that first the restriction of X to C is a tangent vector field on C, because $\overset{\text{for } i=1,2}{X\cdot\nabla f_i} = (\nabla f_1 \times \nabla f_2)\cdot\nabla f_i = 0$

So $(f_i\circ\alpha)'(t) = \nabla f_i(\alpha(t))\cdot\dot\alpha(t) = \nabla f_i(\alpha(t))\cdot X(\alpha(t)) = 0$. So $f_i\circ\alpha$ is constant, thus $\alpha(t)\in C$.

~~To make~~ α map I onto C, ~~first of all~~ iff C ~~must be~~ is connected. The proof is similar to the Thm 1 in Chapter 11, except the construction of rectangle B, we now construct $A = \{p_0 + s_1v_1 + s_2v_2 + ru \mid |s_i| < \varepsilon_i, |r| < \varepsilon_3\}$ where $v_i = \nabla f_i(p_0)$, $u = X(p_0)$. The $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are chosen ~~as follows. First~~, so that $J\vec f(p)$ is fully ranked for all $p\in A$. This is possible because $J\vec F(p_0)$ is fully ranked. Denote $g_r(s_1,s_2) = \vec f(p_0 + s_1v_1 + s_2v_2 + ru)$, then $Jg_r = \begin{pmatrix}\nabla f_1\\ \nabla f_2\end{pmatrix}(v_1, v_2) = \begin{pmatrix}\nabla f_1\\ \nabla f_2\end{pmatrix}(\nabla f_1, \nabla f_2)$. As $\mathrm{rank}(P'P) = \mathrm{rank}(P) = \mathrm{rank}(P')$, $\mathrm{rank}\begin{pmatrix}\nabla f_1\\ \nabla f_2\end{pmatrix}_p = 2$ for $\forall p\in A$. So Jg_r is fully ranked for all $p\in A$. Applying ~~Inverse function Thm~~ Lagrange Mean Value Thm, if r is chosen such that $\exists s_1, s_2$ s.t. $\vec f(p_0 + s_1v_1 + s_2v_2 + ru) = \begin{pmatrix}c_1\\ c_2\end{pmatrix}$, then such s_1, s_2 are unique in $(-\varepsilon_1,\varepsilon_1)\times(-\varepsilon_2,\varepsilon_2)$.

Now let $\gamma(t) = p_0 + h_0(t)u + h_1(t)v_1 + h_2(t)v_2$, where $h_0(t) = (\gamma(t) - p_0)\cdot u/\|u\|^2$, $h_i(t) = (\gamma(t) - p_0)\cdot v_i/\|v_i\|^2$ $(i=1,2)$. h_1, h_2, h_0 are all smooth and $h_1(0) = h_2(0) = h_0(0) = 0$. $h_0'(0) = \dot\gamma(0)\cdot u/\|u\|^2 = 1$ by definition that $\dot\gamma(0) = X(p_0) = u$. So we can choose $t_1 < 0 < t_2$ (small enough), s.t. $h_0'(t) > 0$, set $r_1 = h_0(t_1)$, $r_2 = h_0(t_2)$, then for $\forall r\in(r_1, r_2)$, $\exists!\, t\in(t_1,t_2)$, s.t. $h_0(t) = r$. Now construct $B = \{p_0 + ru + s_1v_1 + s_2v_2 \mid r_1 < r < r_2, |s_i| < \varepsilon_i\ (i=1,2)\}$.

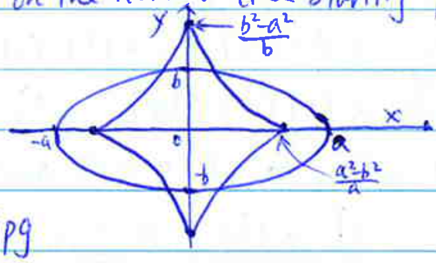
$\forall p_0 + ru + s_1v_1 + s_2v_2 \in B\cap C$, then it's belonging to $B \Rightarrow \exists!\, t\in(t_1,t_2)$, $h_0(t) = r$.

Its belonging to $C \Rightarrow \exists!\, s_1, s_2$ s.t. $p_0 + ru + s_1v_1 + s_2v_2 \in C$. Now that we know $p_0 + h_0(t)u + h_1(t)v_1 + h_2(t)v_2 \in \gamma \subseteq C$, so $s_1 = h_1(t)$, $s_2 = h_2(t)$. So $p_0 + ru + s_1v_1 + s_2v_2 \in \gamma$. i.e. $C\cap B \subseteq \gamma$

16.1 (a) By using the result of Ex 10.4 (b) at $p = (a\cos t, b\sin t)$, the curvature for outward orientation is $k(p) = -ab\left(\frac{a^2}{b^2}x_2^2 + \frac{b^2}{a^2}x_1^2\right)^{-\frac{3}{2}}$, $N = (a^2\sin^2 t + b^2\cos^2 t)^{-\frac{1}{2}}(b\cos t, a\sin t)$.

By applying Thm 1, the focal point on the normal line starting from p is $p + \frac{1}{k(p)}N(p)$
$= \left(\frac{a^2-b^2}{a}\cos^3 t, \frac{b^2-a^2}{b}\sin^3 t\right)$

(b) For example of $a=2, b=1$, see
http://rsise.anu.edu.au/
~xzhang/reading/ex1601.jpg



16.2 (a) Only need to prove that for q sufficiently close to p, $N(p)$ and $N(q)$ are not parallel in R^2. Otherwise, for $\forall k\in Z^+$, $\exists\, q_k\in C\cap\varepsilon(p,\varepsilon)$ (the ε-ball about p), such that $N(p)$ and $N(q_k)$ are parallel. But as N is smooth, $\boxed{\text{So } \|N(p) - N(q_k)\| = 0 \text{ or } 2}$, in the neighbourhood close enough to p, $\|N(p) - N(q_k)\|$ must be less than an arbitrary small positive number. So $N(q_k) = N(p)$, so $\underset{J_N(p+\lambda_k(q_k-p))\cdot}{\cancel{(N(q_k) - N(p))}}\frac{(q_k-p)}{\|q_k-p\|} = 0$ $(\lambda_k\in(0,1))$. As $(q_k-p)/\|q_k-p\|\in S^1$ which is compact, there must be a subsequence of $(q_k - p)/\|q_k - p\|$ which converges to v ($\|v\|=1$). Without loss of generality we assume that subsequence is $\{q_k\}$ itself. Let $k\to\infty$, We have $\nabla_v N = 0$, because $\lim_{k\to\infty}\frac{q_k - p}{\|q_k - p\|} = v$. $\lim_{k\to\infty} p + \lambda_k(q_k - p) = p$ as $q_k \to p$. $\nabla_v N = 0$ contradicts with the assumption that the curvature $k(p) \neq 0$. So for $q \in C$ sufficiently close to p, the normal lines to C at p and q intersects at some point $h(q) \in \mathbb{R}^2$

(b) First derive $h(q)$. $p + S_1 \cdot N(p) = q + S_2 \cdot N(q)$. Suppose there is a local parametrization of C about p: $\alpha(t): I \to C$, $\alpha(t_0) = p$, and suppose I is small enough s.t. $\forall t \in I$, $\alpha(t)$ satisfies (a). So to derive $h(\alpha(t))$, suppose $\alpha(t) + S_2 N(\alpha(t)) = \alpha(t_0) + S_1 N(\alpha(t_0))$ ($S_1, S_2 \in \mathbb{R}$). Multiply both sides by $\dot\alpha(t_0)$ and Notice $N(\alpha(t_0)) \cdot \dot\alpha(t_0) = 0$, so

$\alpha(t)\cdot\dot\alpha(t_0) = \cancel{\alpha(t_0)\dot\alpha(t)} + S_2 N(\alpha(t))\cdot\dot\alpha(t_0) = \alpha(t_0)\dot\alpha(t_0)$. By assumption $N(\alpha(t))$ is not parallel with $\alpha(t_0)$, so $N(\alpha(t))\cdot\alpha(t_0) \neq 0$ So $S_2 = \frac{\alpha(t_0)\cdot\dot\alpha(t_0) - \alpha(t)\dot\alpha(t_0)}{N(\alpha(t))\cdot\dot\alpha(t_0)}$

So $h(\alpha(t)) = \frac{1}{N(\alpha(t))\cdot\dot\alpha(t_0)}\left[\alpha(t)\cdot\left(N(\alpha(t))\cdot\dot\alpha(t_0)\right) + N(\alpha(t))\cdot\left(\alpha(t_0)\dot\alpha(t_0) - \dot\alpha(t_0)\alpha(t)\right)\right]$

Both numerator and denominator $\to 0$ as $t \to t_0$. So using L'Hospital's rule, the derivative of denominator is $\dot{N\circ\alpha}(t)\cdot\dot\alpha(t_0)$ which equals $-k(t_0)\|\dot\alpha(t_0)\|^2$

the derivative of numerator is $\dot\alpha(t)\cdot\left(N(\alpha(t))\cdot\dot\alpha(t_0)\right) + \alpha(t)\left[\dot{N\circ\alpha}(t)\cdot\dot\alpha(t_0)\right]$
$+ \dot{N\circ\alpha}(t)\cdot\left(\alpha(t_0)\cdot\dot\alpha(t_0)\right) - \dot{N\circ\alpha}(t_0)\cdot\left[\dot\alpha(t_0)\cdot\alpha(t)\right] - N(\alpha(t))\cdot\left[\dot\alpha(t_0)\cdot\dot\alpha(t)\right]$

when $t = t_0$, it is equal to $\alpha(t_0)\cdot\left(-k(t_0)\|\dot\alpha(t_0)\|^2\right) - k(t_0)\dot\alpha(t_0)\left(\alpha(t_0)\cdot\dot\alpha(t_0)\right)$
$+ k(t_0)\dot\alpha(t_0)\left[\dot\alpha(t_0)\cdot\alpha(t_0)\right] - N(\alpha(t_0))\cdot\|\dot\alpha(t_0)\|^2$

So $\lim_{t\to t_0} h(\alpha(t)) = -\|\dot\alpha(t_0)\|^2\left(N\circ\alpha(t_0) + k(t_0)\cdot\alpha(t_0)\right) / \left[-k(t_0)\|\dot\alpha(t_0)\|^2\right]$

$= \alpha(t_0) + \frac{1}{k(t_0)}(N\circ\alpha)(t_0)$ $\quad (\alpha(t_0) = p)$

By Thm 1, this is the focal point of C along the normal line through p.

16.3 (a) $\dot\alpha(t) = \varphi'(t) + \frac{1}{k^2(t)}\left[(\dot{N\circ\varphi})(t)\,k(t) - k'(t)\,(N\circ\varphi)(t)\right]$

As $(\dot{N\circ\varphi})(t) = -L_p(\dot\varphi(t)) = -k(t)\dot\varphi(t)$.

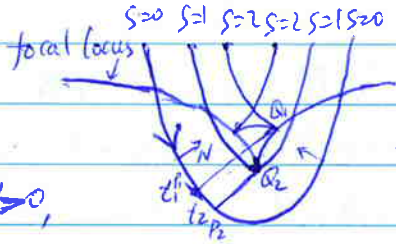
So $\dot\alpha(t) = \frac{-1}{k^2(t)}k'(t)(N\circ\varphi)(t)$. So $\dot\alpha(t) = 0$ iff $k'(t) = 0$

(b) As $\dot\alpha(t)$ is parallel to $(N\circ\varphi)(t)$ and by definition $\alpha(t)$ is on the normal line to Image φ at $\varphi(t)$ and by Thm 1, $\alpha(t)$ is the focal locus of φ. So the latter is tangent at $\alpha(t)$ to the focal locus of φ

(c) The sum is $Q = \int_{t}^{t_2}\|\dot\alpha(t_1)\|\,dt_1 + \|\alpha(t) - \varphi(t)\|$. Suppose $k'(t) = b\|k'(t)\|$ and $k(t) = a\|k(t)\|$ where $a, b \in \{\pm 1\}$ as both $k(t)$ and $k'(t)$ keep their sign for $t \in (t_1, t_2)$. So $\frac{d}{dt}Q = -\|\dot\alpha(t)\| + \frac{d}{dt}\left|\frac{1}{k(t)}\right| = -\|\dot\alpha(t)\| + a\frac{d}{dt}\frac{1}{k(t)}$

$= \frac{-1}{k^2(t)}ab\,k'(t) + a\frac{-1}{k^2(t)}k'(t)$. Notice that if $a\cdot b = 1$ then the conclusion in this exercise doesn't hold. Otherwise if $k'(t)\cdot k(t) < 0$. $\frac{d}{dt}Q = 0$ so Q = constant

An example of $ab=1$ is the ~~fo~~ parabola. If we parametrize by $\varphi(t)=(t,\frac{1}{2}t^2)$ $t\in(-\infty, \cancel{\ } 0)$
then $\dot\varphi(t)=(1,t)$, $N=\frac{1}{\sqrt{t^2+1}}(-t,+1)$, $k(t)=((1+t^2)^{3/2})_{>0}$, $k>0$,
then $|P_1Q_1|+|\text{length of } \alpha\ \alpha(t_1)\to\alpha(t_2)| > |P_2Q_2|+0$. can't be constant.
So we will need $kk' \overset{<}{\cancel{>}} 0$, like what the Figure 16.6 shows



Q_2 is a little left to this point

16.4 (a) Let $\psi(s,t)=\varphi(t)+sN(\varphi(t))$. For each $s<\frac{1}{k(t_0)}$, $\varphi_s(t_0)$ is not a focal point by Thm 1, so $I_s\neq\phi$. If $t_0\in I_s$, then $\varphi_s'(t_0)\neq 0$. As φ_s' is continuous, there must be $\varepsilon>0$ s.t. $\forall t\in(t_0-\varepsilon,t_0+\varepsilon)$, $\varphi_s'(t)\neq 0$, i.e. $t\in I_s$. Thus I_s is open.

(b) Suppose $\varphi(t)$ is unit speed, which doesn't lose generality as the conclusion only takes care of t_0. $\varphi_s(t)=\varphi(t)+sN(\varphi(t))$, $k(t_0)=\ddot\varphi(t_0)\cdot N(\varphi(t_0))$

$\varphi_s'(t)=\varphi'(t)+s(\dot N\circ\varphi)(t)$ As $\varphi_s'(t)\cdot(N\circ\varphi)(t)=(\varphi'(t)+s(\dot N\circ\varphi)(t))\,(N\circ\varphi)(t)$

by definition $\varphi'(t)\cdot(N\circ\varphi)(t)=0$. $\|(N\circ\varphi)(t)\|=\overset{\text{const}}{\cancel{\ }}\Rightarrow (\dot N\circ\varphi)(N\circ\varphi)=0$, so $\varphi_s'(t)\cdot(N\circ\varphi)(t)=0$

~~So $N_s(\varphi_s(t))=N(\varphi(t))$~~. To check the direction, we notice that

$\varphi_s'(t)\cdot\varphi'(t)=(\varphi'(t)+s\cdot(\dot N\circ\varphi)(t))\cdot\varphi'(t)=\|\varphi'(t)\|^2+s(-k\|\varphi'(t)\|^2)$

As $s<\frac{1}{k(t_0)}$. So if $k(t_0)>0$, then $\varphi_s'(t)$ is in the same direction as $\varphi'(t)$, and $N_s(\varphi_s(t))=N(\varphi(t))$. If $k(t_0)<0$, then $N_s(\varphi_s(t))=-N(\varphi(t))$.

1°. $k(t_0)>0$, $k_s(t_0)=\ddot\varphi_s(t_0)\,N_s(\varphi_s(t_0))/\|\varphi_s'(t_0)\|^2$

$$=(\ddot\varphi(t_0)+s\,(\ddot N\circ\varphi)(t_0))\,N(\varphi(t_0))\,/\,\|\varphi_s'(t_0)\|^2$$

$\ddot\varphi(t_0)\cdot N(\varphi(t_0))=k(t_0)$. Besides, as $-(\dot N\circ\varphi)=k\cdot\dot\varphi$,

so $-(\ddot N\circ\varphi)=k'\varphi'+k\cdot\ddot\varphi$, so $-(\ddot N\circ\varphi)(N\circ\varphi)=k\cdot\ddot\varphi(N\circ\varphi)=+k^2$

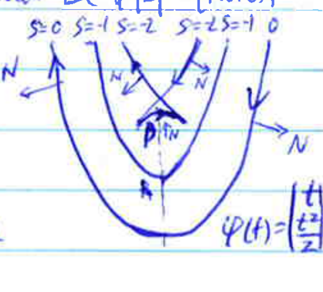
So $k_s(t_0)=(k(t_0)-sk^2(t_0))/\|\varphi_s'(t_0)\|^2$

$\varphi_s'(t_0)=\varphi'(t_0)+s(\dot N\circ\varphi)(t_0)=\varphi'(t_0)+s\cdot(-k(t_0)\varphi'(t_0))$. So $\|\varphi_s'(t_0)\|=|1-sk(t_0)|$

So $k_s(t_0)=(k(t_0)-sk^2(t_0))/(1-sk(t_0))^2=(\frac{1}{k(t_0)}-s)^{-1}$

2° $k(t_0)<0$. $k_s(t_0)=-\ddot\varphi_s(t_0)\,N(\varphi(t_0))/\|\varphi_s'(t_0)\|$ similar to above, we have ~~$\ddot\varphi_s(t_0)=$~~ $k_s(t_0)=-(\frac{1}{k(t_0)}-s)^{-1}$ So we suspect that it should be $|s|<|\frac{1}{k(t_0)}|$ or simply assume $k(t_0)>0$. To double check we are correct, see parabola again and $\varphi(t_0)=(0,0)$, $y=\frac{1}{2}x^2$. $k(0)=-1$, $\varphi(t)=\binom{t}{\frac{1}{2}t^2}$

Let $s=-2<1/k(0)$. at P, the curvature should still be negative, while the conclusion in the textbook exercise insists $k_{-2}(0)=\frac{1}{-1-(-2)}=1>0$



16.5 (a) ~~证~~ $J\Psi|_{t=0} = \left(N^S(\alpha(0)), \dot\alpha(0) + s(N\dot\circ\alpha)(0)\right)$

so $X(s) = J\Psi|_{t=0}\cdot\binom{0}{1} = \dot\alpha(0) + s(N\dot\circ\alpha)(0)$

$X(0) = \dot\alpha(0) = v$. $\dot X(s) = (N\dot\circ\alpha)(0)$ So $\dot X(0) = (N\dot\circ\alpha)(0) = -L_p(v)$

(b) $\ddot X(s) = 0$. So $X(s) = \left(\Psi(s,0), \dot\alpha(0) + s(N\dot\circ\alpha)(0)\right) = (\beta(s), v + sw)$

(c) $X(s) = 0 \Leftrightarrow$ ~~$\dot\alpha(0)$~~ $v = -s\,(N\dot\circ\alpha)(0) = s L_p(v)$

So $\frac{1}{s}$ is a principal curvature and v is a principal curvature direction

By Thm 1, $\Leftrightarrow \alpha(0) + \frac{1}{\frac{1}{s}}\cdot N^S(\alpha(t)) = \alpha(0) + s N^S(\alpha(t)) = \beta(s)$ is focal point of S along β.