(L_p[E_i(p)]\cdot E_j(p)) / \det(E_i(p)\cdot E_j(p)) = (-1)^n\det(J_N^T J_\varphi)/\det(J_\varphi^T J_\varphi)$ (2)

Notice $\begin{pmatrix}J_N^T \\ N\end{pmatrix}(J_\varphi\ N^T) = \begin{pmatrix} J_N^T J_\varphi & J_N^T N^T \\ N J_\varphi & N N^T\end{pmatrix}$ By definition of $N$, $NN^T = 1$. $N J_\varphi = 0$ we have

$\det\begin{pmatrix}J_N^T \\ N\end{pmatrix}\cdot\det\begin{pmatrix}J_\varphi^T \\ N\end{pmatrix} = \det(J_N^T J_\varphi)$ (3)

Likewise $\begin{pmatrix}J_\varphi^T \\ N\end{pmatrix}(J_\varphi\ N^T) = \begin{pmatrix} J_\varphi^T J_\varphi & J_\varphi^T N^T \\ N J_\varphi & NN^T\end{pmatrix}$ hence $\det\begin{pmatrix}J_\varphi^T \\ N\end{pmatrix}\det\begin{pmatrix}J_\varphi^T \\ N\end{pmatrix} = \det(J_\varphi^T J_\varphi)$ (4)

$\frac{(3)}{(4)}$ we have $\det\begin{pmatrix}J_N^T \\ N\end{pmatrix}/\det\begin{pmatrix}J_\varphi^T \\ N\end{pmatrix} = \det(J_N^T J_\varphi)/\det(J_\varphi^T J_\varphi)$ then by (1), (2) we prove

$K(p) = \det A = \det[L_p[E_i(p)]\cdot E_j(p)] / \det(E_i(p)\cdot E_j(p))$

(b) ~~$\nabla_{e_i}E_j = \nabla_{e_i}\left(\frac{\partial\varphi_1}{\partial x_j} \cdots \frac{\partial\varphi_{n+1}}{\partial x_j}\right) = \left(\frac{\partial^2\varphi_1}{\partial x_j\partial x_i}\cdots\frac{\partial^2\varphi_{n+1}}{\partial x_j\partial x_i}\right)$~~ As $E_j\cdot N = 0$ we have

$0 = \nabla_{e_i}(E_j N) = \nabla_{e_i}E_j\, N + E_j(p)\nabla_{e_i}N$, So $\nabla_{e_i}E_j\cdot N(p) = (-\nabla_{e_i}N)\cdot E_j(p) = L_p(E_i(p))\cdot E_j(p)$

So $\det[L_p[E_i(p)]\cdot E_j(p)] / \det(E_i(p)\cdot E_j(p)) = \det[\nabla_{e_i}E_j\, N(p)] / \det(E_i(p)\cdot E_j(p))$.

So if $n$ = even number, then whether using $N$ or $-N$ doesn't matter

For Ex 14.15 – 14.18, there's no need to check $N$ or $-N$.

14.15 $J\varphi = \begin{pmatrix} -a\sin\theta\sin\phi & a\cos\theta\cos\phi \\ a\cos\theta\sin\phi & a\sin\theta\cos\phi \\ 0 & -a\sin\phi \end{pmatrix} = (E_1, E_2)$, $N = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$

$L_p(E_1(p)) = -\nabla_{(1,0)} N = -\frac{\partial N}{\partial\theta} = (\sin\theta\sin\phi, -\cos\theta\sin\phi, 0) = \frac{-1}{a} E_1(p)$

$L_p(E_2(p)) = -\nabla_{(0,1)} N = -\frac{\partial N}{\partial\phi} = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi) = \frac{1}{a} E_2$. So $k(p) = \frac{-1}{a^2}$

14.16 $J\varphi = \begin{pmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \\ 1 & 0 \end{pmatrix}$ $N = (\cos\theta, \sin\theta, 0)$ $L_p(E_1(p)) = (0,0,0) = 0 \cdot E_1(p)$ So $k(p) = 0$

14.17 ~~$L_p(E_1$~~ $J\varphi = \begin{array}{c} \begin{array}{cc} E_1 & E_2 \end{array} \\ \begin{pmatrix} \cos\theta & -t\sin\theta \\ \sin\theta & t\cos\theta \\ 0 & 1 \end{pmatrix} \end{array}$, $N = (\sin\theta, -\cos\theta, t)/\sqrt{t^2+1}$

$L_p(E_1(p)) = (-\sin\theta \cdot t(t^2+1)^{\frac{-3}{2}}, \cos\theta\, t(t^2+1)^{\frac{-3}{2}}, (t^2+1)^{\frac{-3}{2}})$

$L_p(E_2(p)) = (\cos\theta(t^2+1)^{\frac{-1}{2}}, \sin\theta(t^2+1)^{\frac{-1}{2}}, 0)$ $\det[E_i(p) \cdot E_j(p)] = t^2+1$.

$\det[L_p(E_i(p)) \cdot E_j(p)] = \begin{vmatrix} 0 & (t^2+1)^{-1/2} \\ (t^2+1)^{-1/2} & 0 \end{vmatrix} = -(t^2+1)^{-1}$ So $k(p) = -(t^2+1)^{-2}$

14.18 $J\varphi = \begin{pmatrix} \cosh t & 0 \\ \sinh t\cos\theta & -\cosh t\sin\theta \\ \sinh t\sin\theta & \cosh t\cos\theta \end{pmatrix}$ $N = (\sinh t, -\cosh t\cos\theta, -\cosh t\sin\theta)/\sqrt{\cosh 2t}$

Using the fact that $\nabla_{e_i} E_j = (\frac{\partial^2\varphi_1}{\partial x_i \partial x_j}, \cdots, \frac{\partial^2\varphi_{n+1}}{\partial x_i \partial x_j})$,

we have $\nabla_{e_1} E_1 = (\sinh t, \cosh t\cos\theta, \cosh t\sin\theta)$ $\nabla_{e_1} E_2 = \nabla_{e_2} E_1 = (0, -\sinh t\sin\theta, \sinh t\cos\theta)$

$\nabla_{e_2} E_2 = (0, -\cosh t\cos\theta, -\cosh t\sin\theta)$

$\det[E_i(p) \cdot E_j(p)] = \cosh 2t \cdot \cosh^2 t$. $\det[\nabla_{e_i} E_j \cdot N(p)] = -\cosh^2 t/\cosh 2t$

So $k(p) = -(\cosh 2t)^{-2}$

14.19 $J\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2x & 2y & 2z \end{pmatrix}$ $N = (2x, 2y, 2z, -1)/\sqrt{1+4(x^2+y^2+z^2)}$. ~~$L_p(E_i(p)) = \frac{\partial N}{\partial x}$~~ $H\varphi_1 = H\varphi_2 = H\varphi_3 = 0$

$H\varphi_4 = \mathrm{diag}(2,2,2)$ So $\nabla_{e_i} E_i = (0,0,0,2)$ for $i = 1,2,3$. $\nabla_{e_i} E_j = (0,0,0,0)$ for $i \neq j$.

$\det[E_i(p) \cdot E_j(p)] = \begin{vmatrix} 1+4x^2 & 4xy & 4xz \\ 4xy & 1+4y^2 & 4yz \\ 4xz & 4yz & 1+4z^2 \end{vmatrix} = 1+4(x^2+y^2+z^2)$.

$\det[\nabla_{e_i} E_j \cdot N(p)] = -8(1+4(x^2+y^2+z^2))^{-3/2}$, So $k(p) = -8(1+4(x^2+y^2+z^2))^{-5/2}$

14.20(a) $J\varphi = \begin{pmatrix} x' & 0 \\ y'\cos\theta & -y\sin\theta \\ y'\sin\theta & y\cos\theta \end{pmatrix}$ $N = (y', -x'\cos\theta, -x'\sin\theta)/(y'^2+x'^2)^{1/2}$

$H\varphi_1 = \begin{pmatrix} x'' & 0 \\ 0 & 0 \end{pmatrix}$, $H\varphi_2 = \begin{pmatrix} y''\cos\theta & -y'\sin\theta \\ -y'\sin\theta & -y\cos\theta \end{pmatrix}$, $H\varphi_3 = \begin{pmatrix} y''\sin\theta & y'\cos\theta \\ y'\cos\theta & -y\sin\theta \end{pmatrix}$

$\nabla_{e_1} E_1 = (x'', y''\cos\theta, y''\sin\theta)$ $\nabla_{e_1} E_2 = \nabla_{e_2} E_1 = (0, -y'\sin\theta, y'\cos\theta)$. $\nabla_{e_2} E_2 = (0, -y\cos\theta, -y\sin\theta)$

So $\det[E_i(p) \cdot E_j(p)] = \begin{vmatrix} x'^2+y'^2 & 0 \\ 0 & y^2 \end{vmatrix} = y^2(x'^2+y'^2)$

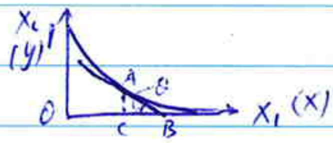
$\det[\nabla_{e_i} E_j \cdot N(p)] = \begin{vmatrix} x''y'-x'y'' & 0 \\ 0 & x'y \end{vmatrix} / (x'^2+y'^2) = (x''y'-x'y'')x'y/(x'^2+y'^2)$

So $k(p) = x'(x''y'-x'y'')/y(x'^2+y'^2)^2$

(b) If $\|\alpha(t)\| = 1$, then $x'^2+y'^2 = 1$ $\ddot\alpha \cdot \dot\alpha = 0$, i.e. $x''x'+y''y' = 0$

So $x'x''y' = -y''y'^2 = -y''(1-x'^2)$. So $k(p) = \frac{1}{y}(-y''+y''x'^2-x'^2y'') = \frac{-1}{y}y''$

14.21 (a) $x'^2+y'^2 = 1-e^{-2t}+e^{-2t} = 1 = \|\dot{\alpha}(t)\|$




(b) $-\tan\theta = y'/x' = -e^{-t}/\sqrt{1-e^{-2t}}$. So $\sin\theta = e^{-t}$

$|AB| = y/\sin\theta = e^{-t}/e^{-t} = 1$.

(c) $k = -y''/y = -e^{-t}/e^{-t} = -1$ by Ex 14.20(b) and $\alpha$ being unit speed.