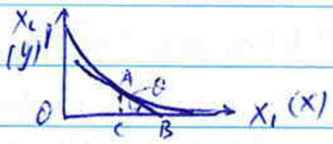
14.21 (a) $x'^2+y'^2 = 1-e^{-2t}+e^{-2t} = 1 = \|\dot\alpha(t)\|^2 = \|\dot\alpha(t)\|$

(b) $-\tan\theta = y'/x' = -e^{-t}/\sqrt{1-e^{-2t}}$. So $\sin\theta = e^{-t}$

$|AB| = y/\sin\theta = e^{-t}/e^{-t} = 1$.

(c) $k = -y''/y = -e^{-t}/e^{-t} = -1$ by Ex 14.20(b) and α being unit speed.



15.1 For $(x_1, x_2, \cdots x_n, 0) \in R^n$. (Equatorial hyperplane). Solve

$\|t(x_1,\cdots,x_n,0)+(1-t)(0,\cdots,0,-1)\|=1$, i.e. $\|(tx_1,\cdots,tx_n,t-1)\|=1$

So $t^2(x_1^2+\cdots+x_n^2)+(t-1)^2=1$. If $t\neq 0$, then $t=2\left(\sum_{i=1}^n x_i^2+1\right)^{-1}$

So $\varphi(x_1,\cdots x_n,0) = \left(2x_1,\cdots,2x_n,1-\sum_{i=1}^n x_i^2\right)/\left(\sum_{i=1}^n x_i^2+1\right)$

15.2 For $(x_1\cdots x_n,-1)$. Solve $\|(1-t)(0,0,\cdots 0,1)+t(x_1\cdots x_n,-1)\|=1$, so $t=4\left(4+\sum_{i=1}^n x_i^2\right)^{-1}$

So $\varphi(x_1\cdots x_n,-1) = \left(4x_1,\cdots,4x_n,\sum_{i=1}^n x_i^2-4\right)/\left(\sum_{i=1}^n x_i^2+4\right)$

15.3 (a) If $v(t)\in f^{-1}(c)$. let $(\alpha(t),s(t)) = \psi_V^{-1}(v(t))$. so

$f(v(t)) = f(\psi(\alpha(t),s(t))) = s(t) = c$. So $v(t) = \psi(\alpha(t),c) = \varphi\circ\alpha + c\cdot N\circ\alpha$.

If $\beta_q(s) = v(t)$, i.e. $\varphi(q)+sN(q) = \varphi(\alpha(t))+cN(\alpha(t))$. then since there is a smooth inverse of $\psi|_V$, so $q=\alpha(t)$. $s=c$. then

$v'(t)\cdot\beta_q'(s) = N(q)\cdot((\varphi\dot\circ\alpha)(t)+c\cdot(N\dot\circ\alpha)(t)) = N(\alpha(t))\left((\varphi\dot\circ\alpha)(t)+c\cdot(N\dot\circ\alpha)(t)\right)$

As $\|N(\alpha(t))\|\equiv 1$ So $N(\alpha(t))\cdot N\dot\circ\alpha(t)=0$. By definition, $(N\circ\alpha)(t)\cdot(\varphi\dot\circ\alpha)(t)=0$

So $v'(t)\cdot\beta_q'(s)=0$. i.e. $f^{-1}(c)$ are everywhere orthogonal to the lines $\beta_q(s)$.

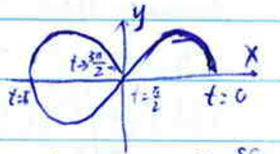
(b) By (a) the vector part of $\nabla f(\psi(q,s)) = \lambda\cdot\beta_q'(s) = \lambda N(q)$.

But $\frac{\partial f}{\partial s}=1$, i.e. $\nabla f\cdot\frac{\partial\psi}{\partial s} = \nabla f\cdot N(q)=1$ So $\lambda=1$ $\nabla f(g)=(g,N(q))$, $g=\psi(q,s)$

15.4 $(x(t),y(t)) = (2\cos t,\sin 2t)$ $t\in(0,\frac{3\pi}{2})$

$(x'(t),y'(t)) = (-2\sin t, 2\cos 2t)\neq(0,0)$ obviously one to one

But when $t\to\frac{3\pi}{2}$, the curve approaches its own point $(0,0)$ crossed at $t=\frac{\pi}{2}$, so not n-surface.



15.5 $\forall(\underset{R^{n+1}}{p},\underset{R^{n+1}}{v})\in T(S)$. $f(p)=c$, $v\cdot N(p)=0$. $J=\begin{pmatrix}\nabla f^T & 0\\ sth. & N(p)\end{pmatrix} = \nabla f\cdot N(p)\neq 0$.

So $T(S)$ is $2n$-surface in R^{2n+2}

15.6 $\forall(p,v)\in T(S)$. $f(p)=c$. $v\cdot N(p)=0$ $v\cdot v=0$ $J=\begin{pmatrix}\nabla f^T & 0\\ \beta & N(p)\\ 0 & 2v\end{pmatrix}$. If $\alpha_1\begin{vmatrix}\nabla f\\ 0\end{vmatrix}+\alpha_2\begin{vmatrix}\beta\\ N(p)\end{vmatrix}+\alpha_3\begin{vmatrix}0\\ 2v\end{vmatrix}=0$

then $\left.\begin{array}{l}\alpha_1\nabla f+\alpha_2\beta=0\\ \alpha_2 N(p)+\alpha_3\cdot 2v=0\end{array}\right\}\Rightarrow \alpha_2=-2v\cdot N(p)=0\Rightarrow\alpha_3=0$

$\Rightarrow$ $\Rightarrow$ $\alpha_1=0$ So independent, thus $T_1(S)$ is $(2n-1)$-surface in R^{2n+2}

15.7 (a) To be in $O(2)$, the matrix must satisfy: $x_1^2+x_2^2-1=0$, $x_1x_3+x_2x_4=0$, $x_3^2+x_4^2-1=0$

$$J=\begin{pmatrix}2x_1 & 2x_2 & 0 & 0\\ x_3 & x_4 & x_1 & x_2\\ 0 & 0 & 2x_3 & 2x_4\end{pmatrix}=(J_1\ J_2\ J_3\ J_4)$$ ~~If $(\alpha_1,\alpha_2,\alpha_3)J=0$~~

$Q=(\det(J_1J_2J_3))^2+(\det(J_1J_3J_4))^2+(\det(J_2J_3J_4))^2+(\det(J_1J_2J_4))^2$

$=16(x_1x_4-x_2x_3)^2\sum_{i=1}^{4}x_i^2$ if $=0$ So ① $\sum x_i^2=0\Rightarrow x_i=0\Rightarrow x_1^2+x_2^2-1\neq 0$ contradiction!

② $x_1x_4=x_2x_3$, So $x_1x_4x_3=x_2x_3^2$, i.e. $-x_2x_4^2=x_2x_3^2$ So $x_2=0\Rightarrow x_1=\pm1\Rightarrow x_4=0$

$\Rightarrow x_3=\pm1\Rightarrow x_1x_3=\pm1$ but $x_2x_4=0\Rightarrow x_1x_3+x_2x_4\neq0$ contradiction

So $Q\neq0$, rank$(J)=3$, $O(2)$ is 1-surface in R^4.

(b) Now $J=\begin{pmatrix}2&0&0&0\\0&1&1&0\\0&0&0&2\end{pmatrix}$ $O(2)_p=\left\{(a,b,c,d)\,\middle|\,J\cdot\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}2&0&0&0\\0&1&1&0\\0&0&0&2\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=0\right\}=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\,\middle|\,\begin{matrix}a=d=0\\b+c=0\end{matrix}\right\}$

Solution 2 Let $\alpha(t)=\begin{pmatrix}\alpha_1(t)\\\alpha_2(t)\end{pmatrix}$, then $\alpha(t)\in O(2)\Leftrightarrow\|\alpha_i(t)\|=1$, $\alpha_i(t)\cdot\alpha_j(t)=0$.

So $\alpha_i'\cdot\alpha_i=0$ So $(a\ b)\begin{pmatrix}1\\0\end{pmatrix}=0\Leftrightarrow a=0$. $(c\ d)\begin{pmatrix}0\\1\end{pmatrix}=0\Leftrightarrow d=0$.

$\alpha_i'\cdot\alpha_j+\alpha_i\cdot\alpha_j'=0\Leftrightarrow(a\ b)\begin{pmatrix}0\\1\end{pmatrix}+(c\ d)\begin{pmatrix}1\\0\end{pmatrix}=0\Leftrightarrow b+c=0$

$\alpha(t_0)=\begin{pmatrix}\alpha_1(t_0)\\\alpha_2(t_0)\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ Let $\alpha'(t_0)=\begin{pmatrix}a&b\\c&d\end{pmatrix}$

So $O(2)_p=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\,\middle|\,\begin{matrix}a=d=0\\b+c=0\end{matrix}\right\}$.

15.8 (a) Prove that J has rank $\frac12 n(n+1)$ by induction on n. For $n=2$ 15.7 has proven it.

Let the $n\times n$ matrix be written as $\begin{pmatrix}\alpha_1\\\vdots\\\alpha_n\end{pmatrix}$, the constraints are $\|\alpha_i\|^2=1$, $\alpha_i\alpha_j=0$ $(i\neq j)$. So Jacobian is

$$\text{rank}\,J_n=\text{rank}\begin{pmatrix}2\alpha_1&&&\\&2\alpha_2&&\\&&\ddots&2\alpha_n\\\alpha_2&\alpha_1&&\\\alpha_n&\cdots&&\alpha_1\\&&\alpha_n&\alpha_{n-1}\end{pmatrix}=\text{rank}\begin{pmatrix}J_{n-1}&0\\0&2\alpha_n\\\alpha_n&\alpha_1\\\vdots&\vdots\\\alpha_n&\alpha_{n-1}\end{pmatrix}$$

~~$=\text{rank}$~~ ($\frac{n(n+1)}{2}$ rows, n^2 columns)

Note the lowest n rows are linearly independent. As if $\exists\,\beta_1\cdots\beta_n\in R$ s.t.

$\beta_n\begin{pmatrix}0\\\vdots\\0\\\alpha_n^T\end{pmatrix}+\beta_1\begin{pmatrix}\alpha_n^T\\0\\\vdots\\0\\\alpha_1^T\end{pmatrix}+\beta_2\begin{pmatrix}0\\\alpha_n^T\\\vdots\\0\\\alpha_2^T\end{pmatrix}+\cdots+\beta_{n-1}\begin{pmatrix}0\\\vdots\\0\\\alpha_n^T\\\alpha_{n-1}^T\end{pmatrix}=0$ So $\beta_i\alpha_n^T=0$ $i=1..n-1$ ①

$\sum_{i=1}^{n}\beta_i\alpha_i^T=0$ ②

As none of the α_i is straight 0, $\beta_i=0$ for $i=1..n-1$ by ①. Then by ② $\beta_n\alpha_n^T=0$ So $\beta_n=0$.

Finally the rows in $(J_{n-1}\ 0)$ (the first $\frac{n(n-1)}{2}$ rows) are independent of the last n rows, because these $\frac{n(n-1)}{2}$ rows all have last n elements straight 0 and none of α_i is straight 0. ($\alpha_1,\cdots,\alpha_n$) are linearly independent

So rank $J_n=n+$ rank $J_{n-1}=n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}$. So $O(n)$ is $\frac{n(n-1)}{2}$ surface in R^{n^2}

(b) Let $\beta=\begin{pmatrix}\beta_1\\\vdots\\\beta_n\end{pmatrix}\in O(n)_p$ then $J\beta=0$, i.e. $\begin{cases}\alpha_i\cdot\beta_i=0 & \Rightarrow\beta_{ii}=0\\\alpha_i\beta_j+\alpha_j\beta_i=0 & \Rightarrow\beta_{ji}+\beta_{ij}=0\end{cases}\Rightarrow O(n)_p=\{P\in R^{n\times n}\mid\forall i,j.\ P_{ij}+P_{ji}=0\}$

If we use the hint in Ex 15.7(b). $\alpha(t)\in O(n)$, $\|\alpha_i(t)\|=1$, $\alpha_i(t)\cdot\alpha_j(t)=0$

So $\alpha_i(t)\cdot\alpha_i'(t)=0$, $\alpha_i(t)\alpha_j'(t)+\alpha_i'(t)\alpha_j(t)=0$

i^{th} element of $\alpha_i'(t)=0$, i^{th} element of $\alpha_j'(t)+j^{th}$ element of $\alpha_i'(t)=0$

which yields the same result/conclusion.

This is because if $\sum_{j}\beta_j\alpha_j=0$, $\beta_i\in R$ then $0=\alpha_i\cdot\sum\beta_j\alpha_j=\beta_i\alpha_i\alpha_i=\beta_i$

15.9 $v\in S_p=\{v\in R^{n+1}_p\mid\nabla f_i(p)\cdot v=0\}\Leftrightarrow\nabla f_i(p)\cdot v=0\ \forall i\Leftrightarrow v\in\ker df_p$.

15.10 (brief proof). Since $J = \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_k \end{pmatrix}$ is fully ranked, so there are k independent columns indexed by $i_1 \cdots i_k$, which form matrix P. So $\det P \neq 0$. Define $\Psi(x_1 \cdots x_n)$ as $\Psi(x_1, \cdots x_{i_1-1}, f_1(x_1 \cdots x_{n+1}), x_{i_1+1}, \cdots, x_{i_k-1}, f_k(x_1 \cdots x_n), x_{i_k+1}, \cdots x_{n+1})$, whose Jacobian J satisfies $\det(J) = \det(P) \neq 0$. Then go on as in proof of Thm 1 by applying inverse function theorem. Finally, $U = \{(u_1 \cdots u_n) \in R^n \mid a_j < u_j < b_j$ for $j < i_1$, $a_{j+1} < u_j < b_{j+1}$ for $j \in [i_1, i_2)$, $\cdots$, $a_{j+k-1} < u_j < b_{j+k-1}$ for $j \in [i_{k-1}, i_k)$, $a_{j+k} < u_j < b_{j+k}$ for $j \geq i_k\}$. and define $\varphi: U \to R^{n+k}$ by

$\varphi(u_1 \cdots u_n) = (\Psi|_V)^{-1}(u_1 \cdots u_{i_1-1}, c_1, u_{i_1+1}, \cdots, u_{i_k-1}, c_k, u_{i_k+1}, \cdots, u_n)$. (elsewhere, just change $n+1$ to $n+k$ in proof of Thm 1)

15.11 (brief proof). Define $U \times R^k \to R^{n+k}$ by $\psi(q, \underbrace{t}_{t_1 \cdots t_k}) = \varphi(q) + \sum_{i=1}^{k} t_i N_i(q)$, where N_i are the vector fields along φ which span the normal space $(\text{Image } d\varphi_q)^\perp$ for each $q \in U$. Then Jacobian $J\psi(p, \underbrace{0 \cdots 0}_{k \text{ 0's}}) = (J\varphi(p), N_1(p) \cdots N_k(p))$ whose determinant $\neq 0$. By the inverse func Thm, there is an open set $V \subset U \times R^k$ about $(p, 0 \cdots 0)$ such that the restriction $\psi|_V$ of ψ to V maps V one to one onto the open set $\psi(V)$, and $(\psi|_V)^{-1}$ is smooth. By shrinking V if necessary, we may assume $V = U_1 \times I^k$ for some open set $U_1 \subseteq U$ containing p and some interval $I \subseteq R$ containing 0. Now define $f: \text{Image } \psi|_V \to R^k$ by $f(\psi(q, t_1 \cdots t_k)) = (t_1, \cdots, t_k)$. f is well defined and is smooth because f is the composition of the smooth map $(\psi|_V)^{-1}$ and projection map $U_1 \times I^k \to I^k$. The level set $f^{-1}(0, \cdots 0)$ is just $\varphi(U_1)$, because $f^{-1}(0) = \{\psi(q, t_1 \cdots t_k) \mid q \in U_1, t_i = 0\} = \{\varphi(q) \mid q \in U_1\}$. Finally we prove that $Jf(g)$ is fully ranked for $g = \psi(q, 0 \cdots 0) \in f^{-1}(0 \cdots 0)$. Let $\alpha_i(s) = \varphi(q) + s \cdot N_i(q)$ then $\nabla f_j(g) \cdot N_i(q)$

$= \nabla f_j(\alpha_i(0)) \cdot \dot{\alpha}_i(0) = \cancel{f_j} (f_j \circ \alpha_i)^{\cdot}(0) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. So

$\underbrace{\begin{pmatrix} \nabla f_1(g) \\ \vdots \\ \nabla f_k(g) \end{pmatrix}}_{\triangleq A} \cdot \underbrace{(N_1(q) \cdots N_k(q))}_{B} = I_k$. ~~By definition rank $(N_1(q) \cdots N_k(q)) = k$~~ To be fast, let's quote a matrix result: $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$

Now $k \leq \min(\text{rank}(A), \text{rank}(B))$. But both $\text{rank}(A)$ and $\text{rank}(B) \leq k$ thus.

$\text{rank}(A) = \text{rank}(B) = k$, i.e., A (Jacobian) is fully ranked.

To prove rank $Jf(g) = k$, another way is: assume $\sum_{i=1}^{k} \beta_i \nabla f_i(g) = 0$, then $\sum_{i=1}^{k} \beta_i (\nabla f_i(g) B) = 0$. But $\nabla f_i(g) \cdot B$ is just the i^{th} row of I_k. So $\beta_i = 0$ for all $i = 1 \cdots k$, i.e. $\{\nabla f_1(g), \cdots, \nabla f_k(g)\}$ are independent. Thus $\varphi(U_1) = f^{-1}(0 \cdots 0)$ is an n-surface in R^{n+k}

~~15.12~~

15.12 (a) $\varphi(p + tv) = \left(2(x_1 + tv_1), \cdots, 2(x_n + tv_n), \sum_{i=1}^{n}(x_i + tv_i)^2 - 1\right) \Big/ \left(1 + \sum_{i=1}^{n}(x_i + tv_i)^2\right)$

$\frac{d}{dt}\Big|_0 \varphi(p + tv) = \left(2v_i\left(\sum_{j=1}^{n} x_j^2 + 1\right) - 4x_i \sum_{j=1}^{n} x_j v_j \text{ for } i = 1 \cdots n,\ -4\sum_{i=1}^{n} x_i v_i\right) \Big/ \left(\sum_{j=1}^{n} x_j^2 + 1\right)^2$

So $\|d\varphi(v)\|^2 = \|\frac{d}{dt}|_0 \varphi(p + tv)\|^2 = 4\left\{\sum_{i=1}^{n}\left[v_i\left(\sum_{j=1}^{n} x_j^2 + 1\right) - 2x_i \sum_{j=1}^{n} x_j v_j\right]^2 + 4\left(\sum_{i=1}^{n} x_i v_i\right)^2\right\} \Big/ \left(\sum_{j=1}^{n} x_j^2 + 1\right)^4$

$= 4\left(\sum_{j=1}^{n} x_j^2 + 1\right)^{-2} \|v\|^2$ So $\lambda(p) = \frac{2}{\|p\|^2 + 1}$

(b) $d\varphi(v) \cdot d\varphi(w) = \frac{1}{4}\left(\|d\varphi(v) + d\varphi(w)\|^2 - \|d\varphi(v) - d\varphi(w)\|^2\right)$ then by linearity of $d\varphi_p$

$= \frac{1}{4}\left(\|d\varphi(v+w)\|^2 - \|d\varphi(v-w)\|^2\right) = \frac{1}{4}\lambda^2(p)\left(\|v+w\|^2 - \|v-w\|^2\right) = \lambda^2(p)\cdot v\cdot w.$

15.13 Let $\tilde{S} = \{q \in S \mid q$ can be joined to p by a continuous curve in $S\}$. Let $S = f^{-1}(c)$.
First $\tilde{S}$ is obviously connected. $\forall q_1, q_2 \in \tilde{S}$, just concatenate their curve joining p will yield a continuous curve between q_1 and q_2. Since $\tilde{S} \subseteq S$, so $\forall q \in \tilde{S}$, $\nabla f(q) \neq 0$. Now we only need to prove that ~~there~~ there is an open set U, s.t. ~~$\tilde{S} = \{x \in U \mid f(x) = c\}$~~. $\tilde{S} = \{x \in U \mid f(x) = c\}$. We mimic the proof of Thm 3.
For each $q \in \tilde{S} \subseteq S$, let $\varphi_q : U_q \to S$ be a local parametrization of S whose image contains q and let $\psi_q : U_q \times R \to R^{n+1}$ be defined by $\psi_q(r,s) = \varphi_q(r) + s\,N(\varphi_q(r))$, where N is the orientation of S. Then as in the proof of Thm 2, we can find an open set V_q about $(\varphi_q^{-1}(q), 0)$ in $U_q \times R$ s.t. $\psi_q|_{V_q}$ maps V_q one to one onto and open set W_q in R^{n+1}, and $(\psi_q|_{V_q})^{-1} : W_q \to V_q$ is smooth. Furthermore by shrinking V_q if neccssary, we may assume that $\psi_q(r,s) \in S$ for $(r,s) \in V_q$ iff $s = 0$. ~~Since V_q is an open set,~~ then for any $u \in W_q \cap S$, let $(\psi_q|_{V_q})^{-1}(u) = (r, 0)$ $r \in U_q'$. ~~there must be a unique~~ must be 0. Since U_q' is open and connected, there is a smooth curve $\alpha(t) : [a,b] \to U_q'$ s.t. ~~$\alpha(t) \in V_q$~~. $\alpha(a) = \varphi_q^{-1}(q)$, $\alpha(b) = r$, (actually we should define $\alpha(t)$ in an open set containing $[a,b]$). Since ~~$\alpha(t) \in V_q$~~

By shrinking V_q further, we may assume that $V_q = U_q' \times I$ where $U_q' \subseteq U_q$, $I \subseteq R$, $0 \in I$, $\varphi_q^{-1}(q) \in U_q'$, U_q' open, I open, U_q' connected

As $\alpha(t) \in U_q'$ and $0 \in I$, we have $\beta(t) \triangleq \psi_q(\alpha(t); 0) \in W_q \cap S$. So $\beta(b) = u$ is connected to $\beta(a) = q$ through a continuous curve on S, so $u \in \tilde{S}$. In other words, for $\forall q \in \tilde{S}$, there is an open set W_q about q, s.t. $W_q \cap S \subseteq \tilde{S}$.
Now we define $U = \bigcup_{q \in \tilde{S}} W_q$ which is open, ~~then $\tilde{S} \subseteq U$ by definition~~.
① $\forall x \in \tilde{S}$, we have $x \in U$, $f(x) = c$. So $x \in \{x \in U \mid f(x) = c\}$, so $\tilde{S} \subseteq \{x \in U \mid f(x) = c\}$
② $\forall x \in \{x \in U \mid f(x) = c\}$, there must be a $q \in \tilde{S}$, s.t. $x \in W_q$. As $x \in S$, so $x \in W_q \cap S \subseteq \tilde{S}$. Thus, $\{x \in U \mid f(x) = c\} \subseteq \tilde{S}$.
Hence $\tilde{S} = \{x \in U \mid f(x) = c\}$, i.e. $\tilde{S}$ is n-surface.

15.14 Suppose $\alpha(t_1) = \alpha(t_2)$ for some $t_1 \neq t_2 \in I$. As the maximal integral curve of X through $\alpha(t_1)$ is unique, ~~then~~ denoted as $\beta(t)$ and $\beta(0) = \alpha(t_1)$, then $\alpha(t) = \beta(t - t_1)$ and $\alpha(t) = \beta(t - t_2)$ for all $t \in I$. Setting $\tau = t_2 - t_1$, we have $\alpha(t) = \beta(t - t_1) = \beta(t + \tau - t_2) = \alpha(t + \tau)$ for all t such that both t and $t + \tau \in I$.
Thus if α is not one to one then it is periodic.
To prove that the maximal integral curve X through $\alpha(t_1)$ is ~~unique~~ constrained in C, we notice

that first the restriction of X to C is a tangent vector field on C, because $X \cdot \nabla f_i = (\nabla f_1 \times \nabla f_2) \cdot \nabla f_i = 0$ for $i=1,2$.

So $(f_i \circ \alpha)'(t) = \nabla f_i(\alpha(t)) \cdot \dot{\alpha}(t) = \nabla f_i(\alpha(t)) \cdot X(\alpha(t)) = 0$. So $f_i \circ \alpha$ is constant, thus $\alpha(t) \in C$.

~~To make~~ α map I onto C, ~~first of all~~ iff C ~~must be~~ is connected. The proof is similar to the Thm 1 in Chapter 11, except the construction of rectangle B, we now construct $A = \{ p_0 + s_1 v_1 + s_2 v_2 + ru \mid |s_i| < \varepsilon_i, |r| < \varepsilon_3 \}$ where $v_i = \nabla f_i(p_0)$, $u = X(p_0)$. The $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are chosen ~~as follows. First~~, so that $J\vec{f}(p)$ is fully ranked for all $p \in A$. This is possible because $J\vec{f}(p_0)$ is fully ranked ~~and~~. Denote $g_r(s_1, s_2) = \vec{f}(p_0 + s_1 v_1 + s_2 v_2 + ru)$, then $Jg_r = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \end{pmatrix} (v_1, v_2) = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \end{pmatrix} (\nabla f_1, \nabla f_2)$. As $\mathrm{rank}(P'P) = \mathrm{rank}(P) = \mathrm{rank}(P')$, $\mathrm{rank}\begin{pmatrix} \nabla f_1 \\ \nabla f_2 \end{pmatrix}_p = 2$ for $\forall p \in A$. So Jg_r is fully ranked for all $p \in A$. Applying Inverse Function Thm, if r is chosen such that $\exists\, s_1, s_2$ s.t. $\vec{f}(p_0 + s_1 v_1 + s_2 v_2 + ru) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, then such s_1, s_2 are unique.

Now let $\gamma(t) = p_0 + h_0(t) u + h_1(t) v_1 + h_2(t) v_2$, where $h_0(t) = (\gamma(t) - p_0) \cdot u / \|u\|^2$, $h_i(t) = (\gamma(t) - p_0) \cdot v_i / \|v_i\|^2$ $(i=1,2)$. h_1, h_2, h_0 are all smooth and $h_1(0) = h_2(0) = h_0(0) = 0$. $h_0'(0) = \dot{\gamma}(0) \cdot u / \|u\|^2 = 1$ by definition that $\dot{\gamma}(0) = X(p_0) = u$. So we can choose $t_1 < 0 < t_2$ (small enough) s.t. $h_0'(t) > 0$, set $r_1 = h_0(t_1)$, $r_2 = h_0(t_2)$, then for $\forall r \in (r_1, r_2)$, $\exists!\, t \in (t_1, t_2)$, s.t. $h_0(t) = r$. Now construct $B = \{ p_0 + ru + s_1 v_1 + s_2 v_2 \mid r_1 < r < r_2, |s_i| < \varepsilon_i \ (i=1,2) \}$.

$\forall\, p_0 + ru + s_1 v_1 + s_2 v_2 \in B \cap C$, then it's belonging to $B \Rightarrow \exists!\, t \in (t_1, t_2)$, $h_0(t) = r$.

Its belonging to $C \Rightarrow \exists!\, s_1, s_2$, s.t. $p_0 + ru + s_1 v_1 + s_2 v_2 \in C$. Now that we know $p_0 + h_0(t) u + h_1(t) v_1 + h_2(t) v_2 \in \gamma \subseteq C$, so $s_1 = h_1(t)$, $s_2 = h_2(t)$. So $p_0 + ru + s_1 v_1 + s_2 v_2 \in \gamma$.

i.e. $C \cap B \subseteq \gamma$